Relations between Sarkisov links of surfaces over a perfect field

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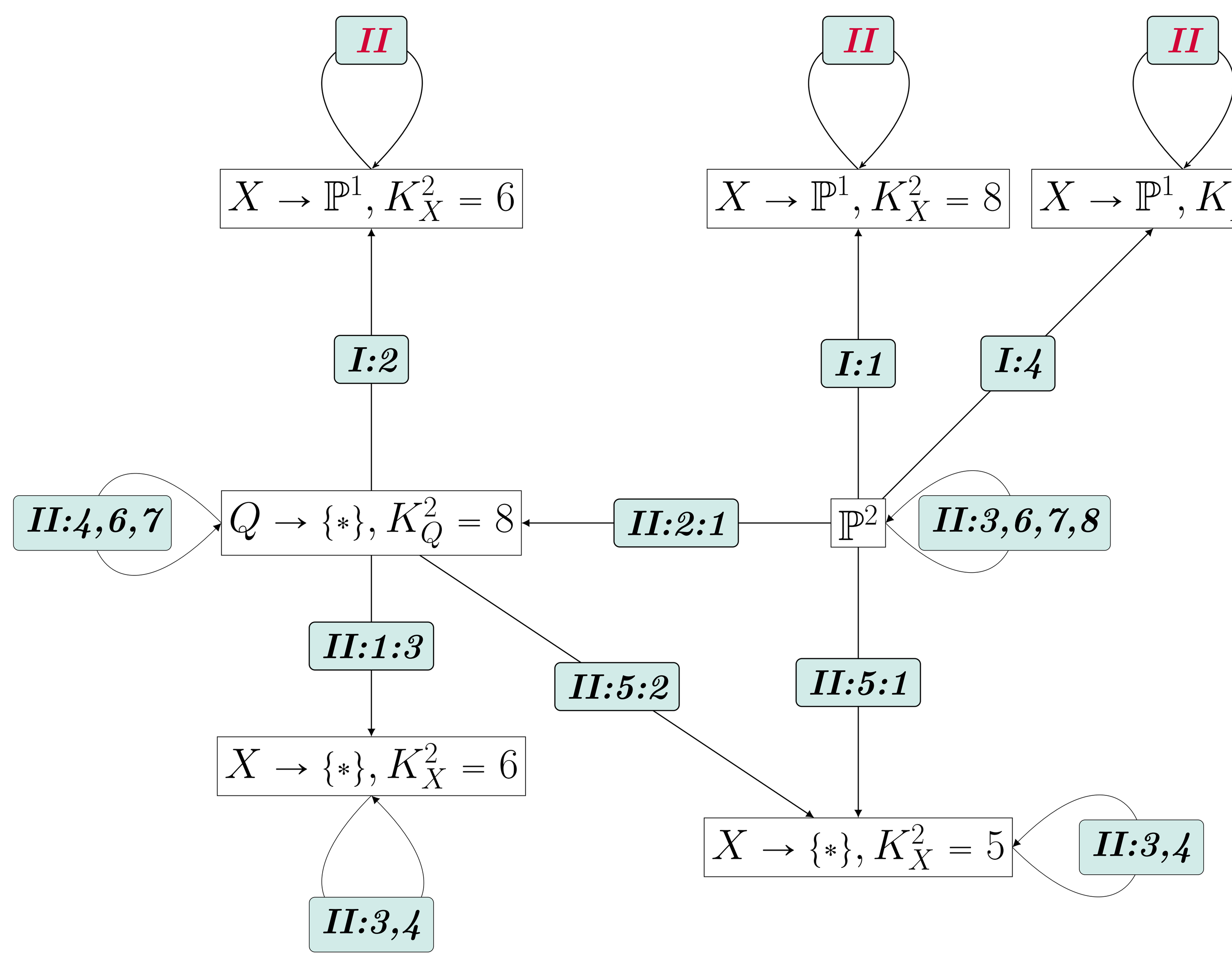
Abstract

The Cremona group $\mathrm{Cr}_n(\mathbf{k}) = \mathrm{Bir}(\mathbb{P}^2)$ is the group of birational transformations of the projective n-space over a field $\mathbf{k}$. In dimension $n = 2$ it is known [CLC13, Lon16] that the Cremona group over any field is not simple. It was recently shown [BLZ19] that the same holds over (subfields of) the field of complex numbers for higher dimensions $n \geqslant 3$. The goal of this article is to adapt the strategy of [BLZ19] to dimension two over perfect fields and find new normal subgroups of $Cr_2(\mathbf{k})$.

1. Introduction

Let $\mathbf{k}$ be a perfect field. (Readers unfamiliar with perfect fields are encouraged to think of $\mathbf{k} = \mathbb{Q}$.) We are interested in birational maps between minimal surfaces and their factorization into "simple" maps. Whereas points can be blown up over algebraically closed fields, over perfect fields one has to blow up an entire orbit of $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$.

Example. *Any birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ can be factorized into links as depicted in the following figure (following the classification of Sarkisov links [Isk96]):*



The $X \to \mathbb{P}^1$ with $K_X^2 = 8$ are Hirzebruch surfaces, and the $Q \to \{\star\}$ are quadrics in $\mathbb{P}^3$. The arrows denote the direction of the link, labels of the form "I:d" denote a blow-up of an orbit of size d and "II:a:b" denotes a link that is given by the blow-up of an orbit of size a, followed by the contraction to an orbit of size b.

Note that not all surfaces that arise in the figure of the example are minimal surfaces. For example, the first Hirzebruch surface $\mathbb{F}_1$ appears, but is not minimal since it is the blow-up of $\mathbb{P}^2$ in one point. However, all surfaces are *Mori fiber spaces* and we are interested in studying "simple" birational maps between Mori fiber spaces, namely *Sarkisov links*.

Definition. *A morphism $\pi\colon X \to B$, where X is a smooth surface and B is smooth, is called a Mori fiber space if the following conditions are satisfied:*

(1) $\dim(B) < \dim(X)$,

(2) $\mathrm{rk}_{\mathbf{k}} \mathrm{Pic}(X/B) = 1$ *(where $\mathrm{Pic}(X/B) = \mathrm{Pic}(X)/\pi^* \mathrm{Pic}(B)$ is the relative Picard group),*

(3) $-K_X \cdot D > 0$ *for all $D \in \mathrm{Pic}(X/B)$.*

There are only two possibilities:

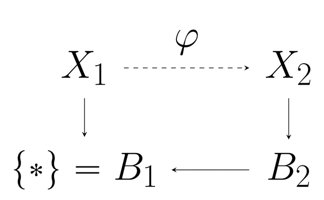
(1) $\dim(B) = 1$, so B is a curve and $X \to B$ is a conic bundle (that is a general fiber is isomorphic to $\mathbb{P}^1$ and any singular fiber is the union of two (-1)-curves intersecting at one point), or

(2) $B = \{*\}$ with $\mathrm{rk}_{\mathbf{k}} \mathrm{Pic}(X) = 1$ and so X is a del Pezzo surface.

Note that over algebraically closed fields, the only Mori fiber spaces are $\mathbb{P}^2$ and ruled surfaces (which are Hirzebruch surfaces if they are rational).

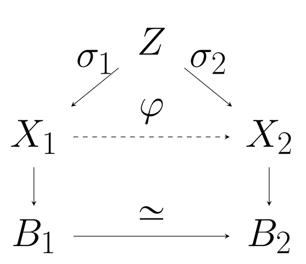
Definition. *A Sarkisov link is a birational map $\varphi\colon X_1 \dashrightarrow X_2$ between two Mori fiber spaces $\pi_i\colon X_i \to B_i$, $i = 1, 2$, that is of one of the following four types:*

Type I



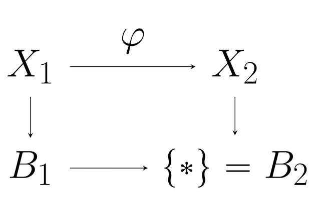
where $\varphi^{-1}\colon X_2 \to X_1$ is the blow up of an orbit.

Type II



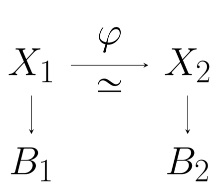
where $\sigma_i\colon Z \to X_i$ is the blow-up of an orbit.

Type III



where $\varphi\colon X_1 \to X_2$ is the blow-up of an orbit.

Type IV



where B_1 and B_2 are curves of genus 0, and φ is an isomorphism that does not preserve the fibration.

Definition. *Let $X \to V$ be a Mori fiber space. We denote by $\mathrm{BirMori}(X)$ the groupoid consisting of all birational maps $\phi\colon Y \dashrightarrow Y'$ where $Y \to W$, $Y' \to W'$ are Mori fiber spaces and Y, Y' are birational to X.*

All birational maps in $\mathrm{BirMori}(X)$ can be factorized by Sarkisov links of type I-IV [Isk96], so Sarkisov links are the "building blocks" of birational maps.

2. Results

In [LZ19] they focused on Bertini involutions, which correspond to links of type II based on an orbit of size 8 in $\mathbb{P}^2$. Our method lies in studying links of type II between conic bundles that have a "large" base orbit. For our purposes, "large" means that the cardinality of the orbit is $\geqslant 16$. (We would like it to mean $\geqslant 8$, but some technicalities deny us this pleasure.)

In fact, links of type I, III, and IV all have base orbits of size $\leqslant 8$. This is the reason why the links of type II between conic bundles are colored in red in the figure of the starting example, since these are the only birational maps that can have base orbits of cardinality $\geqslant 9$.

We will say that a birational map φ has *cardinality* d if the maximal cardinality of orbits in $\mathrm{Bs}(\varphi)$ and $\mathrm{Bs}(\varphi^{-1})$ is d.

Theorem: Generating Relations

Let X be a Mori fiber space. Relations of the groupoid $\mathrm{BirMori}(X)$ are generated by the following relations:

(a) $\varphi_n \circ \cdots \circ \varphi_1 = \mathrm{id}$, where the cardinality of all φ_i is $\leqslant 15$, and

(b) $\alpha_4\alpha_3\alpha_2\alpha_1 = \mathrm{id}$ where $\alpha_i\colon X_{i-1} \dashrightarrow X_i$ are links of type II between conic bundles of cardinality $\geqslant 16$ such that

- α_1^{-1} is a local isomorphism on the fiber on X_1 containing $\mathrm{Bs}(\alpha_2)$,
- α_3 is centered at $\alpha_2(\mathrm{Bs}(\alpha_1^{-1}))$,
- α_4 is centered at $\alpha_3(\mathrm{Bs}(\alpha_2^{-1}))$.

Theorem: Group homomorphism

Let X be a Mori fiber space. There exists a groupoid homomorphism

$$\mathrm{BirMori}(X) \to \mathop{\ast}_{C \in \mathrm{CB}(X)} \bigoplus_{\chi \in \mathrm{M}(C)} \mathbb{Z}/2\mathbb{Z}$$

that sends each Sarkisov link χ of type II between conic bundles that is of cardinality $\geqslant 16$ onto the generator indexed by its associated marked conic bundle, and all other Sarkisov links and all automorphisms of Mori fibre spaces birational to X onto zero.

Moreover it restricts to group homomorphisms

$$\mathrm{Bir}(X) \to \mathop{\ast}_{C \in \mathrm{CB}(X)} \bigoplus_{\chi \in \mathrm{M}(C)} \mathbb{Z}/2\mathbb{Z}, \qquad \mathrm{Bir}(X/W) \to \bigoplus_{\chi \in \mathrm{M}(C)} \mathbb{Z}/2\mathbb{Z}.$$

The homomorphism of the above theorem is interesting only if it is not trivial. If X is rational, the kernel of the homomorphism is a non trivial normal subgroup of the Cremona group.

Example. *Consider the birational map $\varphi\colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ that is given by*

$$[x_0 : x_1; y_0 : y_1] \mapsto [x_0 y_1^d : x_1 p(y_0, y_1); y_0 : y_1],$$

where $p \in \mathbf{k}[y_0, y_1]$ is an irreducible polynomial of degree $d \geqslant 16$. Since $\mathbf{k}$ is perfect, $p(t, 1)$ has d different zeroes $t_1, \ldots, t_d \in \bar{\mathbf{k}}$. So φ is not defined on $[1 : 0; 1 : 0]$ and on the points $p_i = [0 : 1; t_i : 1]$ for $i = 1, \ldots, d$. One can check that φ is the composition of a link of type II $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{F}_d$ centered at the orbit $\{p_1, \ldots, p_d\}$, followed by d links $\mathbb{F}_n \dashrightarrow \mathbb{F}_{n-1}$ of type II of cardinality 1 for $n = d, \ldots, 1$.

Therefore, $\varphi \in \mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^1)$ is mapped onto $(\ldots, 0, 1, 0, \ldots)$ under the group homomorphism of the above theorem, where the 1 comes from the link of type II of cardinality $d \geqslant 16$.

Corollary

For each perfect field $\mathbf{k}$ such that $[\bar{\mathbf{k}} : \mathbf{k}] > 2$, there is a surjective group homomorphism $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2) \to \mathbb{Z}/2\mathbb{Z}$ whose kernel contains $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbf{k})$.

This result was already obtained in [LZ19] if $\mathbf{k}$ has an extension of degree 8, but the normal subgroup they constructed is different. However, it contrasts with [CLC13, Lon16]: The normal subgroups of the Cremona group that they found do not contain any automorphism except the identity.

References

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