On weak homotopy equivalences between mapping spaces

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Abstract

Let S_+^n denote the n-sphere with a disjoint basepoint. We give conditions ensuring that a map $h\colon X \to Y$ that induces bijections of homotopy classes of maps $[S_+^n, X] \cong [S_+^n, Y]$ for all $n \geq 0$ is a weak homotopy equivalence. For this to hold, it is sufficient that the fundamental groups of all path-connected components of X and Y be inverse limits of nilpotent groups. This condition is fulfilled by any map between based mapping spaces $h\colon \mathrm{map}_*(B, W) \to \mathrm{map}_*(A, V)$ if A and B are connected CW-complexes. The assumption that A and B be connected can be dropped if $W = V$ and the map h is induced by a map $A \to B$.

From the latter fact we infer that, for each map f, the class of f-local spaces is precisely the class of spaces orthogonal to f and $f \wedge S_+^n$ for $n \geq 1$ in the based homotopy category. This has useful implications in the theory of homotopical localization.

0 Introduction

It is well known that unbased homotopy classes of maps from spheres are not sufficient to recognize weak homotopy equivalences in general; see Section 1 for details about this claim. Thus, there is no unbased analogue of the Whitehead theorem, stating that a map $h \colon X \to Y$ between connected CW-complexes that induces bijections of based homotopy classes of maps $[S^n, X] \cong [S^n, Y]$ for all n is a homotopy equivalence [14, Theorem V.3.5].

In fact, there is no set of spaces K_α such that maps between CW-complexes inducing bijections of unbased homotopy classes of maps from K_α for all α are necessarily homotopy equivalences. This was proved by Heller in [11, Corollary 2.3]. (Of course, any family of representatives of *all* homotopy types of CW-complexes suffices to recognize homotopy equivalences, but this is a proper class, not a *set*.)

On the other hand, in the homotopy theory praxis it is frequent to encounter situations where one would like to prove that certain maps between function spaces are homotopy equivalences; see e.g. [2], [9]. This can be an arduous task, since function spaces usually fail to be path-connected and their components can be of distinct homotopy types. The results in this article aim to simplify this task whenever possible.

We denote by $[A, X]$ the set of based homotopy classes of maps from A to X, and by A_+ the union of A with a disjoint basepoint. Thus, $[S^n_+, X]$ is identified with the set of unbased homotopy classes of maps from the n-sphere S^n to X. All spaces, including function spaces, are endowed with the compactly generated topology.

Suppose that a map $h \colon X \to Y$ induces bijections $[S^n_+, X] \cong [S^n_+, Y]$ for all n. In Section 1 we prove that such a map h is a weak homotopy equivalence if and only if the induced homomorphism of fundamental groups, $h_* \colon \pi_1(X, x) \to \pi_1(Y, h(x))$, is surjective for every choice of a point $x \in X$. This condition is fulfilled in many important cases, namely

- if $\pi_1(Y, y)$ is nilpotent for all y (see Theorem 1.8 below), or also

- if $\pi_1(X, x)$ and $\pi_1(Y, y)$ are both $H\mathbf{Z}$-local for all x and y (see Theorem 4.2).

The reader is referred to [4] and [5] for a discussion of $H\mathbf{Z}$-local groups.

From these observations we infer the following general result about maps between function spaces.

Theorem 0.1 *Let $h\colon \mathrm{map}_*(B, X) \to \mathrm{map}_*(A, Y)$ be any map of based function spaces, where A and B are connected CW-complexes. Then the following statements are equivalent:*

(1) *h is an integral homology equivalence.*

(2) *h is a weak homotopy equivalence.*

(3) *h induces bijections $[S_+^n, \mathrm{map}_*(B, X)] \cong [S_+^n, \mathrm{map}_*(A, Y)]$ for $n \geq 0$.*

Of course, this is not true if we remove the assumption that A and B be connected, as every space X is homeomorphic to $\mathrm{map}_*(S^0, X)$. However, using other methods, we prove the following.

Theorem 0.2 *Let $f\colon A \to B$ be any map between (not necessarily connected) CW-complexes, and let $h\colon \mathrm{map}_*(B, X) \to \mathrm{map}_*(A, X)$ be induced by f, where X is any space. Then h is a weak homotopy equivalence if and only if it induces bijections*

$$[S_+^n, \mathrm{map}_*(B, X)] \cong [S_+^n, \mathrm{map}_*(A, X)] \quad \text{for } n \geq 0. \tag{0.1}$$

In view of these results, it is tempting to believe that a map $h\colon X \to Y$ inducing bijections $[S_+^n, X] \cong [S_+^n, Y]$ for all n is necessarily an integral homology equivalence. We show that this is not the case, by exhibiting a counterexample in Section 1.

Our main motivation for embarking in this study was Dror Farjoun's approach to homotopical localization [9], [10]. For a map $f\colon A \to B$ of CW-complexes, a space X is called f-local [9] if the map of function spaces $\mathrm{map}_*(B, X) \to \mathrm{map}_*(A, X)$ induced by f is a weak homotopy equivalence. Thus, Theorem 0.2 asserts precisely that unbased homotopy classes of maps from spheres suffice to recognize f-local spaces. Moreover, note that (0.1) can also be written as

$$[B \wedge S_+^n, X] \cong [A \wedge S_+^n, X] \quad \text{for } n \geq 0. \tag{0.2}$$

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The fact that (0.2) characterizes f-local spaces is very useful in certain constructions of homotopy idempotent functors. Indeed, the results contained in a preliminary version of this article have been exploited in [10, p. 14].

Similarly, if A is any CW-complex, then a map $g: X \to Y$ is said to be an A-equivalence if the arrow $\mathrm{map}_*(A, X) \to \mathrm{map}_*(A, Y)$ induced by g is a weak homotopy equivalence [3], [10, Section 2.A]. From Theorem 0.1 it follows that unbased homotopy classes of maps from spheres suffice again to characterize A-equivalences, provided that the space A is connected. This is useful, for instance, in the context of [10, p. 54].

1 Unbased homotopy classes of maps

We keep denoting by X_+ the union of a space X with a disjoint basepoint. Recall from [14, Section III.1] that, if a space Y is path-connected, then for each space X the set $[X_+, Y]$ of unbased homotopy classes of maps from X to Y can be identified with the set of orbits of $[X, Y]$ under the usual action of $\pi_1(Y)$. In particular, $[S^1_+, Y]$ corresponds bijectively with the set of conjugacy classes of elements in $\pi_1(Y)$.

A map $h: X \to Y$ between topological spaces is a weak homotopy equivalence if it induces a bijection of path-connected components $\pi_0(X) \cong \pi_0(Y)$ together with isomorphisms

$$\pi_n(X, x) \cong \pi_n(Y, h(x)) \quad \text{for } n \geq 1 \text{ and every } x \in X. \tag{1.1}$$

Even though it might seem plausible, condition (1.1) cannot be replaced by the condition that h induces bijections

$$[S^n_+, X] \cong [S^n_+, Y] \quad \text{for } n \geq 1. \tag{1.2}$$

The following source of counterexamples is extracted from [11, § 2].

Example 1.1 Let N be a torsion-free group such that any two nontrivial elements are conjugate; embed N into a larger group G with a single nontrivial conjugacy

class of elements as well (this can be done by iterating suitable HNN constructions; see [13, Theorem 6.4.6]). Then the induced map of classifying spaces $h: BN \to BG$ induces bijections $[S_+^n, BN] \cong [S_+^n, BG]$ for all n. However, h is not a weak homotopy equivalence, as it fails to be surjective on the fundamental group.

Constructions of this kind also serve to discard the belief that a map $h: X \to Y$ inducing bijections $[S_+^n, X] \cong [S_+^n, Y]$ for $n \geq 0$ is an integral homology equivalence. Here is a counterexample.

Example 1.2 Let G be the union of an ascending chain of groups

$$N = N_0 \subset N_1 \subset N_2 \subset \ldots$$

where, for each $i \geq 0$, the group N_{2i} has precisely one nontrivial conjugacy class of elements and N_{2i+1} is acyclic. Then the inclusion of N into G induces bijections $[S_+^n, BN] \cong [S_+^n, BG]$ for all n. Yet, BG is acyclic and BN need not be.

We next give conditions under which (1.2) suffices to guarantee that h is a weak homotopy equivalence.

Lemma 1.3 *Let G be any group and let $\varphi: A \to B$ be a $\mathbf{Z}G$-module homomorphism inducing a bijection of orbits. Then φ is an isomorphism.*

Proof. If $\varphi(a) = 0 = \varphi(0)$, then a is in the orbit of 0 and hence $a = 0$. This shows that φ is a monomorphism. Moreover, for every $b \in B$ we may write $b = x \cdot \varphi(a) = \varphi(x \cdot a)$ for some $a \in A$ and $x \in G$, showing that φ is an epimorphism. ♯

Theorem 1.4 *Suppose that a map $h: X \to Y$ induces bijections $[S_+^n, X] \cong [S_+^n, Y]$ for $n \geq 0$. Then h is a weak homotopy equivalence if and only if the induced homomorphism of fundamental groups is surjective on each path-connected component.*

Proof. One implication is obvious. To prove the converse, we may assume without loss of generality that X and Y are path-connected. By assumption, the homomorphism $h_*: \pi_1(X) \to \pi_1(Y)$ induces a bijection of conjugacy classes. Then h_* is a

monomorphism, since $h_*(x) = 1 = h_*(1)$ forces $x = 1$; hence, our additional assumption guarantees that h_* is in fact an isomorphism. Now, for each $n \geq 2$, we have a homomorphism $h_* \colon \pi_n(X) \to \pi_n(Y)$ of $\mathbf{Z}G$-modules, where $G = \pi_1(X) \cong \pi_1(Y)$, and each of these is bijective on orbits. It then follows from Lemma 1.3 that h induces isomorphisms of all homotopy groups. ♯

We denote the lower central series of a group G by $\Gamma^0 G = G$ and $\Gamma^i G = [G, \Gamma^{i-1} G]$ for $i \geq 1$. The proof of the next result is an exercise on commutator calculus and induction, that we omit.

Lemma 1.5 *Suppose that a group homomorphism $\varphi \colon G \to K$ is surjective on conjugacy classes. Then, for every $i \geq 1$, each element $y \in K$ can be written as $y = \gamma_i\, \varphi(\xi_i)$, where $\gamma_i \in \Gamma^i K$ and $\xi_i = \eta_{i-1}\xi_{i-1}$ with $\eta_{i-1} \in \Gamma^{i-1} G$.* ♯

As an immediate consequence, we have

Proposition 1.6 *If a group homomorphism $\varphi \colon G \to N$ is surjective on conjugacy classes and N is nilpotent, then φ is an epimorphism.* ♯

We also record the following variation, which will be used later.

Proposition 1.7 *Suppose given a commutative diagram*

$$
\begin{array}{ccccccccccc}
G & \ldots & \to & G_{i+1} & \overset{\alpha_i}{\to} & G_i & \overset{\alpha_{i-1}}{\to} & \ldots & \to & G_2 & \overset{\alpha_1}{\to} & G_1 \\
\varphi \downarrow & & & \downarrow \varphi_{i+1} & & \downarrow \varphi_i & & & & \downarrow \varphi_2 & & \downarrow \varphi_1 \\
K & \ldots & \to & K_{i+1} & \underset{\beta_i}{\to} & K_i & \underset{\beta_{i-1}}{\to} & \ldots & \to & K_2 & \underset{\beta_1}{\to} & K_1,
\end{array}
$$

where φ is induced by passage to the inverse limit. If all G_i and K_i are nilpotent and φ is surjective on conjugacy classes, then φ is an epimorphism.

Proof. By refining the inverse systems if necessary, we may assume that G_i and K_i have nilpotency class less than or equal to i. Take any element $y \in K$, and denote it by $(y_1, y_2, y_3, \ldots)$, with $y_i \in K_i$, and $\beta_{i-1}(y_i) = y_{i-1}$. We are going to construct an element $x \in G$ such that $\varphi(x) = y$. By Lemma 1.5, we can write $y = \gamma_1\varphi(\xi_1)$ with $\gamma_1 \in \Gamma^1 K$. Set $x_1 = (\xi_1)_1 \in G_1$. Then $\varphi_1(x_1) = y_1$, since $\Gamma^1 K_1$ is trivial. Next, write

$y = \gamma_2 \varphi(\xi_2)$ with $\gamma_2 \in \Gamma^2 K$, $\xi_2 = \eta_1 \xi_1$, $\eta_1 \in \Gamma^1 G$. Set $x_2 = (\xi_2)_2$. Then $\varphi_2(x_2) = y_2$ and, moreover, $\alpha_1(x_2) = x_1$, since $\Gamma^1 G_1$ is trivial. By continuing the same way, we obtain an element $x = (x_1, x_2, x_3, \ldots) \in G$ such that $\varphi_i(x_i) = y_i$ for all i, so that $\varphi(x) = y$. $\sharp$

Note that Propositions 1.6 and 1.7 can also be proved by resorting to Lemma 4.1 below, since every inverse limit of nilpotent groups is $H\mathbf{Z}$-local.

From Theorem 1.4 and Proposition 1.6 we infer the main result of this section:

Theorem 1.8 *Let $h\colon X \to Y$ induce bijections $[S_+^n, X] \cong [S_+^n, Y]$ for $n \geq 0$. Suppose that the fundamental group of each path-connected component of Y is nilpotent. Then h is a weak homotopy equivalence.* $\sharp$

2   Maps between function spaces

Given topological spaces B and X with basepoint, we denote by $\mathrm{map}_*(B, X)$ the space of all based maps from B to X with the compactly generated topology. The space $\mathrm{map}_*(B_+, X)$ of unbased maps is denoted, as usual, by $\mathrm{map}(B, X)$. For a based map $g\colon B \to X$, we denote by $\mathrm{map}_*(B, X)_g$ the path-connected component containing g, and similarly for unbased maps.

We recall from [14, Theorem I.7.8] that, if B is well pointed, then for any X the following sequence is a fibre sequence, where the second arrow is evaluation at the basepoint:

$$\mathrm{map}_*(B, X) \to \mathrm{map}(B, X) \to X. \qquad (2.1)$$

In fact, for every map $g\colon B \to X$ we have a fibre sequence

$$\bigcup_j \mathrm{map}_*(B, X)_j \to \mathrm{map}(B, X)_g \to X, \qquad (2.2)$$

where j ranges over a set of representatives of based homotopy classes of maps such that $j \simeq g$ by an unbased homotopy.

We shall exploit the crucial fact, explained in [12, Theorem II.2.5], that if A is a connected CW-complex of finite dimension, then for every space X the path-connected components of $\mathrm{map}_*(A, X)$ are nilpotent. In view of Theorem 1.8, this

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remark proves Theorem 0.2 in the special case when A is finite-dimensional and connected. In the rest of this section we prove Theorem 0.2 in its full generality.

PROOF OF THEOREM 0.2:

Suppose given a map $f\colon A \to B$ of CW-complexes, not necessarily connected. Let X be an arbitrary space, and assume that the induced map

$$h\colon \mathrm{map}_*(B, X) \to \mathrm{map}_*(A, X)$$

gives rise to a bijection $[B, X] \cong [A, X]$ together with bijections

$$[S_+^n, \mathrm{map}_*(B, X)] \cong [S_+^n, \mathrm{map}_*(A, X)] \quad \text{for } n \geq 1.$$

We can write

$$\mathrm{map}_*(B, X) = \mathrm{map}_*(B_0, X) \times \prod_b \mathrm{map}(B_b, X),$$

where B_0 denotes the basepoint component and a point b has been chosen in each of the other connected components of B; indeed, we denote by B_b the connected component containing b. The same notation is used with A.

We start by showing that there is no restriction in assuming that f induces a bijection of connected components $\pi_0(A) \cong \pi_0(B)$. Firstly, suppose that B has a component B_b which does not intersect $f(A)$. Then the condition $[B, X] \cong [A, X]$ forces $[(B_b)_+, X]$ to be trivial. In addition, the exponential law yields

$$[B, \mathrm{map}_*(S_+^n, X)] \cong [A, \mathrm{map}_*(S_+^n, X)] \tag{2.3}$$

for all $n \geq 1$. Hence, for each $n \geq 1$, the set $[(B_b)_+, \mathrm{map}_*(S_+^n, X)]$ has a single element, and this implies that $\pi_n \mathrm{map}(B_b, X)$ is zero, since all its elements lie in a single orbit under the action of the fundamental group. It follows that $\mathrm{map}(B_b, X)$ is weakly contractible. Secondly, suppose that two components A_a, A_c map into the same component B_b. Then (2.3) forces $\mathrm{map}_*(S_+^n, X)$ to be path-connected for $n \geq 1$. This implies that $[S_+^n, X]$ is trivial for all n; therefore, $\pi_n(X)$ is also trivial for all n, and X is weakly contractible.

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We therefore assume that f induces a bijection $\pi_0(A) \cong \pi_0(B)$. Then h determines a collection of maps

$$\mathrm{map}_*(B_0, X_0) \to \mathrm{map}_*(A_0, X_0)$$

$$\mathrm{map}(B_{f(a)}, X_x) \to \mathrm{map}(A_a, X_x), \tag{2.4}$$

and we are led to showing that each of these is a weak homotopy equivalence. By Theorem 1.4, it suffices to prove that the induced homomorphisms of fundamental groups are surjective. For simplicity of notation, we shall assume from now on that A, B and X are path-connected, and drop most subscripts. Using (2.2), for each choice of a based map $g \colon B \to X$ we obtain a commutative diagram with exact rows

$$
\begin{array}{ccccccc}
\pi_2(X) & \xrightarrow{\varphi_1} & \pi_1\mathrm{map}_*(B,X)_g & \xrightarrow{\varphi_2} & \pi_1\mathrm{map}(B,X)_g & \xrightarrow{\varphi_3} & \pi_1(X) \\
\downarrow= & & \downarrow \varphi_7 & & \downarrow \varphi_8 & & \downarrow= \\
\pi_2(X) & \xrightarrow[\varphi_4]{} & \pi_1\mathrm{map}_*(A,X)_{gf} & \xrightarrow[\varphi_5]{} & \pi_1\mathrm{map}(A,X)_{gf} & \xrightarrow[\varphi_6]{} & \pi_1(X).
\end{array}
$$

Lemma 2.1 *Suppose given a commutative diagram where the rows are exact,*

$$
\begin{array}{ccccc}
M & \rightarrowtail & G & \twoheadrightarrow & Q \\
\downarrow \varphi' & & \downarrow \varphi & & \downarrow \bar{\varphi} \\
N & \rightarrowtail & K & \twoheadrightarrow & R.
\end{array}
$$

Then the following hold:

(a) *If φ is surjective on conjugacy classes, so is $\bar{\varphi}$.*

(b) *If $N \subseteq \mathrm{im}\,\varphi$ and $\bar{\varphi}$ is an epimorphism, then φ is an epimorphism.*

(c) *If N is nilpotent, $\bar{\varphi}$ is an epimorphism, and φ is surjective on conjugacy classes, then φ is an epimorphism.*

(d) *If $\bar{\varphi}$ is an isomorphism and φ is surjective on conjugacy classes, then φ' is surjective on conjugacy classes.*

(e) *If $N \subseteq \mathrm{im}\,\varphi$ and $\bar{\varphi}$ is surjective on conjugacy classes, then φ is surjective on conjugacy classes.*

Proof. Parts (a) and (b) are straightforward. In order to prove (c), we show that $N \subseteq \operatorname{im} \varphi$ and apply (b). Thus, pick any element $y \in N$. By assumption, we may write $y = z\varphi(u)z^{-1}$ with $z \in K$ and $u \in G$. Then $\bar{z} = \bar{\varphi}(\bar{v})$ for some $v \in G$, and $y_1 = z\varphi(v)^{-1}$ belongs to N. Hence, if we set $x_0 = vuv^{-1}$, then we have

$$y = y_1 \varphi(vuv^{-1})y_1^{-1} = [y_1, \varphi(x_0)]\varphi(x_0),$$

where both y_1 and $\varphi(x_0)$ belong to N. By arguing as in Lemma 1.5, we find that $y = \gamma_i \varphi(\xi_i)$ for all $i \geq 1$, with $\gamma_i \in \Gamma^i N$, which finishes the argument.

We next prove (d). As in the previous part, start with an element $y \in N$ and write it as $y = y_1 \varphi(vuv^{-1})y_1^{-1}$ with $y_1 \in N$ and $u, v \in G$. Now the injectivity of $\bar{\varphi}$ ensures that $vuv^{-1} \in M$, as required. Part (e) is straightforward. ♯

In our situation, the assumption that f induces a bijection $[B, X] \cong [A, X]$ guarantees that the arrow $\operatorname{im} \varphi_3 \to \operatorname{im} \varphi_6$ is an isomorphism. Since φ_8 is surjective on conjugacy classes, the restriction

$$\operatorname{im} \varphi_2 \to \operatorname{im} \varphi_5$$

is surjective on conjugacy classes, by part (d) of the above lemma. Furthermore, the commutative diagram

$$
\begin{array}{ccccc}
\operatorname{im} \varphi_1 & \rightarrowtail & \pi_1 \operatorname{map}_*(B, X)_g & \twoheadrightarrow & \operatorname{im} \varphi_2 \\
\downarrow & & \downarrow \varphi_7 & & \downarrow \\
\operatorname{im} \varphi_4 & \rightarrowtail & \pi_1 \operatorname{map}_*(A, X)_{gf} & \twoheadrightarrow & \operatorname{im} \varphi_5,
\end{array}
$$

in view of part (e) of the above lemma, shows that φ_7 is surjective on conjugacy classes. This argument reduces our problem to the case of based mapping spaces.

Denote by A^i the ith skeleton of A, and similarly for B. Assuming that f is a cellular map, there is a commutative diagram with exact rows,

$$
\begin{array}{ccccc}
\varprojlim{}^1 \pi_2 \operatorname{map}_*(B^i, X)_g & \rightarrowtail & \pi_1 \operatorname{map}_*(B, X)_g & \twoheadrightarrow & \varprojlim \pi_1 \operatorname{map}_*(B^i, X)_g \\
\downarrow & & \varphi_7 \downarrow & & \downarrow \\
\varprojlim{}^1 \pi_2 \operatorname{map}_*(A^i, X)_{gf} & \rightarrowtail & \pi_1 \operatorname{map}_*(A, X)_{gf} & \twoheadrightarrow & \varprojlim \pi_1 \operatorname{map}_*(A^i, X)_{gf}.
\end{array}
$$

For every i, the spaces $\operatorname{map}_*(B^i, X)_g$ and $\operatorname{map}_*(A^i, X)_{gf}$ are nilpotent. By assumption, φ_7 is surjective on conjugacy classes. Thus, parts (a) and (c) of Lemma 2.1,

together with Proposition 1.7, imply that φ_7 is in fact surjective, hence completing the proof of Theorem 0.2. ♯

3   Characterizing local spaces

The half-smash product $X \rtimes Y$ of two spaces is a standard notation for $X \wedge Y_+$ (cf. [10, Section 2.D]). For a map $f \colon A \to B$, a space X is called f-local if $\operatorname{map}_*(f, X)$ is a weak homotopy equivalence [9]. Since $[S_+^n, \operatorname{map}_*(A, X)] \cong [A \rtimes S^n, X]$, Theorem 0.2 can be reformulated as follows. This answers in the affirmative a question posed in [6, p. 15].

Corollary 3.1 *Let $f \colon A \to B$ be any map between CW-complexes. Then a space X is f-local if and only if f induces a bijection $[B, X] \cong [A, X]$ together with bijections $[B \rtimes S^n, X] \cong [A \rtimes S^n, X]$ for $n \geq 1$.* ♯

In a more categorical language, this result implies the following. If $\mathcal{C}$ is any category, we say that an object X and a morphism $f \colon A \to B$ are orthogonal, as in [1] or [8], if the map of sets $\mathcal{C}(B, X) \to \mathcal{C}(A, X)$ induced by f is bijective. A class of objects $\mathcal{D}$ is called a small-orthogonality class [1, Section 1.C] if there is a set of morphisms f_α such that $\mathcal{D}$ is precisely the class of objects orthogonal to all f_α. Thus, Corollary 3.1 yields:

Corollary 3.2 *For each map f between CW-complexes, the class of f-local spaces is a small-orthogonality class in the based homotopy category.* ♯

Indeed, a space X is f-local if and only if it is orthogonal to the set consisting of f and $f \rtimes S^n$ for $n \geq 1$. This remark sheds light on Dror Farjoun's argument in [9] or [10, Section 1.B], where it is shown that the class of f-local spaces is reflective in the based homotopy category for every map f, i.e., that f-localization exists for all spaces.

4 Homology equivalences of function spaces

The possibility of the following improvement of our previous results was suggested by Dror Farjoun. The reader is referred to [4] for the definition and properties of $H\mathbf{Z}$-localization, i.e., localization with respect to ordinary integral homology. Recall that a space L is $H\mathbf{Z}$-local if every integral homology equivalence $W \to V$ of CW-complexes induces a bijection of based homotopy classes of maps $[V, L] \cong [W, L]$, and a group G is $H\mathbf{Z}$-local if and only if it is isomorphic to the fundamental group of an $H\mathbf{Z}$-local space.

Lemma 4.1 *If a group homomorphism $\varphi\colon G \to K$ between $H\mathbf{Z}$-local groups is surjective on conjugacy classes, then it is an epimorphism.*

Proof. The assumption that φ is surjective on conjugacy classes implies that the induced homomorphism of abelianizations, $\varphi_*\colon H_1(G) \to H_1(K)$, is surjective. According to [5, Corollary 2.13], a homomorphism between $H\mathbf{Z}$-local groups which becomes surjective after abelianizing is itself surjective.   ♯

In view of Theorem 1.4, we have

Theorem 4.2 *Suppose that a map $h\colon X \to Y$ induces bijections $[S^n_+, X] \cong [S^n_+, Y]$ for $n \geq 0$. If the fundamental groups of all path-connected components of X and Y are $H\mathbf{Z}$-local, then h is a weak homotopy equivalence.*   ♯

We can now prove Theorem 0.1 as a corollary.

PROOF OF THEOREM 0.1:

Let $h\colon \mathrm{map}_*(B, X) \to \mathrm{map}_*(A, Y)$ be any map between function spaces, where A and B are now assumed to be connected (and this is essential). As before, denote by A^i the ith skeleton of A and similarly for B. The space $\mathrm{map}_*(A, Y)$ is weakly equivalent to the inverse limit of the spaces $\mathrm{map}_*(A^i, Y)$, under the inclusions $A^i \hookrightarrow A^{i+1}$. Since each space $\mathrm{map}_*(A^i, Y)$ is a disjoint union of nilpotent spaces for $i \geq 1$, it follows from [4, § 12] that the space $\mathrm{map}_*(A, Y)$ is $H\mathbf{Z}$-local. Of course, we can argue the same way with $\mathrm{map}_*(B, X)$. If h is an integral homology equivalence, then,

since its source and target are $H\mathbf{Z}$-local spaces, h is a weak homotopy equivalence. The converse implication is well known, as it is also the fact that a weak homotopy equivalence induces bijections of unbased homotopy classes of maps from all spheres. To prove the converse of the latter claim in our case, observe that the fundamental group of each path-component of $\mathrm{map}_*(A, Y)$ or $\mathrm{map}_*(B, X)$ is an $H\mathbf{Z}$-local group, so that Theorem 4.2 applies. ♯

Recall that a map $g\colon X \to Y$ is said to be an A-equivalence if $\mathrm{map}_*(A, g)$ is a weak homotopy equivalence; cf. [3], [10, Section 2.A]. As a corollary of Theorem 0.1, we obtain the following.

Corollary 4.3 *Let A be any connected CW-complex. Then a map $g\colon X \to Y$ is an A-equivalence if and only if it induces a bijection $[A, X] \cong [A, Y]$ together with bijections $[A \rtimes S^n, X] \cong [A \rtimes S^n, Y]$ for $n \geq 1$.* ♯

Since $[A \rtimes S^n, X] \cong [A, \mathrm{map}(S^n, X)]$, the latter condition can of course be reformulated in terms of iterated free loops of g.

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