LIMIT-SKETCHABLE INFINITY CATEGORIES

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Abstract. A presentable ∞-category is defined as an accessible localization of an ∞-category of presheaves over some small ∞-category. In ordinary category theory, the Representation Theorem of Adámek and Rosický establishes that locally presentable categories are equivalent to categories of models of limit sketches. In this article, we prove an analogous result for ∞-categories, by showing that an ∞-category is presentable if and only if it is limit-sketchable. As a special case, we exhibit the sketchability of the ∞-category of complete dendroidal Segal spaces, which serves as a model for ∞-operads.

1. Introduction

The concept of locally presentable categories, introduced by Gabriel and Ulmer [7], emerged as an abstraction from the notion of presentations by generators and relations A locally presentable category is a cocomplete category that is generated through filtered colimits by a set of compact objects. By relaxing the requirement for cocompleteness to only include the presence of filtered colimits, the broader notion of accessible categories is obtained.

A *sketch* in the sense of Bastiani and Ehresmann [4] is defined as a small category together with a set of cones and a set of cocones in it. A model of a sketch Σ is a functor from Σ to the category of sets that sends each specified cone to a limit cone and each specified cocone to a colimit cocone. In this article, we focus on limit sketches, i.e., sketches that only include cones.

There are numerous examples in the literature of categories modeled on some limit sketch, including colored operads and models of any Lawvere theory. Through categorical logic, the models of limit sketches can be shown to be equivalent to the models of essentially algebraic theories. A pivotal result by Adámek and Rosický [1] highlights the significance of sketches, revealing that locally presentable categories are precisely categories sketchable by a limit sketch.

The primary objective of this article is to extend the equivalence between limit sketchability and presentability to ∞-categories. Joyal [9, 10, 11] and Lurie [12] explored presentability and accessibility in ∞-categories, paralleling classical definitions even though with subtle distinctions. For example, defining accessible categories as those equivalent to a higher Ind-category arises as a natural choice, though less conventional in classical contexts. In Section 2, we introduce the necessary concepts of higher category theory, culminating in discussions on presentability and accessibility.

The theory of sketches was first explored in homotopical contexts using the formalism of Quilen model categories [3, 5, 15], by considering enriched sketches and their models up to weak equivalence. In [15], Rosický studied models of finite weigthed limit sketches in combinatorial monoidal model categories, and proved that their models are again combinatorial —hence locally presentable— under mild assumptions. In the framework of ∞-categories, limit sketches first appeared in the work of Joyal [9] on quasicategories. In [12], Lurie showed that the ∞-category of models of a higher Lawvere theory (i.e., a sketch with only product cones) is presentable.

In Section 3, we expand upon Joyal's results and present a collection of examples, some of which are new, showcasing that many well-known ∞-categories are limit-sketchable. In Section 4, we write down a proof of the equivalence between models of higher limit sketches and presentable ∞-categories. Our proof relies on ideas from [1] adapted to the higher categorical context. To conclude, we collect some instances of this equivalence, including the fact that the ∞-category of complete dendroidal Segal spaces is limit-sketchable, hence presentable.

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2. Preliminaries

2.1. **Quasi-categories.** In this work, we implicitly use the formalism of quasi-categories [6, 9, 12] for ∞-category theory. Thus, we use the term ∞-*groupoid* to refer to a Kan complex. Every ∞-category $\mathcal{C}$ has a collection of objects $\mathcal{C}_0$, and we denote the fact that x is an object of $\mathcal{C}$ by writing $x \in \mathcal{C}$.

If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, the simplicial set $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D})_n = \mathbf{sSet}(\mathcal{C} \times \Delta^n, \mathcal{D})$$

is an ∞-category, which we call the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$. Here **sSet** denotes the category of simplicial sets and $\Delta^n = \mathbf{\Delta}(-, [n])$, where $\mathbf{\Delta}$ is the simplex category. We denote an object $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})_0$ by $F \colon \mathcal{C} \to \mathcal{D}$. A natural transformation $\alpha \colon \mathcal{C} \times \Delta^1 \to \mathcal{D}$ between two functors $F, G \colon \mathcal{C} \to \mathcal{D}$ is a 1-simplex of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ whose restriction to $\mathcal{C} \times \{0\}$ is F and whose restriction to $\mathcal{C} \times \{1\}$ is G.

Given two objects x, y of an ∞-category $\mathcal{C}$, we denote by $\mathrm{Map}_{\mathcal{C}}(x, y)$ the ∞-groupoid of maps (or *mapping space*) from x to y, which is defined by the following pullback of simplicial sets:

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(x,y) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \\ \Delta^0 \times \Delta^0 & \xhookrightarrow{(x,y)} & \mathcal{C} \times \mathcal{C}, \end{array}$$

where the right-hand map is obtained by applying $\mathrm{Fun}(-, \mathcal{C})$ to the map $(d_0, d_1) \colon \Delta^1 \to \Delta^0 \times \Delta^0$. We denote by $f \colon x \to y$ the fact that $f \in \mathrm{Map}_{\mathcal{C}}(x, y)_0$. A mapping space of a functor category $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G)$ has as 0-simplices the natural transformations between F and G, and we denote these by $\alpha \colon F \Rightarrow G$.

Every ∞-category $\mathcal{C}$ has a *homotopy category* $\mathrm{ho}(\mathcal{C})$, with the same objects as $\mathcal{C}$ and with $\mathrm{ho}(\mathcal{C})(x, y) = \pi_0 \mathrm{Map}_{\mathcal{C}}(x, y)$. Given a morphism $f \colon x \to y$, we denote by $[f]$ the corresponding morphism in $\mathrm{ho}(\mathcal{C})$. A morphism $f \colon x \to y$ in $\mathcal{C}$ is an *isomorphism* if it is invertible in $\mathrm{ho}(\mathcal{C})$.

As is common in the literature, we make two exceptions to this nomenclature: the isomorphisms between ∞-categories and ∞-groupoids will be called *equivalences*. In this article, we call an object or a map *unique* when it is unique up to isomorphism, i.e., the space it inhabits is contractible. For example, the inverse of an isomorphism is unique. Given two composable morphisms $f \colon x \to y$ and $g \colon y \to z$ in $\mathcal{C}$, there exists a unique morphism $h \colon x \to z$ such that $[h] = [g] \circ [f]$. Composition can also be studied at the level of mapping spaces, where there is a unique (up to natural isomorphism) composition functor

$$- \circ - : \mathrm{Map}_{\mathcal{C}}(y, z) \times \mathrm{Map}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Map}_{\mathcal{C}}(x, z)$$

defined by the construction given in [14, § 45.6]. It has the expected properties; namely, it is associative up to homotopy and it matches with the composition defined in $\mathrm{ho}(\mathcal{C})$.

2.2. **Cardinality assumptions.** The main concepts studied in this article are related to sizes of ∞-categories. For an infinite cardinal κ, an ∞-groupoid $\mathcal{X}$ is called κ-*small* if $\pi_n(\mathcal{X})$ has cardinality smaller than κ for all $n \geq 0$. An ∞-category is called *locally* κ-*small* if all its mapping spaces are κ-small ∞-groupoids. Furthermore, an ∞-category is called κ-*small* if it is locally κ-small and its set of isomorphism classes of objects has cardinality smaller than κ. This definition is found with the name of essentially small ∞-category in some references such as [12], but we follow the conventions of [2, 6].

We assume the existence of a strongly inaccessible cardinal κ, and call small sets (or sometimes just sets) the sets with cardinality smaller than κ. An ∞-category will be called *small* (resp. *locally small*) if it is κ-small (resp. locally κ-small). The locally small ∞-category of all small ∞-groupoids is denoted by $\mathcal{S}$, and the one of all small ∞-categories is denoted by ∞-**Cat**.

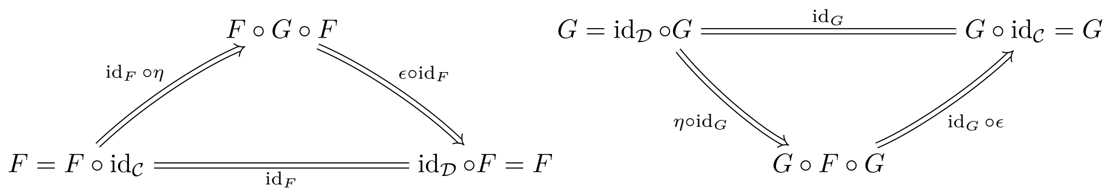
If $\mathcal{K}$ is small and $\mathcal{C}$ is locally small, then $\mathrm{Fun}(\mathcal{K}, \mathcal{C})$ is a locally small ∞-category [12, Example 5.4.1.8]. Throughout this paper, unless explicitly specified, all ∞-categories are assumed to be locally small. In the case where we need some ∞-category which is not necessarily locally small, it will be called a *large* ∞-category.

2.3. **Adjunctions and the Yoneda lemma.** We say that a functor $F \colon \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if for every object $y \in \mathcal{D}$ there exists an object $x \in \mathcal{C}$ together with an isomorphism $y \cong Fx$. We say that F is *fully faithful* if the map

$$\mathrm{Map}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(Fx, Fy)$$

is an equivalence for every pair of objects $x, y \in \mathcal{C}$. We say that $\mathcal{K}$ is a *full subcategory* of $\mathcal{C}$ if there is a fully faithful functor $J \colon \mathcal{K} \to \mathcal{C}$. Sometimes we refer to J as the *inclusion* of $\mathcal{K}$ into $\mathcal{C}$.

Adjunctions between functors can be generalized to the setting of ∞-categories using the following characterization of the unit and the counit. Given two functors $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$, we say that F is a *left adjoint* to G (or that G is a *right adjoint* to F) if there exist natural transformations $\eta \colon \mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\epsilon \colon F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$ such that the following diagrams commute:



In this case, we refer to η as the *unit* of the adjunction and to ϵ as the *counit* of the adjunction.

Every simplicial set X has an opposite simplicial set X^{op} [6, Definition 1.5.7], defined for each $n \in \mathbb{N}$ by $(X^{\mathrm{op}})_n = X_n$,

$$d_i^{\mathrm{op}} = d_{n-i} : (X^{\mathrm{op}})_n \to (X^{\mathrm{op}})_{n-1} \quad \text{and} \quad s_i^{\mathrm{op}} = s_{n-i} : (X^{\mathrm{op}})_n \to (X^{\mathrm{op}})_{n+1}.$$

The opposite simplicial set satisfies that $(X^{\mathrm{op}})^{\mathrm{op}} = X$. If a simplicial set $\mathcal{C}$ is an ∞-category, then the opposite simplicial set $\mathcal{C}^{\mathrm{op}}$ is also an ∞-category with the same objects as $\mathcal{C}$, but with $\mathrm{Map}_{\mathcal{C}^{\mathrm{op}}}(x, y) = \mathrm{Map}_{\mathcal{C}}(y, x)^{\mathrm{op}}$. We call $\mathcal{C}^{\mathrm{op}}$ the *opposite* ∞*-category* of $\mathcal{C}$.

For any small ∞-category $\mathcal{K}$, we denote $\mathrm{PSh}(\mathcal{K}) = \mathrm{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{S})$ and call it the ∞-category of *presheaves* on $\mathcal{K}$.

Theorem 2.1 (Yoneda Embedding [12, Proposition 5.1.3.1])**.** *For every ∞-category $\mathcal{C}$, there exists a unique functor*

$$h_\bullet \colon \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

such that $h_x(y) \cong \mathrm{Map}_{\mathcal{C}}(y, x)$ for all $x, y \in \mathcal{C}$. This functor $h_\bullet$ is fully faithful.

We refer to $h_\bullet$ as the *covariant Yoneda embedding*.

Theorem 2.2 (Yoneda Lemma [12, Lemma 5.5.2.1])**.** *Let $\mathcal{K}$ be a small ∞-category and $x \in \mathcal{K}$ be any object. For any functor $F \colon \mathcal{K}^{\mathrm{op}} \to \mathcal{S}$, there is an isomorphism*

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{S})}(h_x, F) \cong Fx.$$

Conversely, the covariant Yoneda embedding applied to the opposite of an ∞-category $\mathcal{C}$ yields a unique (and fully faithful) functor

$$h^\bullet \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{S})$$

such that $h^x(y) \cong \mathrm{Map}_{\mathcal{C}}(x, y)$ for all $x, y \in \mathcal{C}$. We refer to $h^\bullet$ as the *contravariant Yoneda embedding*. It satisfies the dual of the Yoneda Lemma: for every functor $G \colon \mathcal{K} \to \mathcal{S}$, there is an isomorphism

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{K}, \mathcal{S})}(h^x, G) \cong Gx.$$

2.4. **Limits and colimits.** Let $\mathcal{K}$ be a small ∞-category. A $\mathcal{K}$*-diagram* in an ∞-category $\mathcal{C}$ is a functor $\mathcal{K} \to \mathcal{C}$. For any small ∞-category $\mathcal{K}$ and any object $x \in \mathcal{C}$, the constant diagram $\Delta x \colon \mathcal{K} \to \mathcal{C}$ sends all objects of $\mathcal{K}$ to x and higher morphisms of $\mathcal{K}$ to higher identities over x, i.e., the iterated application of the first degeneracy over x. By applying $\mathrm{Fun}(-, \mathcal{C})$ to the terminal map $\mathcal{K} \to \Delta^0$, we obtain the *diagonal functor* $\Delta \colon \mathcal{C} \to \mathrm{Fun}(\mathcal{K}, \mathcal{C})$, which sends any object $x \in \mathcal{C}$ to the constant diagram Δx, and any morphism $f \colon x \to y$ to a natural transformation $\Delta f \colon \Delta x \to \Delta y$ defined by post-composition with f.

Definition 2.3. Let $\mathcal{K}$ be a small ∞-category and $\mathcal{C}$ an ∞-category. A $\mathcal{K}$*-cone* in $\mathcal{C}$ is a triple (D, x, α), where $D \colon \mathcal{K} \to \mathcal{C}$ is a $\mathcal{K}$-diagram, $x \in \mathcal{C}$ is the *apex* and $\alpha \colon \Delta x \Rightarrow D$ is a natural transformation. Moreover, a $\mathcal{K}$-cone (D, x, α) is a *limit $\mathcal{K}$-cone* if, for all $y \in \mathcal{C}$, the map

$$\mathrm{Map}_{\mathcal{C}}(y, x) \xrightarrow{\Delta} \mathrm{Map}_{\mathrm{Fun}(\mathcal{K}, \mathcal{C})}(\Delta y, \Delta x) \xrightarrow{\alpha \circ -} \mathrm{Map}_{\mathrm{Fun}(\mathcal{K}, \mathcal{C})}(\Delta y, D)$$

is an equivalence of ∞-groupoids.

A $\mathcal{K}$-cocone in $\mathcal{C}$ is a $\mathcal{K}$-cone in the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$, and a *colimit cocone* is a limit cone in $\mathcal{C}^{\mathrm{op}}$. The following results show that the apex of a limit cone is unique in the infinity categorical sense.

Lemma 2.4. *Let $\mathcal{C}$ be an ∞-category, $\mathcal{K}$ a small ∞-category, $D\colon \mathcal{K} \to \mathcal{C}$ a diagram, and $f\colon x \to y$ a morphism in $\mathcal{C}$. If there exists a $\mathcal{K}$-cone $\alpha\colon \Delta y \to D$ and we denote by $\beta\colon \Delta x \to D$ a composition of α with $\Delta f\colon \Delta x \to \Delta y$, then any two of the following three properties implies the third:*

1. *α exhibits y as a limit of the diagram D.*
2. *β exhibits x as a limit of the diagram D.*
3. *The morphism $f\colon x \to y$ is an isomorphism.*

Proof. By definition of f, α and β, the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{Map}_{\mathcal{C}}(z,y) & \xrightarrow{\Delta_{z,y}} & \mathrm{Map}_{\mathrm{Fun}(\mathcal{K},\mathcal{C})}(\Delta z, \Delta y) & \xrightarrow{\alpha\circ -} & \\ {\scriptstyle f\circ -}\uparrow & & \uparrow{\scriptstyle \Delta f\circ -} & & \mathrm{Map}_{\mathrm{Fun}(\mathcal{K},\mathcal{C})}(\Delta z, D). \\ \mathrm{Map}_{\mathcal{C}}(z,x) & \xrightarrow{\Delta_{z,x}} & \mathrm{Map}_{\mathrm{Fun}(\mathcal{K},\mathcal{C})}(\Delta z, \Delta x) & \xrightarrow{\beta\circ -} & \end{array}$$

The commutativity of the right triangle follows from the definition of β as a composition, and the commutativity of the left square follows by the definition of the functor Δ on morphisms as a natural transformation. The covariant Yoneda functor applied to f is $h_f = f \circ$ -, and because $h_\bullet$ is fully faithful by Theorem 2.1, f is an isomorphism if and only if $f \circ$ - is an isomorphism. Thus, we have a commutative triangle with $f \circ$ -, $\alpha \circ \Delta_{z,y}$ and $\beta \circ \Delta_{z,x}$, where, by the two-out-of-three property, the desired result follows. □

Proposition 2.5. *Let $\mathcal{C}$ be an ∞-category and let $D\colon \mathcal{K} \to \mathcal{C}$ be a diagram. If D has a (co)limit with apex $y \in \mathcal{C}$, then an object $x \in \mathcal{C}$ is an apex of a (co)limit of D if and only if it is isomorphic to y.*

Proof. Let $D\colon \mathcal{K} \to \mathcal{C}$ be a diagram which has a limit with apex $y \in \mathcal{C}$ and natural transformation $\beta\colon \Delta y \to D$. Assume that an object $x \in \mathcal{C}$ is another apex of a limit cone $\alpha\colon \Delta x \to D$ of D. By the natural property of a limit cone applied to x and the cone α, there exists a morphism $f\colon x \to y$ such that α is a composition of β with $\Delta f\colon \Delta x \to \Delta y$. Then, by Lemma 2.4, f must be an isomorphism. Conversely, assume given an isomorphism $f\colon x \to y$. Then, by Lemma 2.4, the composition of α with $\Delta f\colon \Delta x \to \Delta y$ is a limit cone with apex x. The case of colimits follows by applying the same argument to the opposite ∞-category. □

Hence, when it is clear from the context, we refer to the unique apex of a limit $\mathcal{K}$-cone with base $D\colon \mathcal{K} \to \mathcal{C}$ as $\lim_{\mathcal{K}} D$ (or simply $\lim D$). In the dual case, we denote the unique apex of a colimit as $\mathrm{colim}_{\mathcal{K}} D$ (or simply $\mathrm{colim}\, D$).

An ∞-category $\mathcal{C}$ is *(co)complete* if, for every small ∞-category $\mathcal{K}$, each diagram $D\colon \mathcal{K} \to \mathcal{C}$ admits a (co)limit. If a functor preserves terminal objects and all fiber products, it is called *left exact* (or *lex*). The preservation of terminal objects and all fiber products is equivalent to the preservation of all finite limits, where a limit is finite if its base has a finite index ∞-category.

Let $\mathcal{A}$ and $\mathcal{K}$ be small ∞-categories and κ be a regular cardinal. A $\mathcal{K}$-diagram, $\mathcal{K}$-cone or limit $\mathcal{K}$-cone is *κ-small* if $\mathcal{K}$ is κ-small as an ∞-category. A functor $F\colon \mathcal{A} \to \mathcal{S}$ is called *κ-continuous* if it preserves κ-small limits. In particular, it is called *continuous* if it is $\aleph_0$-continuous. We denote by $\mathrm{Cont}_\kappa \mathcal{A}$ the full subcategory of $\mathrm{Fun}(\mathcal{A}, \mathcal{S})$ spanned by all κ-continuous functors.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which admits a right adjoint $G : \mathcal{D} \to \mathcal{C}$. For every small ∞-category $\mathcal{K}$, the functor F preserves colimit $\mathcal{K}$-cocones and the functor G preserves limit $\mathcal{K}$-cones [12, Proposition 5.2.3.5].

Proposition 2.6 ([12, Proposition 5.1.3.2])**.** *Let $\mathcal{C}$ be an ∞-category. The covariant Yoneda embedding $h_\bullet\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ preserves all limits that exist in $\mathcal{C}$.*

As a direct consequence, the contravariant Yoneda embedding sends colimits in $\mathcal{C}$ to limits in $\mathcal{S}$. Given an object $c \in \mathcal{C}$, define the *evaluation functor ev_c* : $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\Delta^0, \mathcal{C}) \cong \mathcal{C}$ as the functor resulting from applying the contravariant functor $\mathrm{Fun}(-, \mathcal{D})$ to the morphism $c\colon 1 \to \mathcal{C}$ in **sSet**.

Proposition 2.7 ([12, Proposition 5.1.2.3])**.** *Let $\mathcal{K}$ and $\mathcal{A}$ be small ∞-categories and let $\mathcal{C}$ be an ∞-category which admits $\mathcal{K}$-indexed colimits. Then:*

(i) $\mathrm{Fun}(\mathcal{A}, \mathcal{C})$ *admits $\mathcal{K}$-indexed colimits.*

(ii) *For every diagram* $D\colon \mathcal{K} \to \mathrm{Fun}(\mathcal{A},\mathcal{C})$*, a cone* $\alpha\colon \Delta F \Rightarrow D$ *is a colimit cocone if and only if, for each* $a \in \mathcal{A}$*, the induced cocone* $\alpha_a\colon \Delta\,\mathrm{ev}_a(F) \Rightarrow (\mathrm{ev}_a \circ D)$ *is a colimit cocone.*

The dual of Proposition 2.7 also holds for limits. As a direct conclusion of this fact, the ∞-category of presheaves over any small ∞-category is complete and cocomplete. Furthermore, part (ii) of Proposition 2.7 and its dual implies that the functor $\mathrm{ev}_a : \mathrm{Fun}(\mathcal{A},\mathcal{C}) \to \mathcal{C}$ preserves limits and colimits for all $a \in \mathcal{A}$.

Lemma 2.8. *Let* $\mathcal{C}$ *be an* ∞*-category and let* $x \in \mathcal{C}$*. The image of the covariant Yoneda functor* $h_x\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ *preserves limits in* $\mathcal{C}^{\mathrm{op}}$*, i.e.,* h_x *sends colimits to limits.*

Proof. By Theorem 2.1, h_x is equivalent to the composite of $h_\bullet\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})$ with $\mathrm{ev}_x\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}) \to \mathcal{S}$. In addition, $h_\bullet$ and ev_x preserve limits, by Propositions 2.6 and 2.7 respectively. Therefore, h_x preserves limits. □

2.5. **Localizations.** A functor $L\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is a *reflective localization* if it has a fully faithful right adjoint $J\colon \mathcal{D} \to \mathcal{C}$. Hence, L is a reflective localization if and only if $\mathcal{D}$ embeds as a reflective subcategory into $\mathcal{C}$, that is, for every object $x \in \mathcal{C}$ there exists an object $x' \in \mathcal{D}$ and a map $r\colon x \to Jx'$ such that the pre-composition map

$$\mathrm{Map}_{\mathcal{C}}(r,z)\colon\ \mathrm{Map}_{\mathcal{C}}(Jx',z) \longrightarrow \mathrm{Map}_{\mathcal{C}}(x,z)$$

is an equivalence of ∞-groupoids for all $z \in \mathcal{D}$.

If L is a reflective localization, then the counit of the adjunction $\epsilon\colon LJ \to \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism. The endofunctor $JL\colon \mathcal{C} \to \mathcal{C}$ is called a *reflector* on $\mathcal{C}$.

The terminology around reflective localizations is not consistent in the literature, where some authors [12] call localization what we call here a reflective localization. We choose to follow the convention of [2], using the more general term localization to mean the following: a functor $L\colon \mathcal{C} \to \mathcal{D}$ is a *localization* if there exists a class of morphisms S such that L inverts S and the induced functor

$$(-) \circ L\colon\ \mathrm{Fun}(\mathcal{D},\mathcal{E}) \longrightarrow \mathrm{Fun}^S(\mathcal{C},\mathcal{E})$$

is an equivalence for every ∞-category $\mathcal{E}$, where $\mathrm{Fun}^S(\mathcal{C},\mathcal{E})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathcal{E})$ spanned by functors inverting S. As shown in the following proposition, every reflective localization is a localization, but the converse is not true in general.

Proposition 2.9. *Every reflective localization* $L\colon \mathcal{C} \to \mathcal{D}$ *with right adjoint* J *is a localization at the class of maps* $S = \{\eta_x\colon x \to JLx \mid x \in \mathcal{C}\}$*.*

Proof. First, we want to show that L inverts S, i.e., $L\eta_x$ is an isomorphism for all $x \in \mathcal{C}$. By the adjunction identities of $L \dashv J$, the map $L\eta_x\colon Lx \to LJLx$ has a left inverse, namely ϵ_{Lx}. Furthermore, since L is a localization, the counit of the adjunction $\epsilon\colon LJ \to \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism. Using the inverse of ϵ, we find that ϵ_{Lx} is also a right inverse of $L\eta_x$, and hence $L\eta$ is an isomorphism.

Therefore, the induced composition functor factors through $\mathrm{Fun}^S(\mathcal{C},\mathcal{E})$:

$$(-) \circ L\colon\ \mathrm{Fun}(\mathcal{D},\mathcal{E}) \longrightarrow \mathrm{Fun}^S(\mathcal{C},\mathcal{E}).$$

We want to show that $(-) \circ L$ is an equivalence. On one hand, the counit isomorphism $LJ \cong \mathrm{id}_{\mathcal{D}}$ implies that

$$((-) \circ J) \circ ((-) \circ L) \cong ((-) \circ LJ) \cong ((-) \circ \mathrm{id}_{\mathcal{D}}) \cong \mathrm{id}_{\mathrm{Fun}(\mathcal{D},\mathcal{E})}\,.$$

On the other hand, we need to prove that $((-) \circ L) \circ ((-) \circ J) \cong \mathrm{id}_{\mathrm{Fun}^S(\mathcal{C},\mathcal{E})}$. Observe that, for all $F \in \mathrm{Fun}^S(\mathcal{C},\mathcal{E})$,

$$(((-) \circ L) \circ ((-) \circ J))(F) \cong ((-) \circ JL)(F) \cong F \circ JL \cong F,$$

because F inverts $\eta_x\colon x \to JLx$ for all $x \in \mathcal{C}$, i.e., it inverts $\eta\colon \mathrm{id}_{\mathcal{C}} \to JL$. Therefore, $(-) \circ L$ is an equivalence and L is a localization. □

Let S be a class of maps in an ∞-category $\mathcal{C}$. An object $z \in \mathcal{C}$ is S*-local* if, for every $f\colon x \to y$ in S, there is an equivalence of ∞-groupoids induced by composition with f:

$$f^*\colon\ \mathrm{Map}_{\mathcal{C}}(y,z) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{C}}(x,z).$$

We denote by $\mathrm{Loc}(\mathcal{C}, S)$ the full subcategory of $\mathcal{C}$ spanned by S-local objects. A morphism $f\colon x \to y$ is an *S-equivalence* if, for every S-local object z, composition with f induces an equivalence of ∞-groupoids

$$f^*\colon \mathrm{Map}_{\mathcal{C}}(y,z) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{C}}(x,z).$$

In general, $\mathrm{Loc}(\mathcal{C}, S)$ need not be reflective.

A class of maps S in $\mathcal{C}$ is called *strongly saturated* if it contains the isomorphisms of $\mathcal{C}$ and has the two-out-of-three property, and the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by the maps in S is closed under colimits.

The smallest strongly saturated class in $\mathcal{C}$ is the one containing only the isomorphisms of $\mathcal{C}$. Since the intersection of strongly saturated classes is a strongly saturated class, for every collection S of morphisms the intersection of all the strongly saturated classes containing S is the smallest strongly saturated class $\overline{S}$ containing S. We call it the strongly saturated class *generated* by S. If S is a set, then $\overline{S}$ is said to be of *small generation.*

2.6. **Presentablility.** Let κ be a regular cardinal. An ∞-category $\mathcal{K}$ is *κ-filtered* if the colimits of shape $\mathcal{K}$ on $\mathcal{S}$ commute with all κ-small limits in $\mathcal{S}$, i.e., if the functor

$$\operatorname*{colim}_{\mathcal{K}}\colon \mathrm{Fun}(\mathcal{K}, \mathcal{S}) \longrightarrow \mathcal{S}$$

preserves κ-small limits. A diagram $F\colon \mathcal{K} \to \mathcal{C}$ where $\mathcal{K}$ is a κ-filtered ∞-category is called a *κ-filtered diagram*, and a *κ-filtered colimit* is a colimit over a κ-filtered diagram. An object $x \in \mathcal{C}$ is *κ-compact* if $h^x\colon \mathcal{C} \to \mathcal{S}$ preserves κ-filtered colimits. We denote by $\mathcal{C}^\kappa$ the full subcategory of $\mathcal{C}$ spanned by κ-compact objects.

If $\mathcal{C}$ is a cocomplete ∞-category, we say that a class of objects $G \subseteq \mathrm{Obj}(\mathcal{C})$ *generates $\mathcal{C}$ under colimits* if every object in $\mathcal{C}$ is the colimit of a diagram with objects in G. If $\mathcal{C}$ is not cocomplete but admits κ-filtered colimits for some regular cardinal κ, we say that G *generates $\mathcal{C}$ under κ-filtered colimits* if every object in $\mathcal{C}$ is the colimit of a κ-filtered diagram with objects in G. For example, for every small ∞-category $\mathcal{A}$, the image of the Yoneda embedding $h_\bullet\colon \mathcal{A} \to \mathrm{PSh}(\mathcal{A})$ generates $\mathrm{PSh}(\mathcal{A})$ under colimits.

As in the case of ordinary categories, there are two equivalent characterizations of accessible ∞-categories: one with filtered colimits and compact objects, and another with Ind-objects.

Definition 2.10. For a regular cardinal κ, an ∞-category $\mathcal{C}$ is *κ-accessible* if $\mathcal{C}$ is locally small, it admits κ-filtered colimits, the full subcategory $\mathcal{C}^\kappa \subseteq \mathcal{C}$ of κ-compact objects is small, and $\mathcal{C}^\kappa$ generates $\mathcal{C}$ under κ-filtered colimits. Furthermore, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is *κ-accessible* if $\mathcal{C}$ is κ-accessible and F preserves κ-filtered colimits.

We say that an ∞-category $\mathcal{C}$ (resp. a functor $F\colon \mathcal{C} \to \mathcal{D}$) is *accessible* if it is κ-accessible for some regular cardinal κ.

Definition 2.11. Let $\mathcal{A}$ be a small ∞-category and κ a regular cardinal. The ∞-category of Ind-*objects* $\mathrm{Ind}_\kappa(\mathcal{A})$ is the full subcategory of $\mathrm{PSh}(\mathcal{A})$ spanned by those functors $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{S}$ for which there exists a small κ-filtered ∞-category $\mathcal{K}$ and a diagram $p\colon \mathcal{K} \to \mathcal{A}$ such that F is a colimit of the composition $h_\bullet \circ p\colon \mathcal{K} \to \mathrm{PSh}(\mathcal{A})$.

Lemma 2.12 ([12, Proposition 5.3.5.4])**.** *Let $\mathcal{A}$ be a small ∞-category and let κ be a regular cardinal. Then:*

(i) $\mathrm{Ind}_\kappa(\mathcal{A}) \subseteq \mathrm{Cont}_\kappa(\mathcal{A})$.
(ii) *If $\mathcal{A}$ admits κ-small colimits, then* $\mathrm{Ind}_\kappa(\mathcal{A}) = \mathrm{Cont}_\kappa(\mathcal{A})$.

Theorem 2.13 ([12, Proposition 5.4.2.2])**.** *Let $\mathcal{C}$ be an ∞-category. The following are equivalent:*

(i) *$\mathcal{C}$ is an accessible ∞-category.*
(ii) *$\mathcal{C}$ is equivalent to* $\mathrm{Ind}_\kappa \mathcal{A}$ *for some regular cardinal κ and some small ∞-category $\mathcal{A}$.*

Proposition 2.14 ([12, Proposition 5.4.7.7])**.** *If a functor between accessible ∞-categories has a left or right adjoint functor, then it is itself accessible.*

Examples of accessible ∞-categories include $\mathcal{S}$ [12, Example 5.4.2.7] and any accessible category. In addition, if $\mathcal{A}$ is a small ∞-category and $\mathcal{C}$ is an accessible ∞-category, then $\mathrm{Fun}(\mathcal{A}, \mathcal{C})$ is accessible [12, Proposition 5.4.4.3]. In particular, $\mathrm{PSh}(\mathcal{A})$ and $\mathrm{Fun}(\mathcal{A}, \mathcal{S})$ are accessible for every small ∞-category $\mathcal{A}$.

Definition 2.15. An ∞-category is *presentable* if it is accessible and cocomplete.

As in the case of accessibility, there are other equivalent definitions of presentability. Most notably, presentability can be characterized in terms of a certain type of localization of an ∞-category of presheaves. Thus, we say that a reflective localization $L\colon \mathcal{C} \to \mathcal{D}$ is an *accessible* reflective localization if the right adjoint to L is an accessible functor. As a consequence of Proposition 2.14, we have the following characterization of accessible reflective localizations:

Proposition 2.16 ([12, Proposition 5.5.1.2])**.** *Let $\mathcal{C}$ be an accessible ∞-category and $L\colon \mathcal{C} \to \mathcal{D}$ a reflective localization with fully faithful right adjoint $J\colon \mathcal{D} \hookrightarrow \mathcal{C}$. Then, the following are equivalent:*

(i) *J is an accessible functor.*
(ii) *$\mathcal{D}$ is accessible.*
(iii) *The reflector $JL\colon \mathcal{C} \to \mathcal{C}$ is an accessible functor.*

Therefore, any reflective localization between accessible ∞-categories is accessible.

Theorem 2.17 (Simpson [16] and Lurie [12, Theorem 5.5.1.1])**.** *The following are equivalent:*

(i) *$\mathcal{C}$ is a presentable ∞-category.*
(ii) *$\mathcal{C}$ is equivalent to $\operatorname{Ind}_\kappa \mathcal{A}$ for some regular cardinal κ and some small ∞-category $\mathcal{A}$ which admits κ-small colimits.*
(iii) *$\mathcal{C}$ is equivalent to an accessible reflective localization of the ∞-category of presheaves $\operatorname{PSh}(\mathcal{A})$ on some small category $\mathcal{A}$.*

Examples of presentable ∞-categories include $\mathcal{S}$, any ∞-topos, and any presentable category. If $\mathcal{A}$ is a small ∞-category and $\mathcal{C}$ is a presentable ∞-category, then $\operatorname{Fun}(\mathcal{A}, \mathcal{C})$ is presentable [12, Proposition 5.5.3.6]. In particular, $\operatorname{PSh}(\mathcal{A})$ and $\operatorname{Fun}(\mathcal{A}, \mathcal{S})$ are presentable for every small ∞-category $\mathcal{A}$. Furthermore, every presentable ∞-category is complete and cocomplete [12, Corollary 5.5.2.4].

In addition, presentable ∞-categories provide a convenient ambient for localization. As explained in the previous section, every reflective localization induces a reflector which inverts a class of morphisms. Conversely, if we choose a set of morphism S_0 in a presentable ∞-category, then the following result proves that there exists a reflective localization inverting the strongly saturated class S generated by S_0.

Theorem 2.18 ([12, Proposition 5.5.4.15])**.** *Let $\mathcal{C}$ be a presentable ∞-category and S_0 a set of morphisms in $\mathcal{C}$. Let S be the strongly saturated class of morphisms generated by S_0 and $\operatorname{Loc}(\mathcal{C}, S_0)$ the full subcategory of $\mathcal{C}$ consisting of S_0-local objects. Then, $\operatorname{Loc}(\mathcal{C}, S_0)$ is presentable and the inclusion $\operatorname{Loc}(\mathcal{C}, S_0) \subseteq \mathcal{C}$ has a left adjoint L. Furthermore, for every $f\colon x \to y$ in $\mathcal{C}$, the following are equivalent:*

(i) *f is an S_0-equivalence.*
(ii) *f belongs to S.*
(iii) *The induced morphism Lf is an equivalence.*

Therefore, $L\colon \mathcal{C} \to \operatorname{Loc}(\mathcal{C}, S_0)$ is a reflective localization and S is the class of maps inverted by L.

3. Limit sketches

Definition 3.1. Let $\mathcal{A}$ be a small ∞-category and $\mathcal{C}$ be a complete ∞-category. A *limit sketch* $\Sigma = (\mathcal{A}, \mathcal{L})$ is a pair consisting of a small ∞-category $\mathcal{A}$ and a set of cones $\mathcal{L}$ in $\mathcal{A}$. A *model* of a limit sketch Σ in $\mathcal{C}$ is a functor $F\colon \mathcal{A} \to \mathcal{C}$ that sends cones of $\mathcal{L}$ to limit cones in $\mathcal{C}$.

The ∞-category of all models of Σ in $\mathcal{C}$ is a subcategory of $\operatorname{Fun}(\mathcal{A}, \mathcal{C})$ which is denoted by $\operatorname{Mod}(\Sigma, \mathcal{C})$. If $\mathcal{C}$ is not specified, it is assumed to be $\mathcal{S}$, and the ∞-category of all models of Σ in $\mathcal{S}$ is denoted by $\operatorname{Mod}(\Sigma)$. We say that an ∞-category is *limit-sketchable* if it is equivalent to $\operatorname{Mod}(\Sigma)$ for some limit sketch Σ.

A functor $F\colon \mathcal{A} \to \mathcal{S}$ is a model of a limit sketch Σ if and only if, for every $\mathcal{K}$-cone $(D, x, \alpha) \in \mathcal{L}$ with base $D\colon \mathcal{K} \to \mathcal{A}$, apex $x \in \mathcal{A}$ and natural transformation $\alpha\colon \Delta x \Rightarrow D$, the map

$$t\colon Fx \longrightarrow \lim(F \circ D)$$

induced by the limit in $\mathcal{S}$ is an equivalence of ∞-groupoids.

Example 3.2 (Pointed objects)**.** Let $\mathbb{2}$ be the nerve of the small category generated by a morphism $a\colon 0 \to 1$, and let $\mathcal{C}$ be an ∞-category with terminal object $1_{\mathcal{C}}$. Let $\Sigma = (\mathbb{2}, \mathcal{L})$ be the limit sketch with set of cones $\mathcal{L}$ consisting of the empty diagram $D\colon \emptyset \to \mathbb{2}$, the apex $0 \in \mathbb{2}$, and the unique natural transformation $\alpha\colon \Delta 0 \Rightarrow D$. A model $F\colon \mathbb{2} \to \mathcal{C}$ of Σ in $\mathcal{C}$ sends

$$(a\colon 0 \to 1) \longmapsto (f\colon x_0 \to x_1),$$

and it also sends the only cone of $\mathcal{L}$ to a limit cone of the diagram $F \circ D\colon \emptyset \to \mathcal{C}$. It follows that $x_0 \cong \lim(F \circ D) \cong 1_{\mathcal{C}}$. Therefore, each model F exhibits an object $x_1 \in \mathcal{C}$ as a pointed object $f\colon 1_{\mathcal{C}} \to x_1$ of $\mathcal{C}$, and $\mathrm{Mod}(\Sigma, \mathcal{C})$ can be viewed as the ∞-category of pointed objects of $\mathcal{C}$. In particular, $\mathrm{Mod}(\Sigma)$ is the ∞-category of pointed spaces.

Example 3.3 (Morphisms)**.** Take $\mathbb{2}$ as in the previous example, and let $\Sigma = (\mathbb{2}, \emptyset)$ be the trivial limit sketch over $\mathbb{2}$. A model $F\colon \mathbb{2} \to \mathcal{C}$ of Σ in any ∞-category $\mathcal{C}$ exhibits a morphism of $\mathcal{C}$. Therefore, $\mathrm{Mod}(\Sigma, \mathcal{C}) = \mathrm{Fun}(\mathbb{2}, \mathcal{C})$ is the ∞-category of morphisms in $\mathcal{C}$. The same construction can be carried out with any small ∞-category $\mathcal{A}$; hence $\mathrm{Mod}((\mathcal{A}, \emptyset), \mathcal{C}) = \mathrm{Fun}(\mathcal{A}, \mathcal{C})$ is limit-sketchable.

Example 3.4 (Pullback diagrams)**.** Let $\mathcal{A}$ be the nerve of the small category generated by a commutative square

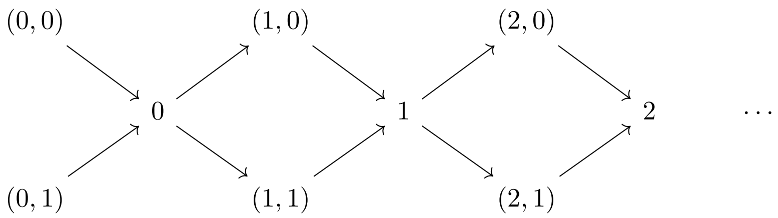
$$\begin{array}{ccc} 3 & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 0. \end{array}$$

Consider a limit sketch $\Sigma = (\mathcal{A}, \mathcal{L})$ with set of cones $\mathcal{L}$ consisting of the inclusion diagram $D\colon \{1 \to 0 \leftarrow 2\} \to \mathcal{A}$, the apex $3 \in \mathcal{A}$, and the natural transformation $\alpha\colon \Delta 3 \Rightarrow D$. A model $F\colon \mathcal{A} \to \mathcal{C}$ in a complete ∞-category $\mathcal{C}$ for the sketch Σ sends

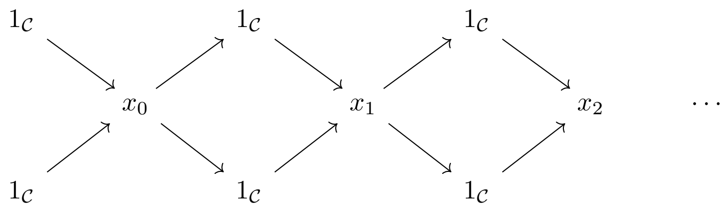
$$\begin{array}{ccc} 3 & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 0 \end{array} \quad \overset{F}{\longmapsto} \quad \begin{array}{ccc} x_3 & \longrightarrow & x_2 \\ \downarrow & \lrcorner & \downarrow \\ x_1 & \longrightarrow & x_0 \end{array}$$

where the image commutative square is a pullback diagram. Therefore, each model of Σ corresponds to a pullback diagram in $\mathcal{C}$.

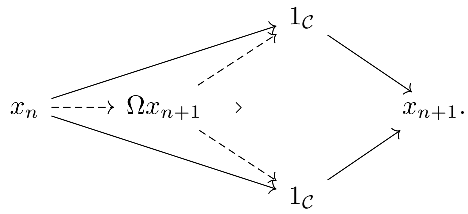
Example 3.5 (Pre-spectrum and spectrum objects)**.** Let $\mathcal{A}$ be the nerve of the small category with objects $\mathbb{N} \sqcup (\mathbb{N} \times \{0,1\})$ and generating morphisms $f_{i,j}\colon i \to (i+1, j)$ and $g_{i,j}\colon (i,j) \to i$ for every $i \in \mathbb{N}$ and $j \in \{0,1\}$, i.e., the category of the following shape:



Consider the limit sketch $\Sigma = (\mathcal{A}, \mathcal{L})$ with set of cones $\mathcal{L}$ consisting of, for each $i \in \mathbb{N}$ and $j \in \{0,1\}$, the empty diagram $D_{i,j}\colon \emptyset \to \mathcal{A}$, the apex $(i,j) \in \mathcal{A}$, and the unique natural transformation $\alpha_{i,j}\colon \Delta(i,j) \Rightarrow D_{i,j}$. A model $F\colon \mathcal{A} \to \mathcal{C}$ in a complete ∞-category $\mathcal{C}$ for the sketch Σ is a diagram



where each (i,j) is replaced by the terminal object $1_{\mathcal{C}}$ of $\mathcal{C}$, and a sequence of objects $x_n \in \mathcal{C}$ is selected. Giving a model of Σ amounts to choosing pointed objects $1_{\mathcal{C}} \to x_n$ for all $n \in \mathbb{N}$ and maps $x_n \to \Omega x_{n+1}$ (by the universal property of the pullback):



Hence, each model of Σ is, by definition, a *pre-spectrum object*, and $\mathrm{Mod}(\Sigma, \mathcal{C})$ is the ∞-category of pre-spectrum objects in $\mathcal{C}$.

If we want to obtain spectrum objects, we need to add more cones to Σ. Consider a limit sketch $\Sigma' = (\mathcal{A}, \mathcal{L} \sqcup \mathcal{L}')$ where $\mathcal{L}'$ consists of, for each $n \in \mathbb{N}$, a diagram

$$\begin{array}{rccc} D_n\colon & \{1 \to 0 \leftarrow 2\} & \longrightarrow & \mathcal{A} \\ & 1 \to 0 \leftarrow 2 & \longmapsto & (n,1) \to n \leftarrow (n,0); \end{array}$$

the apex $n-1 \in \mathcal{A}$, and a natural transformation $\beta_n\colon \Delta(n-1) \Rightarrow D_n$. A model for Σ' is a pre-spectrum object in $\mathcal{C}$ such that

$$x_n \simeq \text{ pullback of } \{1_{\mathcal{C}} \to x_{n+1} \leftarrow 1_{\mathcal{C}}\} \simeq \Omega x_{n+1}.$$

In other words, a model for Σ' is a spectrum object, and $\mathrm{Mod}(\Sigma', \mathcal{C})$ is the ∞-category of spectrum objects in $\mathcal{C}$. In particular, $\mathrm{Mod}(\Sigma')$ is the ∞-category of spectra $\mathbf{Sp}$.

In the next examples, we view categories such as the simplex category $\mathbf{\Delta}$, the category $\mathbf{\Gamma}$ of finite pointed sets and pointed maps, or the tree category $\mathbf{\Omega}$ defined in [8, § 3.2] as ∞-categories by passing to their respective nerves. In all the sketches $(\mathcal{A}, \mathcal{L})$ discussed in this section, except Example 3.15, the corresponding ∞-category $\mathcal{A}$ is the nerve of a small category.

Example 3.6 (Pre-category objects and Segal spaces)**.** Define a limit sketch $\Sigma = (\mathbf{\Delta}^{\mathrm{op}}, \mathcal{L})$ with set of cones $\mathcal{L}$ consisting of, for each $n \in \mathbb{N}$,

- an index category W_n generated by

$$\begin{array}{ccccccccccc} & (0,0) & & (0,1) & & & & (0,n-1) & \\ \overset{l_0}{\swarrow} & & \overset{r_0}{\searrow}\ \overset{l_1}{\swarrow} & & \overset{r_1}{\searrow} & \cdots & \overset{l_{n-1}}{\swarrow} & & \overset{r_{n-1}}{\searrow} \\ (1,0) & & (1,1) & & (1,2) & & (1,n-1) & & (1,n); \end{array}$$

- a diagram $D_n\colon W_n \to \mathbf{\Delta}^{\mathrm{op}}$ sending $(0,i)$ to $[0]$, $(1,i)$ to $[1]$, l_i to δ_0, and r_i to δ_1,
- and a natural transformation $\alpha_n\colon \Delta[n] \to D_n$ with apex $[n]$ defined by composition with the unique morphism from $[n]$ to $[0]$ or $[1]$.

A model $F\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ of Σ in a complete ∞-category $\mathcal{C}$ sends the cones of $\mathcal{L}$ to limit cones in $\mathcal{C}$, i.e., it is equivalent to a simplicial object F in $\mathcal{C}$ such that there is an equivalence

$$F_n \xrightarrow{\sim} F_1 \times_{F_0} F_1 \times_{F_0} \cdots \times_{F_0} F_1.$$

Therefore, $\mathrm{Mod}(\Sigma, \mathcal{C})$ is the ∞-category of *pre-category* objects in $\mathcal{C}$. If $\mathcal{C} = \mathcal{S}$, then $\mathrm{Mod}(\Sigma)$ is the ∞-category of Segal spaces.

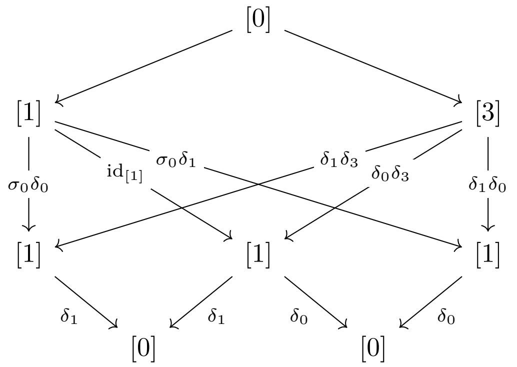
Example 3.7 (Univalent category objects and complete Segal spaces)**.** Let $\mathcal{C}$ be a complete ∞-category and $\Sigma = (\mathbf{\Delta}^{\mathrm{op}}, \mathcal{L})$ be the sketch of Example 3.6. By the characterization found in [13, § 5.5] and [13, Proposition 6.4], a Segal space F is a complete Segal space if and only if the following is a pullback square in $\mathcal{S}$:

(1)
$$\begin{array}{ccc} F_0 & \longrightarrow & F_3 \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle g} \\ F_1 & \xrightarrow{f} & F_1 \times^{d_1,d_1}_{F_0} F_1 \times^{d_0,d_0}_{F_0} F_1 \end{array}$$

where $f = (s_0 d_0, id_{F_1}, s_0 d_1)$ and $g = (d_1 d_3, d_0 d_3, d_1 d_0)$.

Define a sketch $\Sigma' = (\mathbf{\Delta}^{\mathrm{op}}, \mathcal{L}')$ where $\mathcal{L}'$ is the union of $\mathcal{L}$ with the cone represented by the following diagram:

(2)



with apex $[0]$. A model $F\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ of Σ' exhibits a pre-category object in $\mathcal{C}$ and the image of (2) is a limit cone, which is equivalent to the pullback square (1). Therefore, $\mathrm{Mod}(\Sigma', \mathcal{C})$ is the ∞-category of *univalent category objects* in $\mathcal{C}$. In the particular case when $\mathcal{C} = \mathcal{S}$, we have that $\mathrm{Mod}(\Sigma')$ is the ∞-category of complete Segal spaces.

Example 3.8 (Monoid objects and A_∞-spaces/rings)**.** Let $\mathcal{C}$ be a complete ∞-category and $\Sigma = (\mathbf{\Delta}^{\mathrm{op}}, \mathcal{L})$ be the sketch of Example 3.6. Define a sketch $\Sigma' = (\mathbf{\Delta}^{\mathrm{op}}, \mathcal{L}')$ where $\mathcal{L}'$ is the union of $\mathcal{L}$ and a cone with empty diagram and apex $[0]$. Each model of Σ' is a pre-category object F in $\mathcal{C}$ such that $F_0 \simeq 1_{\mathcal{C}}$. Hence, $\mathrm{Mod}(\Sigma', \mathcal{C})$ is the ∞-category of monoid objects in $\mathcal{C}$. In the case when $\mathcal{C} = \mathcal{S}$, we have that $\mathrm{Mod}(\Sigma')$ is the ∞-category of A_∞-spaces. If $\mathcal{C} = \mathbf{Sp}$, then $\mathrm{Mod}(\Sigma', \mathbf{Sp})$ is the ∞-category of A_∞-ring spectra.

Example 3.9 (Groupoid objects)**.** Let $\mathcal{C}$ be a complete ∞-category and $\Sigma = (\mathbf{\Delta}^{\mathrm{op}}, \mathcal{L})$ be the sketch of Example 3.6. Define a sketch $\Sigma' = (\mathbf{\Delta}^{\mathrm{op}}, \mathcal{L}')$ where $\mathcal{L}'$ is the union of $\mathcal{L}$ with a diagram

$$\begin{array}{rccc} D\colon & \{1 \to 0 \leftarrow 2\} & \longrightarrow & \mathbf{\Delta}^{\mathrm{op}} \\ & (1 \to 0 \leftarrow 2) & \longmapsto & \Big([1] \xrightarrow{\delta_0} [0] \xleftarrow{\delta_0} [1]\Big) \end{array}$$

and a natural transformation α with apex $[2]$ defining the following commutative square:

$$\begin{array}{ccc} [2] & \xrightarrow{\delta_1} & [1] \\ {\scriptstyle \delta_0}\downarrow & & \downarrow{\scriptstyle \delta_0} \\ [1] & \xrightarrow[\delta_0]{} & [0]. \end{array}$$

A model of Σ' defines a pre-category object and sends these squares to pullback squares. Therefore, $\mathrm{Mod}(\Sigma', \mathcal{C})$ is the ∞-category of groupoid objects in $\mathcal{C}$.

Example 3.10 (Group objects and grouplike A_∞-spaces)**.** Following the construction used in Example 3.8 but replacing the sketch of pre-categories with the one of groupoids, it follows that $\mathrm{Mod}(\Sigma', \mathcal{C})$ is the ∞-category of group objects in $\mathcal{C}$. In the particular case when $\mathcal{C} = \mathcal{S}$, we have that $\mathrm{Mod}(\Sigma')$ is the ∞-category of grouplike A_∞-spaces.

Example 3.11 (Commutative monoid objects and E_∞-spaces/rings)**.** Let $\mathbf{\Gamma}$ be the category of finite pointed sets and pointed maps, where every object is isomorphic to a set $[n]$ pointed by $0 \in [n]$. For each $1 \leq k \leq n$, there is a pointed map $\delta_k\colon [n] \to [1]$ defined by

$$\delta_k(i) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

A $\mathbf{\Gamma}$-object in an ∞-category $\mathcal{C}$ is a map $E\colon \mathbf{\Gamma} \to \mathcal{C}$. If $\mathcal{C}$ has finite products, we can take the product of the morphisms $E(\delta_k)\colon E_n \to E_1$, which we denote by

$$p_n\colon E_n \longrightarrow \prod_{k=1}^{n} E_1.$$

By definition, E is a commutative monoid object if p_n is invertible for every $n \geq 0$.

Consider a sketch $\Sigma = (\mathbf{\Gamma}, \mathcal{L})$, where the set of cones $\mathcal{L}$ consists of, for each $n \in \mathbb{N}$, a diagram

$$\begin{array}{rccc} D_n\colon & \coprod_{k=1}^{n} \{k\} & \longrightarrow & \mathbf{\Gamma} \\ & k & \longmapsto & [1], \end{array}$$

the apex $[n] \in \mathbf{\Gamma}$, and the natural transformation $\delta^n_\bullet\colon \Delta[n] \Rightarrow D_n$ induced by δ_k at each object k. Therefore, $\mathrm{Mod}(\Sigma, \mathcal{C})$ is the ∞-category of commutative monoid objects in $\mathcal{C}$. If $\mathcal{C} = \mathcal{S}$, then $\mathrm{Mod}(\Sigma)$ is the ∞-category of E_∞-spaces, and if $\mathcal{C} = \mathbf{Sp}$, then $\mathrm{Mod}(\Sigma, \mathbf{Sp})$ is the ∞-category of E_∞-ring spectra.

Example 3.12 (Abelian group objects and infinite loop spaces)**.** Let $\mathcal{C}$ be a complete ∞-category and $\Sigma = (\mathbf{\Gamma}, \mathcal{L})$ be the sketch of Example 3.11. Consider the functor

$$\begin{array}{rccc} i\colon & \mathbf{\Delta}^{\mathrm{op}} & \longrightarrow & \mathbf{\Gamma} \\ & [n] & \longmapsto & \mathrm{Hom}_{\mathbf{sSet}_*}(\Delta^n_+, S^1) \end{array}$$

where S^1 is the pointed simplicial circle $\Delta^1/\partial\Delta^1$. Since $\mathcal{C}$ is complete, the map

$$i^*\colon \mathrm{Fun}(\mathbf{\Gamma}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})$$

sends every commutative monoid object $E \in \mathrm{Mod}(S, \mathcal{C}) \subseteq \mathrm{Fun}(\mathbf{\Gamma}, \mathcal{C})$ to its underlying monoid $i^*E\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$. We say that E is an *abelian group object* if i^*E is a group.

Define a sketch $\Sigma' = (\mathbf{\Gamma}, \mathcal{L}')$ where $\mathcal{L}'$ is the union of $\mathcal{L}$ and a cone consisting of a diagram $i \circ D_{\mathrm{Grp}}$, where

$$\begin{array}{rccc} D_{\mathrm{Grp}}\colon & \{1 \to 0 \leftarrow 2\} & \longrightarrow & \mathbf{\Delta}^{\mathrm{op}} \\ & (1 \to 0 \leftarrow 2) & \longmapsto & \left([1] \xrightarrow{\delta_0} [0] \xleftarrow{\delta_0} [1]\right), \end{array}$$

and a natural transformation with apex $i[2] \in \mathbf{\Gamma}$ defining the following commutative square:

$$\begin{array}{ccc} i[2] & \xrightarrow{i(\delta_1)} & i[1] \\ {\scriptstyle i(\delta_0)}\downarrow & & \downarrow{\scriptstyle i(\delta_0)} \\ i[1] & \xrightarrow[i(\delta_0)]{} & i[0]. \end{array}$$

A model of S' defines an abelian group object by sending these squares to pullback squares. Therefore, $\mathrm{Mod}(\Sigma', \mathcal{C})$ is the ∞-category of abelian group objects in $\mathcal{C}$. In the particular case when $\mathcal{C} = \mathcal{S}$, we have that $\mathrm{Mod}(\Sigma')$ is the ∞-category of infinite loop spaces.

Example 3.13 (Dendroidal Segal spaces)**.** Let $\mathbf{\Omega}$ be the tree category of Moerdijk–Weiss [8, §3.2]. Given two trees T_1 and T_2 sharing an edge e which is a leaf of T_1 and the root of T_2, the *grafting* $T_1 \cup_e T_2$ is the pushout of T_1 and T_2 along the common edge e.

Define a limit sketch $\Sigma = (\mathbf{\Omega}^{\mathrm{op}}, \mathcal{L})$ with the set $\mathcal{L}$ consisting of, for each tree $T \in \mathbf{\Omega}$ and each decomposition of T as a grafting of subtrees $T = T_1 \circ_e T_2$, a cone with apex T represented by the following pushout in $\mathbf{\Omega}$:

$$\begin{array}{ccc} \eta & \xrightarrow{e} & T_1 \\ {\scriptstyle e}\downarrow & \ulcorner & \downarrow \\ T_2 & \dashrightarrow & T. \end{array}$$

A model for the sketch Σ is equivalent to a dendroidal space $X\colon \mathbf{\Omega}^{\mathrm{op}} \to \mathcal{S}$ such that the squares of the form

$$\begin{array}{ccc} X(T) & \longrightarrow & X(T_1) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle e^*} \\ X(T_2) & \xrightarrow[e^*]{} & X(\eta) \end{array}$$

are pullbacks for any tree T and any decomposition of T as a grafting of subtrees $T = T_1 \circ_e T_2$. By [8, Lemma 12.7], this condition is equivalent to claiming that X is a dendroidal Segal space, and hence $\mathrm{Mod}(\Sigma)$ is the ∞-category of dendroidal Segal spaces.

Example 3.14 (Complete dendroidal Segal spaces)**.** Consider the inclusion $J\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Omega}^{\mathrm{op}}$ sending $[n]$ to the linear tree L_n. The induced map

$$J^*\colon\ \mathrm{Fun}(\mathbf{\Omega}^{\mathrm{op}}, \mathcal{S}) \longrightarrow \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})$$

sends every dendroidal space X to its underlying simplicial space J^*X. By [8, Remark 12.15], a dendroidal Segal space $X\colon \mathbf{\Omega}^{\mathrm{op}} \to \mathcal{S}$ is complete if and only if its underlying simplicial space J^*X is complete.

Let $\Sigma_{dS} = (\mathbf{\Omega}^{\mathrm{op}}, \mathcal{L}_{dS})$ be the sketch of Example 3.13, which models dendroidal Segal spaces, and let $(D, [0], \alpha)$ be the cone added in Example 3.7, which models the completeness condition on simplicial spaces. Define a sketch $\Sigma = (\mathbf{\Omega}^{\mathrm{op}}, \mathcal{L})$ where $\mathcal{L}$ is the union of $\mathcal{L}_{dS}$ and a cone consisting of a diagram $J \circ D$ and a natural transformation $J \circ \alpha$ with apex $J[0] = L_0 \in \mathbf{\Omega}^{\mathrm{op}}$. A model of Σ is a dendroidal space $X\colon \mathbf{\Omega}^{\mathrm{op}} \to \mathcal{S}$ such that the map

$$(J^*X)[0] = XJ[0] \longrightarrow \lim(X \circ J \circ D) = \lim((J^*X) \circ D)$$

is an equivalence. This condition is equivalent to imposing that the underlying simplicial space J^*X be complete, according to Example 3.7. Hence, X is a model of Σ if and only if it is a complete dendroidal Segal space, and $\mathrm{Mod}(\Sigma)$ is the ∞-category of complete dendroidal Segal spaces.

Example 3.15 (∞-Sheaves)**.** Let $\mathcal{A}$ be a small ∞-category, and let $\mathcal{A}_{/x}$ denote the slice category over an object $x \in \mathcal{A}$. A *sieve* on an object $x \in \mathcal{A}$ is a full subcategory $\mathcal{D}_x \subseteq \mathcal{A}_{/x}$ closed under precomposition with morphisms in $\mathcal{A}_{/x}$. For S a sieve on $x \in \mathcal{A}$ and $f\colon y \to x$ a morphism, the pullback sieve f^*S on y is the sieve spanned by the morphisms into y that become equivalent to a morphism in S after composition with f.

A *Grothendieck topology* $\mathcal{T}$ on an ∞-category $\mathcal{A}$, as defined in [12, § 6.2.2], is an assignment to each object $x \in \mathcal{A}$ of a collection $\mathcal{T}_x$ of sieves on x, called *covering sieves*, such that:

(1) For each $x \in \mathcal{A}$, the trivial sieve $\mathcal{A}_{/x} \subseteq \mathcal{A}_{/x}$ on x is a covering sieve.
(2) If S is a covering sieve on x and $f\colon y \to x$ is a morphism, then the pullback sieve f^*S is a covering sieve on y.
(3) For a covering sieve S on x and any sieve R on x, if the pullback sieve f^*R is covering for every $f \in S$, then R itself is covering.

By [12, Proposition 6.2.2.5], there is a natural bijection between sieves on x in $\mathcal{A}$ and equivalence classes of monomorphisms $U \to h_x$ in $\mathrm{PSh}(\mathcal{A})$, where h_x is the Yoneda functor, as in Theorem 2.1, and a morphism $U \to V$ is a *monomorphism* if it is a (-1)-truncated object of $\mathrm{PSh}(\mathcal{A})_{/V}$.

Let $S(\mathcal{T})$ be the class of monomorphisms in $\mathrm{PSh}(\mathcal{A})$ corresponding to the covering sieves of $\mathcal{T}$. A presheaf $F \in \mathrm{PSh}(\mathcal{A})$ is an ∞*-sheaf* with Grothendieck topology $\mathcal{T}$ if it is an $S(\mathcal{T})$-local object, i.e., if for every map $f\colon U \to h_x$ in $S(\mathcal{T})$, the morphism

$$Fx \simeq \mathrm{Map}_{\mathrm{PSh}(\mathcal{A})}(h_x, F) \longrightarrow \mathrm{Map}_{\mathrm{PSh}(\mathcal{A})}(U, F)$$

is an equivalence. In fact, this last condition can be rewritten in terms of limits as follows.

Lemma 3.16. *Let $\{u_i \to x\}_{i \in I}$ be a family of morphisms of $\mathcal{A}$ that generate the covering sieve corresponding to a monomorphism $\eta\colon U \to h_x$, and let $U_\bullet$ be the underlying simplicial object of the Čech nerve of the induced map $\coprod_{i \in I} h_{u_i} \to h_x$. Then, a presheaf F is η-local if and only if the induced map $Fx \longrightarrow \lim F(U_\bullet)$ is an equivalence.*

Proof. Let I be a set, and $\{u_i \to x\}_{i \in I}$ be a family of morphisms of $\mathcal{C}$ that generate the covering sieve corresponding to a monomorphism $\eta\colon U \to h_x$ of $S(\mathcal{T})$. By [12, Lemma 6.2.3.18], $f\colon U \to h_x$ can be identified with the (-1)-truncation of the induced map $\coprod_{i \in I} h_{u_i} \to h_x$ in $\mathrm{PSh}(\mathcal{A})_{/h_x}$. Since $\mathrm{PSh}(\mathcal{A})$ is an ∞-topos, by [12, Proposition 6.2.3.4], the (-1)-truncation of a map $p\colon V \to h_x$ can be identified with the map $\operatorname{colim} V_\bullet \to h_x$, where $V_\bullet$ is the underlying simplicial object of the Čech nerve of p. Hence, $f\colon U \to h_x$ can be identified with a map $\operatorname{colim} U_\bullet \to h_x$, where $U_\bullet$ is the underlying simplicial object of the Čech nerve of the induced map $\coprod_{i \in I} h_{u_i} \to h_x$. By Lemma 2.8, a presheaf F is η-local if and only if the following map is an equivalence:

$$\begin{aligned} Fx \simeq \mathrm{Map}_{\mathrm{PSh}(\mathcal{A})}(h_x, F) \longrightarrow \mathrm{Map}_{\mathrm{PSh}(\mathcal{A})}(U, F) &\simeq \mathrm{Map}_{\mathrm{PSh}(\mathcal{A})}(\operatorname{colim} U_\bullet, F) \\ &\simeq \lim \mathrm{Map}_{\mathrm{PSh}(\mathcal{A})}(U_\bullet, F) \simeq \lim F(U_\bullet), \end{aligned}$$

which proves the statement. □

Let $\mathcal{A}$ be a small ∞-category with a Grothendieck topology $\mathcal{T}$. Let $\Sigma = (\mathcal{A}^{\mathrm{op}}, \mathcal{L})$ be a limit sketch where the set $\mathcal{L}$ consists of, for each covering sieve generated by a family $\{u_i \to x\}_{i \in I}$, a cone over the underlying simplicial object of the Čech nerve of the induced map $\coprod_{i\in I} h_{u_i} \to h_x$, with apex $x \in \mathcal{A}$. A model of Σ is a presheaf $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{S}$ such that

$$Fx \longrightarrow \lim F(U_\bullet)$$

is an equivalence for every $(U_\bullet, x)$ in $\mathcal{L}$. By Lemma 3.16, this condition is equivalent to claiming that F is an ∞-sheaf, and hence $\mathrm{Mod}(\Sigma)$ is equivalent to the ∞-category $\mathrm{Sh}(\mathcal{A}, \mathcal{T})$.

4. Representation theorem

Our goal in this section is to generalize the well-known characterization of presentable categories as limit-sketchable categories to the higher setting. Thus, we aim to prove that an ∞-category is presentable if and only if it is equivalent to the ∞-category of models of a limit sketch.

Theorem 4.1 (Sketch Representation Theorem)**.** *An ∞-category $\mathcal{C}$ is presentable if and only if it is limit-sketchable.*

Proof. We first want to prove that if $\mathcal{C}$ is presentable, then there exists some limit sketch Σ such that $\mathcal{C} \simeq \mathrm{Mod}(\Sigma)$. By Theorem 2.17, $\mathcal{C}$ being presentable is equivalent to the existence of some small ∞-category $\mathcal{A}$ such that $\mathcal{A}$ admits κ-small colimits and $\mathcal{C} \simeq \mathrm{Ind}_\kappa \mathcal{A}$ for some regular cardinal κ. Since $\mathcal{A}$ admits κ-small colimits, using Lemma 2.12, we obtain that $\mathrm{Ind}_\kappa \mathcal{A}$ is equal to $\mathrm{Cont}_\kappa(\mathcal{A}^{\mathrm{op}})$.

Consider a sketch $\Sigma = (\mathcal{A}^{\mathrm{op}}, \mathcal{L})$ where $\mathcal{L}$ is the set of all limit cones of κ-small diagrams in $\mathcal{A}^{\mathrm{op}}$. It follows directly from the definitions that $\mathrm{Cont}_\kappa(\mathcal{A}^{\mathrm{op}})$ is the ∞-category of models of Σ, i.e., the category of functors preserving all limit cones of κ-small diagrams in $\mathcal{A}^{\mathrm{op}}$. Observe that $\mathcal{L}$ is well-defined as a set because $\mathcal{A}$ is small, which implies that there is only a set of distinct limit cones of κ-small diagrams in $\mathcal{A}^{\mathrm{op}}$. Therefore, $\mathcal{C}$ is equivalent to the ∞-category of models of Σ.

Now let us consider the reverse implication. Given any ∞-category $\mathcal{C}$ which is equivalent to $\mathrm{Mod}(\Sigma)$ for a limit sketch $\Sigma = (\mathcal{A}, \mathcal{L})$, we want to prove that $\mathrm{Mod}(\Sigma)$ is presentable, which directly implies that $\mathcal{C}$ is presentable, since equivalences of ∞-categories preserve presentability.

First, observe that the ∞-category $\mathrm{Fun}(\mathcal{A}, \mathcal{S})$ is presentable. Our goal is to find a set of morphisms M such that $\mathrm{Mod}(\Sigma) = \mathrm{Loc}(\mathrm{Fun}(\mathcal{A}, \mathcal{S}), M)$, i.e., such that $\mathrm{Mod}(\Sigma)$ is precisely the full subcategory of $\mathrm{Fun}(\mathcal{A}, \mathcal{S})$ consisting of M-local objects. If such a set exists, then, by Theorem 2.18, we may conclude that $\mathrm{Mod}(\Sigma)$ is presentable.

The cones in the set $\mathcal{L}$ have the form (D_i, x_i, α^i), where $D_i\colon \mathcal{K}_i \to \mathcal{A}$ is a diagram, $x_i \in \mathcal{A}$ is an apex, and $\alpha^i\colon \Delta x_i \Rightarrow D_i$ is a cone. For every such $\mathcal{K}_i$-cone $(D_i, x_i, \alpha^i) \in L$, by whiskering $h^\bullet$ with α^i we obtain a natural transformation $h^\bullet \cdot \alpha^i\colon h^\bullet \circ D_i \to \Delta h^{x_i}$. Since $\mathrm{Fun}(\mathcal{A}, \mathcal{S})$ is cocomplete, there exists a colimit $\beta^i\colon h^\bullet \circ D_i \to \Delta \operatorname{colim}(h^\bullet \circ D_i)$, which induces a morphism $m_i\colon \operatorname{colim} h^{D_i} \to h^{x_i}$ such that $[\beta^i] \circ [m_i] = [h^{\alpha^i}]$, i.e., the following diagram commutes:

$$\begin{array}{ccccc} D_i & & h^\bullet \circ D_i & & \\ \Uparrow & & \| & \searrow^{\beta^i} & \\ \alpha^i & -h^\bullet\to & h^\bullet\cdot\alpha^i & & \\ \| & & \Downarrow & & \\ \Delta x_i & & \Delta h^{x_i} & \xleftarrow[\Delta m_i]{} & \Delta \operatorname{colim}(h^\bullet \circ D_i). \end{array}$$

We pick the collection of morphisms $M = \{m_i\}_i$. Since there is one m_i for each cone in $\mathcal{L}$, we have that M is a set of the same cardinality as $\mathcal{L}$.

Now $\mathrm{Mod}(\Sigma) \subset \mathrm{Fun}(\mathcal{A}, \mathcal{S})$ is the full subcategory of functors sending cones of $\mathcal{L}$ to limit cones in $\mathcal{S}$. We want to prove that $\mathrm{Mod}(\Sigma)$ coincides with the full subcategory of $\mathrm{Fun}(\mathcal{A}, \mathcal{S})$ consisting of M-local objects. Given a functor $F\colon \mathcal{A} \to \mathcal{S}$, we need to show that F sends cones of $\mathcal{L}$ to limit cones in $\mathcal{S}$ if and only if F is M-local. Recall that F is M-local if, for any $m_i \in M$, the induced map by the covariant Yoneda embedding

$$h_F(m_i) \cong \mathrm{Map}_{\mathrm{Fun}(\mathcal{A},\mathcal{S})}(m_i, F)\colon\ \mathrm{Map}_{\mathrm{Fun}(\mathcal{A},\mathcal{S})}\left(h^{x_i}, F\right) \longrightarrow \mathrm{Map}_{\mathrm{Fun}(\mathcal{A},\mathcal{S})}\left(\operatorname*{colim}_{k\in\mathcal{K}_i} h^{D_i(k)}, F\right)$$

is an equivalence of ∞-groupoids. On the other hand, a functor $F\colon A \to \mathcal{S}$ sends cones of $\mathcal{L}$ to limit cones in $\mathcal{S}$ if $(F \circ D_i, Fx_i, F\alpha^i)$ is a limit cone in $\mathcal{S}$, i.e., for each $\mathcal{K}_i$-cone $(D_i, x_i, \alpha^i) \in \mathcal{L}$

and each $y \in \mathcal{S}$, the induced map

$$\theta^i_y \colon \operatorname{Map}_{\mathcal{S}}(y, Fx_i) \longrightarrow \operatorname{Map}_{\operatorname{Fun}(\mathcal{K},\mathcal{S})}(\Delta y, F \circ D_i)$$

is an equivalence of ∞-groupoids. By Proposition 2.5, this condition is equivalent to asking that $t_i \colon Fx_i \to \lim(F \circ D_i)$ be an equivalence of ∞-groupoids, where t_i is the induced morphism in the following diagram:

$$\begin{array}{ccccc}
D_i & & F \circ D_i & & \\
\Uparrow & & \Uparrow & \overset{\ell^i}{\nwarrow\!\!\nwarrow} & \\
\alpha^i & \xrightarrow{\;F\;} & F\cdot\alpha^i & & \\
\| & & \| & & \\
\Delta x_i & & \Delta F x_i & \overset{}{\underset{\Delta t_i}{\dashrightarrow}} & \Delta \lim(F \circ D_i).
\end{array}$$

Our goal is to compare when t_i and m_i are equivalences. To this end, we need a third natural construction, which will be used as a bridge. Consider again a $\mathcal{K}_i$-cone $(D_i, x_i, \alpha^i) \in cL$. By whiskering $H = (h_F \circ h^\bullet) \cong \operatorname{Map}_{[\mathcal{A},\mathcal{S}]}(h^\bullet, F)$ with α^i we obtain a natural transformation

$$H \cdot \alpha^i \colon \Delta \operatorname{Map}_{[\mathcal{A},\mathcal{S}]}(h^{x_i}, F) \Rightarrow \operatorname{Map}_{[\mathcal{A},\mathcal{S}]}(h^\bullet \circ D_i, F).$$

Since $\operatorname{Fun}(\mathcal{A}, \mathcal{S})$ is cocomplete, there exists a limit

$$\gamma^i \colon \Delta \lim \operatorname{Map}_{[\mathcal{A},\mathcal{S}]}(h^\bullet \circ D_i, F) \Rightarrow \operatorname{Map}_{[\mathcal{A},\mathcal{S}]}(h^\bullet \circ D_i, F),$$

which induces a morphism

$$n_i \colon \operatorname{Map}_{[\mathcal{A},\mathcal{S}]}(h^{x_i}, F) \longrightarrow \lim \operatorname{Map}_{[\mathcal{A},\mathcal{S}]}(h^\bullet \circ D_i, F)$$

such that $[\gamma^i] \circ [n_i] = [H \cdot \alpha^i]$, i.e., the following diagram commutes:

$$\begin{array}{ccccc}
D_i & & \operatorname{Map}(h^\bullet \circ D_i, F) & & \\
\Uparrow & & \Uparrow & \overset{\gamma^i}{\nwarrow\!\!\nwarrow} & \\
\alpha^i & \xrightarrow{\;H\;} & H\cdot\alpha^i & & \\
\| & & \| & & \\
\Delta x_i & & \Delta \operatorname{Map}(h^{x_i}, F) & \underset{\Delta n_i}{\dashrightarrow} & \Delta \lim \operatorname{Map}(h^\bullet \circ D_i, F).
\end{array}$$

We prove that t_i is an isomorphism if and only if n_i is an isomorphism. By the contravariant Yoneda lemma, there exist natural morphisms

$$y_{F,D_i(k)} \colon \operatorname{Map}(h^{D_i(k)}, F) \longrightarrow F(D_i(k))$$

for any $k \in \mathcal{K}$. By composing $y_{F,D_i(k)}$ with γ^i_k, we obtain a new cone of $F \circ D_i$. By the universal property of the limit, there exists a commuting morphism $\tau_i \colon \lim \operatorname{Map}(h^\bullet \circ D_i, F) \to \lim(F \circ D_i)$. In addition, by Lemma 2.4, τ_i must be an isomorphism.

$$\begin{array}{ccccccc}
\operatorname{Map}(h^{D_i(k)}, F) & & & \xrightarrow[\cong]{\;y_{F,D_i(k)}\;} & & & F(D_i(k)) \\
{\scriptstyle H\cdot\alpha^i}\uparrow & \overset{\gamma^i}{\nwarrow} & & & & \overset{\ell^i}{\nearrow} & \uparrow{\scriptstyle F\cdot\alpha^i} \\
\operatorname{Map}(h^{x_i}, F) & \underset{n_i}{\dashrightarrow} & \lim \operatorname{Map}(h^\bullet \circ D_i, F) & \underset{\tau_i}{\dashrightarrow} & \lim(F \circ D_i) & \underset{t_i}{\dashleftarrow} & Fx_i.
\end{array}$$

Consider the composite of τ_i and n_i, and the contravariant Yoneda lemma map

$$y_{F,x_i} \colon \operatorname{Map}(h^{x_i}, F) \longrightarrow Fx_i.$$

These two morphisms form a diagram together with t_i, and furthermore the diagram commutes, because of the naturality of the contravariant Yoneda lemma. Then, thanks to τ_i being an isomorphism, and the two-out-of-three property, n_i is an isomorphism if and only if t_i is one:

$$\begin{array}{ccc}
\operatorname{Map}(h^{x_i}, F) & \xrightarrow[\cong]{\;y_{F,x_i}\;} & Fx_i \\
& {\scriptstyle \tau_i \circ n_i}\searrow \qquad \swarrow{\scriptstyle t_i} & \\
& \lim(F \circ D_i). &
\end{array}$$

Finally, we prove that n_i is an isomorphism if and only if $\mathrm{Map}(m_i, F)$ is an isomorphism. By Lemma 2.8, $\mathrm{Map}(\mathrm{colim}(h^\bullet \circ D_i), F)$ is a limit of the diagram $\mathrm{Map}(h^\bullet \circ D_i, F)$. By the uniqueness of limits up to isomorphism, there exists an isomorphism

$$\sigma\colon\ \mathrm{Map}(\mathrm{colim}(h^\bullet \circ D_i), F) \longrightarrow \lim\left(\mathrm{Map}(h^\bullet \circ D_i, F)\right).$$

Since the two objects are limits of the same diagram, the universal morphisms n_i and $\mathrm{Map}(m_i, F)$ from $\mathrm{Map}(h^{x_i}, F)$ must commute. Therefore, by the two-out-of-three property, the following diagram commutes:

$$\begin{array}{ccc}
h^{x_i} & \xleftarrow{\quad m_i \quad} & \mathrm{colim}(h^\bullet \circ D_i) \\
{\scriptstyle h_F}\downarrow & & \downarrow{\scriptstyle h_F} \\
\mathrm{Map}(h^{x_i}, F) & \xrightarrow{\mathrm{Map}(m_i,F)} & \mathrm{Map}(\mathrm{colim}(h^\bullet \circ D_i), F) \\
& \searrow^{n_i} & {\scriptstyle \sigma}\,\vdots\,{\scriptstyle \cong} \\
& & \lim \mathrm{Map}(h^\bullet \circ D_i, F).
\end{array}$$

Consequently, $\mathrm{Mod}(\Sigma) = \mathrm{Loc}(\mathrm{Fun}(\mathcal{A}, \mathcal{S}), M)$, and it follows from Theorem 2.18 that $\mathrm{Mod}(\Sigma)$ is presentable. □

Example 4.2. Every ∞-topos is a left-exact accessible reflective localization of $\mathrm{PSh}(\mathcal{A})$ for some small ∞-category $\mathcal{A}$. Therefore, Theorem 4.1 implies that every ∞-topos is limit-sketchable.

The ∞-category $\mathrm{Sh}(\mathcal{A}, \mathcal{T})$ of sheaves on a small ∞-category $\mathcal{A}$ equipped with a Grothendieck topology $\mathcal{T}$ is a special case. A more explicit sketch whose ∞-category of models is equivalent to $\mathrm{Sh}(\mathcal{A}, \mathcal{T})$ has been given in Example 3.15.

Corollary 4.3. *If $\mathcal{C}$ is a presentable ∞-category, then, for every limit sketch $\Sigma = (\mathcal{A}, \mathcal{L})$, the ∞-category of models $\mathrm{Mod}(\Sigma, \mathcal{C})$ over $\mathcal{C}$ is presentable.*

Proof. Assume that $\mathcal{C}$ is κ-presentable for a regular cardinal κ. By Theorem 4.1, if $\mathcal{C}$ is presentable, it is equivalent to $\mathrm{Cont}_\kappa\, \mathcal{B}$ for some small ∞-category $\mathcal{B}$. Consider the sketch $\Sigma' = (\mathcal{A} \times \mathcal{B}, \mathcal{L}')$ where $\mathcal{L}'$ consists of a set of diagrams $D \times D'$, with D the diagrams of $\mathcal{L}$ and D' the κ-small diagrams in $\mathcal{B}$, and a set of natural transformations $N \times N'$, with N the cones of $\mathcal{L}$ and N' the limit cones of all the diagrams of D' in $\mathcal{B}$. Then it follows that $\mathrm{Mod}(\Sigma') = \mathrm{Mod}(\Sigma, \mathcal{C})$, and, by Theorem 4.1, we infer that $\mathrm{Mod}(\Sigma, \mathcal{C})$ is presentable. □

From the examples given in Section 3, we can conclude, using Theorem 4.1, that the following full subcategories of any presentable ∞-category $\mathcal{C}$ are presentable: pointed objects, spectrum objects, pre-category objects, univalent category objects, monoid objects, groupoid objects, group objects, commutative monoid objects, and abelian group objects.

Consequently, the following ∞-categories are presentable: pointed ∞-groupoids, spectra, Segal spaces, complete Segal spaces, A_∞-spaces, grouplike A_∞-spaces, A_∞-ring spectra, E_∞-spaces, infinite loop spaces, and E_∞-ring spectra.

Although these categories are extensively discussed in various forms throughout the literature, their sketchability is seldom explicitly addressed. The initial examples in Section 3 of this article draw on similar examples from unpublished work of Joyal [10, 11]. Our treatment of complete Segal spaces, dendroidal Segal spaces, and complete dendroidal Segal spaces in Examples 3.7, 3.13 and 3.14 is new.

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