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On structures preserved by idempotent transformations of groups and homotopy types

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Abstract. Recent work on homotopical localizations has revealed that many structures and properties are preserved under idempotent functors in the category of groups and in the homotopy category of spaces. In this article we collect a number of findings in this direction, including new results by the author and coauthors. Background and proofs are given throughout. We discuss in particular the preservation of asphericity of spaces under localization at sets of primes, aiming to the study of localizations of infra-nilmanifolds.

0. Introduction

Idempotent functors have been studied in various contexts. Category theorists also refer to them as idempotent triples or idempotent monads. Key references to us are the articles by Deleanu–Frei–Hilton [**21**], Freyd–Kelly [**29**], and especially Adams' monograph [**2**]. The latter discusses idempotent functors in homotopy theory as a suitable tool for the study of homological localizations.

More recent developments in homotopy theory and group theory have focused on properties which are preserved by idempotent functors. That is, if a functor L is naturally equivalent to LL and an object X has a certain property, one inspects if LX shares the same property. In this article we collect many examples and counterexamples. In doing this, we relate results about groups with results about topological spaces, thus exposing new instances of the rich interplay between group theory, homological algebra, and homotopy theory.

In Section 2 and Section 3 we refer to results by Farjoun [**24**], Libman [**34**], the author, and others, showing that the following classes of groups are closed under idempotent functors (among perhaps many other classes): abelian groups; nilpotent groups of class two or less; bounded abelian groups; finite abelian groups; divisible abelian groups; rings; commutative rings; fields; modules over rings. Remarkably, if M is any module over a ring R and L is any idempotent functor on abelian groups, then LM admits a canonical LR-module structure.

It is still unknown whether or not the class of nilpotent groups is closed under arbitrary idempotent functors. However, it has been shown by O'Sullivan in [**41**]

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that the classes of solvable groups and metabelian groups are not. Likewise, as shown by Libman in [**34**], the classes of finite groups and torsion abelian groups are not closed under arbitrary idempotent functors.

In order to relate group-theoretical structures with homotopy theory in our context, one has to consider functors in the category of CW-complexes that are homotopy functors (i.e., preserve homotopy equivalences) and are idempotent when viewed as functors in the homotopy category. The motivating examples are localizations at sets of primes [**31**] and, more generally, homological localizations. To localize a space X with respect to a homology theory E_* is to give a map $X \to L_E X$ which is a terminal E_*-equivalence out of X. This is always possible, by results in [**7**]. It then follows that a map $X \to Y$ is an E_*-equivalence if and only if the induced map $L_E X \to L_E Y$ is a homotopy equivalence. If X is 1-connected and E_* is chosen to be ordinary homology with $\mathbb{Z}_P$ coefficients (the integers localized at a set of primes P), then $L_E X$ is the P-localization X_P of X in the sense of [**31**]. Thus, $H_n(X;\mathbb{Z}_P) \cong H_n(X_P;\mathbb{Z}_P)$ for all n, and $\pi_n(X_P) \cong \pi_n(X) \otimes \mathbb{Z}_P$ for all n. For spaces with nontrivial fundamental group, the interaction of P-localization with the fundamental group is much more transparent if P-localization is defined by means of homology with suitable twisted coefficients, as explained in [**14**] and recalled in Section 5 below. Still, the effect of P-localization on the higher homotopy groups of nonnilpotent spaces is not fully understood.

As shown in the article, the following classes of spaces are closed under homotopy idempotent functors: connected spaces; H-spaces; associative or commutative H-spaces; loop spaces; generalized Eilenberg–Mac Lane spaces. It is not known whether or not the class of 1-connected spaces is closed under arbitrary idempotent functors, but it is known that higher connectivity is not preserved in general, according to results by Mislin [**38**] or Neisendorfer [**39**].

We discuss in greater detail the class of *aspherical* spaces, that is, connected spaces X whose homotopy groups $\pi_n(X)$ vanish for $n \geq 2$. Examples are given showing that this class of spaces is not closed under homotopy idempotent functors in general, and recall from [**14**] a purely algebraic, necessary and sufficient condition for an aspherical space X in order that its localization X_P at a given set of primes P be again aspherical.

Compact flat Riemannian manifolds and more generally infra-nilmanifolds are important examples of aspherical spaces. Since fundamental groups of infra-nilmanifolds are virtually nilpotent, the methods of [**12**] are suitable for the study of their localizations at primes. A study of the preservation of asphericity of infra-nilmanifolds under localizations has been carried out by Descheemaeker in [**22**].

The list of structures and properties that are preserved by idempotent functors is surely much longer than the list given in this article. More importantly, the list of conceptual principles that explain the occurrence of specific examples should be enlarged, as these general principles might apply to other relevant categories.

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1. About idempotent functors

In order to illustrate what we mean by an idempotent functor, let us consider the functor L which sends every abelian group A to the abelian group $LA = A\otimes\mathbb{Q}$, where $\mathbb{Q}$ denotes the rationals, and the tensor product is meant over $\mathbb{Z}$. Then $LA \cong LLA$. A more subtle fact is that there are two natural isomorphisms from LA to LLA, namely $a\otimes x \mapsto a\otimes x\otimes 1$ and $a\otimes x\mapsto a\otimes 1\otimes x$. However, they coincide, since if we write $x = n/m$ where n and m are integers, then $m(x\otimes 1) = n\otimes 1 = 1\otimes n = m(1\otimes x)$ in $\mathbb{Q}\otimes\mathbb{Q}$, so $x\otimes 1 = 1\otimes x$.

Now we recall the following definitions from [**2**] and [**25**]. Given any category $\mathcal{C}$, a *coaugmented functor* is a pair (L,η) consisting of a functor $L\colon\mathcal{C}\to\mathcal{C}$ and a natural transformation $\eta\colon \mathrm{Id}\to L$, called a *coaugmentation*. Thus, for every object X there is a morphism $\eta_X\colon X\to LX$ and for every morphism $\alpha\colon X\to Y$ there is another morphism $L\alpha\colon LX\to LY$ such that $\eta_Y\circ\alpha = L\alpha\circ\eta_X$.

A coaugmented functor (L,η) is *idempotent* if, for every X, the two morphisms η_{LX} and $L\eta_X$ from LX to LLX coincide and are isomorphisms. This is slightly redundant, as we next explain. We follow the usual convention of denoting by ηL the natural transformation assigning to every object X the morphism η_{LX} and by $L\eta$ the natural transformation assigning $L\eta_X$ to X.

PROPOSITION 1.1. *Given a functor L in a category $\mathcal{C}$ and a natural transformation $\eta\colon\mathrm{Id}\to L$, if both ηL and $L\eta$ are isomorphisms, then $\eta L = L\eta$.*

PROOF. Since η is a natural transformation, we have $\eta_{LX}\circ\eta_X = L\eta_X\circ\eta_X$ for every object X. Applying L to this expression we obtain $L\eta_{LX}\circ L\eta_X = LL\eta_X\circ L\eta_X$. Since $L\eta_X$ is an isomorphism by assumption, we have $L\eta_{LX} = LL\eta_X$. Hence,

$$\eta_{LLX}\circ\eta_{LX} = L\eta_{LX}\circ\eta_{LX} = LL\eta_X\circ\eta_{LX} = \eta_{LLX}\circ L\eta_X,$$

and, since η_{LLX} is an isomorphism, we conclude that $\eta_{LX} = L\eta_X$, for every X, as claimed. $\square$

For simplicity, we often say that L is an "idempotent functor" if the associated coaugmentation η is clear from the context. Whenever we discuss natural transformations $\phi\colon L_1\to L_2$ between idempotent functors, we implicitly assume that ϕ is compatible with the coaugmentations η_1 and η_2, that is, $\phi\circ\eta_1 = \eta_2$.

If L is idempotent in a category $\mathcal{C}$ and for every X we pick $\mu_X\colon LLX\to LX$ which is inverse to $L\eta_X$, then μ is in fact a natural transformation, since for each $\alpha\colon X\to Y$ we have

$$\mu_Y\circ LL\alpha\circ L\eta_X = \mu_Y\circ L\eta_Y\circ L\alpha = L\alpha = L\alpha\circ\mu_X\circ L\eta_X,$$

and $L\eta_X$ is an isomorphism by assumption. In addition,

$$\mu L = (L\eta L)^{-1} = L(\eta L)^{-1} = L\mu,$$

so (L,η,μ) is an *idempotent triple* in $\mathcal{C}$, also called an *idempotent monad*, which is a well-known concept in category theory; see [**3**], [**13**], [**21**], [**36**]. The coaugmentation η is also called the *unit* of the monad and μ is called the *multiplication* of the monad.

Proposition 1.2. *Let $L: \mathcal{C} \to \mathcal{C}$ be a functor in any category. Then the following statements are equivalent:*
(a) *L is an idempotent functor with coaugmentation η.*
(b) *For every object X there is a morphism $\eta_X: X \to LX$ which is initial among all morphisms from X to objects isomorphic to LY for some Y.*

Proof. First, suppose that L is idempotent with coaugmentation η. Given any morphism $f: X \to LY$, there is a unique $g: LX \to LY$ such that $g \circ \eta_X = f$. Namely, we may take $g = (\eta_{LY})^{-1} \circ Lf$ and, if another morphism h satisfies $h \circ \eta_X = f$, then $\eta_{LY} \circ h = Lh \circ \eta_{LX} = Lh \circ L\eta_X = Lf$, so $h = g$.

Conversely, suppose that (b) holds. Then L becomes a functor by defining Lf for every morphism $f: X \to Y$ as the unique morphism such that $Lf \circ \eta_X = \eta_Y \circ f$. In this way η automatically becomes a natural transformation. Therefore, $L\eta_X \circ \eta_X = \eta_{LX} \circ \eta_X$ for all X, and this implies that $L\eta_X = \eta_{LX}$. For each X, pick the only morphism $\mu_X: LLX \to LX$ such that $\mu_X \circ \eta_{LX}$ is the identity of LX. Then (b) also implies that $\eta_{LX} \circ \mu_X$ is equal to the identity of LLX. Hence, η_{LX} is an isomorphism and the proof is complete. □

The statement (b) in Proposition 1.2 is called the universal property of idempotent functors. It is in fact their crucial feature, and it will be used many times in this article. This property is also quoted by saying that L is a *reflection* onto the full subcategory $\mathcal{D}$ of objects isomorphic to LY for some Y. Such objects are called *L-local*.

In other words, if L is viewed as a functor from $\mathcal{C}$ to the subcategory $\mathcal{D}$ of L-local objects, then it is left adjoint to the inclusion $J: \mathcal{D} \hookrightarrow \mathcal{C}$. This means that there is a natural bijection

$$\mathcal{D}(LX, D) \cong \mathcal{C}(X, JD), \tag{1.1}$$

for all X in $\mathcal{C}$ and D in $\mathcal{D}$, which is induced by the natural morphism $\eta_X: X \to LX$. This follows of course from Proposition 1.2.

One should think of $\eta_X: X \to LX$ as a "best approximation" of X by an L-local object. For example, the abelianization functor is a reflection from the category of groups onto the full subcategory of abelian groups. The example $LA = A \otimes \mathbb{Q}$ given above is a reflection from the category of abelian groups onto the full subcategory of rational vector spaces. More generally, if P is any set of primes and $\mathbb{Z}_P$ is the subring of $\mathbb{Q}$ consisting of reduced fractions n/m where no prime factor of m belongs to P, then $LA = A \otimes \mathbb{Z}_P$ is a reflection onto the subcategory of $\mathbb{Z}_P$-modules, which are also called P-local abelian groups.

Other examples of idempotent functors in group theory and in homotopy theory are described in [**2**], [**13**], or [**15**], together with many useful properties.

A morphism $f: X \to Y$ is called an *L-equivalence* if $Lf: LX \to LY$ is an isomorphism. Thus, the coaugmentation morphism $\eta_X: X \to LX$ is an L-equivalence for every X. Every L-equivalence between L-local objects is an isomorphism. Consequently, if $f: X \to Y$ is an L-equivalence and Y is L-local, then $Y \cong LX$.

The L-local objects and the L-equivalences are *orthogonal* in the following sense. If X is L-local and $f: A \to B$ is an L-equivalence, then for every $\alpha: A \to X$ there is a unique $\beta: B \to X$ such that $\beta \circ f = \alpha$. Moreover, as explained in [**2**] or [**15**], an object is L-local if and only if it is orthogonal to all L-equivalences, and a morphism is an L-equivalence if and only if it is orthogonal to all L-local objects.

As we next show, the class of L-local objects is closed under limits (if they exist in $\mathcal{C}$) and retracts. An object Y is said to be a *retract* of another object X if morphisms $j\colon Y \to X$ (*injection*) and $r\colon X \to Y$ (*retraction*) are given with $r \circ j = \mathrm{id}_Y$. Similarly, the class of L-equivalences is closed under colimits and retracts. A colimit of morphisms is defined as the induced morphism from the colimit of the domains to the colimit of the targets. A morphism $g\colon A \to B$ is a retract of another morphism $f\colon X \to Y$ if morphisms $j_1\colon A \to X$, $j_2\colon B \to Y$, $r_1\colon X \to A$, $r_2\colon Y \to B$ are given with $j_2 \circ g = f \circ j_1$, $r_2 \circ f = g \circ r_1$, $r_1 \circ j_1 = \mathrm{id}_A$, and $r_2 \circ j_2 = \mathrm{id}_B$.

Proposition 1.3. *For every idempotent functor L, the class of L-local objects is closed under limits and retracts, and the class of L-equivalences is closed under colimits and retracts.*

Proof. Let $\mathcal{I}$ be any small category and $F\colon \mathcal{I} \to \mathcal{C}$ be any functor (this is usually called a *diagram* in $\mathcal{C}$). Suppose that $F(i)$ is L-local for all $i \in \mathcal{I}$, and suppose that $Y = \lim F$ exists. Let $f\colon A \to B$ be any L-equivalence and $\alpha\colon A \to Y$ any morphism. Let $\alpha_i\colon A \to F(i)$ be the composites of α with the limit morphisms $Y \to F(i)$. Since each $F(i)$ is L-local, there is a unique morphism $\beta_i\colon B \to F(i)$ such that $\beta_i \circ f = \alpha_i$, for each i. These define together a unique morphism $\beta\colon B \to Y$ such that $\beta \circ f = \alpha$. This shows that Y is orthogonal to all L-equivalences and hence it is L-local.

Now let Y be a retract of an L-local object X, with injection $j\colon Y \to X$ and retraction $r\colon X \to Y$. Let $f\colon A \to B$ be any L-equivalence and let $\alpha\colon A \to Y$ be any morphism. Since X is L-local, there is a unique $g\colon B \to X$ such that $g \circ f = j \circ \alpha$. Then $\beta = r \circ g$ satisfies $\beta \circ f = \alpha$. If γ also satisfies $\gamma \circ f = \alpha$, then $j \circ \gamma \circ f = j \circ \alpha$ and hence $j \circ \gamma = g$, so $\gamma = r \circ g = \beta$. This proves that Y is L-local.

The arguments for L-equivalences are analogous. □

If L is idempotent, then we know from (1.1) that $L\colon \mathcal{C} \to \mathcal{D}$ is left adjoint to the inclusion $J\colon \mathcal{D} \hookrightarrow \mathcal{C}$. Therefore, if the category $\mathcal{C}$ is complete and cocomplete, then L preserves colimits and J preserves limits; see [**36**]. The fact that J preserves limits can be viewed as a part of Proposition 1.3. The fact that L preserves colimits tells us the following. For any functor $F\colon \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is a small category,

$$L(\operatorname{colim}_{\mathcal{C}} F) \cong \operatorname{colim}_{\mathcal{D}} LF \cong L(\operatorname{colim}_{\mathcal{C}} LF), \tag{1.2}$$

since the colimit of a functor taking values in $\mathcal{D}$ is obtained by picking the colimit of the same functor in $\mathcal{C}$ and next applying L to the outcome. This does *not* imply that colimits of L-local objects are L-local. For example, the free product of two uniquely divisible groups is not uniquely divisible in general; this concept is discussed in the next section.

2. Idempotent functors on groups

In this section and in the next one we specialize to the category of groups. The following is an important source of idempotent functors. Let $\varphi\colon A \to B$ be any group homomorphism. We say that a group G is *φ-local* if it is orthogonal to φ. That is, G is φ-local if for every homomorphism $\alpha\colon A \to G$ there is a unique homomorphism $\beta\colon B \to G$ such that $\beta \circ \varphi = \alpha$. Then there exists a reflection L_φ (called localization with respect to φ, or *φ-localization*) onto the full subcategory of φ-local groups. The construction of L_φ is a special case of the well-known "orthogonal-reflection construction," which is described in [**1**]; see also [**15**]. For simplicity, we call φ-equivalences the L_φ-equivalences.

For example, consider multiplication by a prime p on the group of integers, $\mu_p\colon \mathbb{Z} \to \mathbb{Z}$. Then a group G is μ_p-local if and only if for every $x \in G$ there is a unique $y \in G$ such that $y^p = x$. Such a group G is called *uniquely p-divisible.* More generally, for a set of primes P, we say that G is uniquely P-divisible if it is uniquely p-divisible for all $p \in P$. The complement of a set of primes P is denoted by P', and it is customary to call *P-local* the uniquely P'-divisible groups. Thus, the P-local abelian groups are precisely the $\mathbb{Z}_P$-modules, as we said in the previous section. The P-localization functor is defined in the category of groups as localization with respect to the free product of the homomorphisms μ_p for all primes $p \in P'$. That is, the P-localization of a group G is a homomorphism $G \to G_P$ which is initial among all homomorphisms from G to uniquely P'-divisible groups. This is precisely the localization defined by Ribenboim in [**43**] or [**44**]. This functor restricts to the full subcategory of nilpotent groups, and its behaviour on nilpotent groups was thoroughly described in [**31**].

Abelianization is localization with respect to the projection $\mathbb{Z} * \mathbb{Z} \twoheadrightarrow \mathbb{Z} \times \mathbb{Z}$. Many other idempotent functors encountered in practice take the form L_φ for some homomorphism φ. In fact, as explained in [**17**], the question of whether *every* idempotent functor on groups is of the form L_φ for some φ cannot be answered using the usual ZFC axioms of set theory (Zermelo–Fraenkel axioms with the Axiom of Choice). Specifically, if one admits the validity of Vopěnka's principle [**32**] (which cannot be proved using ZFC), then every idempotent functor on groups is of the form L_φ for some φ. Vopěnka's principle is implied by several large-cardinal principles; see [**1**]. On the other hand, if one assumes that all cardinals are non-measurable (which is consistent with ZFC), then an idempotent functor which is not L_φ for any φ is displayed in [**17**].

By definition, every homomorphism φ is a φ-equivalence, since it is certainly orthogonal to all φ-local groups. Hence, we have the following useful remark.

LEMMA 2.1. *Let $\varphi\colon A \to B$ be a group homomorphism. Suppose that for every homomorphism $\alpha\colon A \to B$ there is a unique endomorphism $\beta\colon B \to B$ such that $\beta \circ \varphi = \alpha$. Then $B \cong L_\varphi A$.*

PROOF. The stated condition says that B is φ-local, and we know that φ is a φ-equivalence. Our claim follows. □

This situation is not infrequent. Keep in mind the example of the inclusion $j\colon \mathbb{Z} \hookrightarrow \mathbb{Q}$. In fact, if L is any idempotent functor with coaugmentation η, then for every group G the homomorphism $\eta_G\colon G \to LG$ fulfills the condition stated in Lemma 2.1. Hence, $LG \cong L_\varphi G$ where $\varphi = \eta_G$. Of course this does not say that L is of the form L_φ for all groups G, since η_G varies, but it does tell us that every property which is preserved by functors of the form L_φ will automatically be preserved by arbitrary idempotent functors (regardless of whether we are willing to assume large-cardinal principles or not).

The following result was known to several people with different approaches and motivations, although we are not aware that it appeared in the literature before [**16**] or [**34**]. The argument given here is due to Farjoun.

THEOREM 2.2. *Let L be any idempotent functor in the category of groups, and let A be any abelian group. Then LA is abelian.*

PROOF. Consider the coaugmentation homomorphism $\eta_A\colon A \to LA$. Fix any element $a \in A$. Let $\tau_a\colon LA \to LA$ be conjugation by $\eta_A(a)$; that is, $\tau_a(x) = \eta_A(a)^{-1} x\, \eta_A(a)$. Then $\tau_a(\eta_A(b)) = \eta_A(b)$ for all $b \in A$, since A is abelian. By Proposition 1.2, the equality $\tau_a \circ \eta_A = \eta_A$ implies that τ_a is the identity homomorphism, for every $a \in A$. Now pick any element $x \in LA$ and let $\tau_x\colon LA \to LA$ be conjugation by x. Then we know from the first part of the argument that $\tau_x(\eta_A(a)) = \eta_A(a)$ for every $a \in A$. Again from Proposition 1.2 it follows that τ_x is the identity homomorphism, for every $x \in LA$, and this means that LA is abelian. □

As a consequence of this result, every idempotent functor L in the category of groups restricts to an idempotent functor in the full subcategory of abelian groups, which we denote with the same letter L. The L-local abelian groups are precisely the abelian groups which are L-local in the category of groups, and the L-equivalences between abelian groups are the homomorphisms which are L-equivalences when viewed in the category of groups.

Nilpotent groups of class two are sent to nilpotent groups of class less than or equal to two by all idempotent functors. This fact was discovered in 1997 by Dwyer and Farjoun (unpublished). The first published proof was given by Libman in [**34**]. Here is an alternative argument found by the author.

THEOREM 2.3. *If L is any idempotent functor in the category of groups and N is a nilpotent group of class two, then LN is nilpotent of class less than or equal to two.*

PROOF. The assumption that N is nilpotent of class 2 tells us that the commutator subgroup $[N, N]$ is in the center of N. From this fact it follows that, for any fixed element $a \in N$, the map $f_a(x) = [a, x]$ is a group homomorphism from N to itself. (Let us make the convention that $x^y = yxy^{-1}$ and $[x, y] = xyx^{-1}y^{-1}$, and let us abbreviate η_N to η for simplicity of notation.) Let $F_a\colon LN \to LN$ be the unique homomorphism such that $F_a \circ \eta = \eta \circ f_a$. We are going to show that $F_a(y) = [\eta(a), y]$ for all $y \in LN$. To this aim, fix any element $b \in N$ and define a homomorphism $\phi_{a,b}\colon LN \to LN$ by $\phi_{a,b}(y) = \eta(b)\, F_a(y)\, \eta(b)^{-1}$. Then, for each $x \in N$ we have

$$\phi_{a,b}(\eta(x)) = \eta(b)\, F_a(\eta(x))\, \eta(b)^{-1} = \eta(b[a, x]b^{-1}) = \eta([a, x]) = F_a(\eta(x)).$$

From this equation and the universal property of η it follows that $\phi_{a,b}$ and F_a are identical, so that $[F_a(y), \eta(b)] = 1$ for any choice of a, b, and y. Using again the universal property of η, we infer that the subgroup $\operatorname{Im} F_a$ is in the center of LN for each $a \in N$. Next, consider the map $\psi_a\colon LN \to LN$ defined as $\psi_a(y) = F_a(y)\, y$. Since $\operatorname{Im} F_a$ is in the center of LN, this map ψ_a is a group homomorphism. Moreover, for every $x \in N$ we have

$$\psi_a(\eta(x)) = \eta([a, x])\, \eta(x) = \eta(a)\, \eta(x)\, \eta(a)^{-1}$$

and this implies that $\psi_a(y) = \eta(a)y\, \eta(a)^{-1}$ for all $y \in LN$. Hence, $F_a(y) = [\eta(a), y]$, as claimed. Since $\operatorname{Im} F_a$ is in the center of LN, we have $[[\operatorname{Im} \eta, LN], LN] = 1$.

The next step is to show that $[[LN, LN], \operatorname{Im} \eta] = 1$. This is a consequence of Witt's identity,

$$[[a^{-1}, b], c]^a\, [[c^{-1}, a], b]^c\, [[b^{-1}, c], a]^b = 1.$$

(We owe this remark to O'Sullivan.) Indeed, for any choice of $x \in N$, $y \in LN$, and $z \in LN$, we have

$$[[y, z], \eta(x)]^{y^{-1}} \; [[\eta(x)^{-1}, y^{-1}], z]^{\eta(x)} \; [[z^{-1}, \eta(x)], y^{-1}]^{z} = 1.$$

Since we already know that the second and third factors are trivial, the first factor is also trivial, as we needed. Now the universal property of η tells us that $[[LN, LN], LN] = 1$, and this implies that LN is nilpotent of class less than or equal to 2. □

So far, this result has not been extended to higher nilpotency classes. On the other hand, we know that the same statement is false for metabelian groups. Indeed, O'Sullivan has proved in [**41**] that the localization of a free metabelian group of rank two or higher at any proper subset of primes fails to be metabelian.

THEOREM 2.4. *Let L be any idempotent functor on abelian groups. Suppose that A is an abelian group such that $nA = 0$ for some integer n. Then $nLA = 0$.*

PROOF. Let $\mu_n\colon A \to A$ be the homomorphism defined as $\mu_n(a) = na$ for all $a \in A$, and define $\mu_n\colon LA \to LA$ in the same way. Let $\eta_A\colon A \to LA$ be the coaugmentation. Then $\eta_A(na) = n\eta_A(a)$ for all $a \in A$, so $\eta_A \circ \mu_n = \mu_n \circ \eta_A$. Since μ_n is the zero homomorphism in A by assumption, we also have $\eta_A \circ \mu_n = 0$. Hence, by Proposition 1.2, μ_n is the zero homomorphism in LA, as claimed. □

In other words, if A is a commutative group of finite exponent n, then LA also has a finite exponent m which divides n. This is false in general for noncommutative groups. Indeed, in Example 3.4 of [**34**] it is shown that if Σ_n denotes the symmetric group on n letters, then the inclusion $j\colon \Sigma_n \hookrightarrow \Sigma_{n+1}$ induces a bijection

$$\mathrm{Hom}(\Sigma_{n+1}, \Sigma_{n+1}) \cong \mathrm{Hom}(\Sigma_n, \Sigma_{n+1})$$

if $n \geq 7$. It follows, by Lemma 2.1, that $L_j\Sigma_n \cong \Sigma_{n+1}$, so exponents need not be preserved by idempotent functors.

Libman proved in [**35**] that the class of finite groups is not preserved by idempotent functors, by displaying a representation

$$\sigma\colon A_n \to SO_{n-1}(\mathbb{R})$$

where A_n is the alternating group with n even and $n \geq 10$, to which Lemma 2.1 applies, yielding $L_\sigma A_n \cong SO_{n-1}(\mathbb{R})$. This result was improved by Göbel, Rodríguez and Shelah in [**30**], by showing that for every given cardinal α and every nonabelian finite simple group G there is an idempotent functor L such that LG has cardinality higher than α, assuming the validity of the Generalized Continuum Hypothesis.

One might inquire if the class of torsion abelian groups is preserved by idempotent functors (a group A is called *torsion* if all its elements have finite order). This is false, as shown also by Libman with the following counterexample in [**34**]. If we let S be the direct sum of $\mathbb{Z}/p$ for all primes p, and T the cartesian product of $\mathbb{Z}/p$ for all primes p, then the embedding $j\colon S \hookrightarrow T$ induces a bijection $\mathrm{Hom}(T, T) \cong \mathrm{Hom}(S, T)$. Therefore, by Lemma 2.1, we have $L_jS \cong T$. But S is torsion and T is not, since $(1, 1, 1, \dots)$ has infinite order.

The following argument was given in [**34**] to show that divisibility is preserved by idempotent functors on abelian groups.

Lemma 2.5. *Let G be any group and let L be any idempotent functor, with coaugmentation η. If the image of $\eta_G\colon G \to LG$ is L-local, then η_G is surjective.*

Proof. The projection $\pi\colon G \twoheadrightarrow \operatorname{Im}\eta_G$ is an L-equivalence, since every homomorphism $\alpha\colon G \to Y$ where Y is L-local factors to $\operatorname{Im}\eta_G$, and it does so uniquely since π is an epimorphism. Therefore, the inclusion $\operatorname{Im}\eta_G \hookrightarrow LG$ is an L-equivalence between L-local groups, hence surjective. □

Theorem 2.6. *If L is any idempotent functor on abelian groups and D is any divisible abelian group, then LD is also divisible. Moreover, the localization homomorphism $\eta_D\colon D \to LD$ is surjective.*

Proof. The image of η_D is an epimorphic image of D and hence it is divisible. Therefore, it is a direct summand in LD. Since the class of L-local groups is closed under retracts, $\operatorname{Im}\eta_D$ is L-local. By Lemma 2.5, η_D is an epimorphism, so LD is divisible. □

Let $\mathbb{Z}(p^\infty)$ be the Prüfer group of p-roots of unity, i.e., the direct limit of $\mathbb{Z}/p^n$, where p is any prime. Theorem 2.6 implies that either $L\mathbb{Z}(p^\infty) = 0$ or $\mathbb{Z}(p^\infty)$ is L-local, since $\mathbb{Z}(p^\infty)$ has no proper quotients.

Here is a naïve list of classes of groups which could be studied, attempting to decide whether or not each of these classes of groups is preserved by arbitrary idempotent functors:

1. Finite p-groups, where p is any prime.
2. Perfect groups, i.e., groups whose abelianization is zero.
3. Simple groups.
4. (Not necessarily abelian) divisible groups.

3. Preservation of higher structures

In this section we present a purely group-theoretical formulation of basic ideas from [**16**]. Hence, although several results in this section are new, former collaboration with Rodríguez and Tai is to be acknowledged.

We first describe how to compute idempotent transformations of arbitrary finitely generated abelian groups. Thus, let L be any idempotent functor on groups, with coaugmentation η.

Theorem 3.1. *For each prime p, either $\mathbb{Z}/p$ is L-local or $L\mathbb{Z}/p = 0$.*

Proof. By Theorem 2.4, $L\mathbb{Z}/p$ is either zero or an abelian group of exponent p; that is, a $\mathbb{Z}/p$-vector space. Hence, if $L\mathbb{Z}/p \neq 0$, then $\mathbb{Z}/p$ is a retract of $L\mathbb{Z}/p$, so $\mathbb{Z}/p$ is L-local by Proposition 1.3. □

Lemma 3.2. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be any short exact sequence of groups. If LN is trivial, then the projection $G \twoheadrightarrow Q$ is an L-equivalence.*

Proof. Given any homomorphism $\alpha\colon G \to Y$ where Y is L-local, we have $\alpha(N) = 1$ and therefore α factors uniquely through Q. This says that the projection $G \twoheadrightarrow Q$ is orthogonal to all L-local groups and hence it is an L-equivalence. □

Theorem 3.3. *If $\mathbb{Z}/p^n$ is L-local for some $n \geq 2$, then $\mathbb{Z}/p^k$ is also L-local for every $k < n$.*

Proof. If $L\mathbb{Z}/p = 0$, then $L\mathbb{Z}/p^n = 0$ for all n by induction, applying Lemma 3.2 to the short exact sequences

$$\mathbb{Z}/p \rightarrowtail \mathbb{Z}/p^n \twoheadrightarrow \mathbb{Z}/p^{n-1}.$$

Hence, if $\mathbb{Z}/p^n$ is L-local for some $n \geq 2$, then $\mathbb{Z}/p$ is L-local, by Theorem 3.1. Now argue downwards with the short exact sequences

$$\mathbb{Z}/p^{k-1} \rightarrowtail \mathbb{Z}/p^k \twoheadrightarrow \mathbb{Z}/p,$$

using the fact that the kernel of any homomorphism between L-local groups is L-local, by Proposition 1.3. □

Theorem 3.4. *If $A = \mathbb{Z}/p^n$ for some prime p and a positive integer n, then either $LA = 0$ or $LA \cong \mathbb{Z}/p^k$ with $1 \leq k \leq n$. Moreover, $\eta_A \colon A \to LA$ is surjective.*

Proof. Suppose that LA is nonzero. Then, by Theorem 2.4, LA is annihilated by p^k for some $k \leq n$. This implies that $\mathbb{Z}/p^k$ is a retract of LA and hence $\mathbb{Z}/p^k$ is L-local. The image of η_A is a cyclic subgroup of LA of order less than or equal to p^k, hence $\operatorname{Im}\eta_A$ is L-local, by Theorem 3.3. Using Lemma 2.5, we conclude that η_A is surjective. □

Lemma 3.5. *Given arbitrary groups G and H, the product homomorphism $\eta_G \times \eta_H \colon G \times H \to LG \times LH$ is an L-equivalence.*

Proof. Let $\alpha \colon G \times H \to Y$ be any homomorphism where Y is L-local. Define $j_G \colon G \to G \times H$ by by $j_G(x) = (x, 1)$ and $j_H \colon H \to G \times H$ by $j_H(y) = (1, y)$. Then there is a unique $\beta_G \colon LG \to Y$ such that $\beta_G \circ \eta_G = \alpha \circ j_G$ and a unique $\beta_H \colon LH \to Y$ such that $\beta_H \circ \eta_H = \alpha \circ j_H$. Define $\beta \colon LG \times LH \to Y$ by $\beta(x, y) = \beta_G(x)\beta_H(y)$. Then $\beta \circ (\eta_G \times \eta_H) = \alpha$, but some argument is needed to justify that β is a group homomorphism. Since $\operatorname{Im} j_G$ and $\operatorname{Im} j_H$ commute in $G \times H$, we have $[\operatorname{Im}(\alpha \circ j_G), \operatorname{Im}(\alpha \circ j_H)] = 1$ in Y. Hence, $[\operatorname{Im}(\beta_G \circ \eta_G), \operatorname{Im}(\beta_H \circ \eta_H)] = 1$. Using twice the universal property of η, we first infer that $[\operatorname{Im}\beta_G, \operatorname{Im}(\beta_H \circ \eta_H)] = 1$ and then $[\operatorname{Im}\beta_G, \operatorname{Im}\beta_H] = 1$, as needed. Finally, if $\gamma \colon LG \times LH \to Y$ also satisfies $\gamma \circ (\eta_G \times \eta_H) = \alpha$, then $\gamma(x, 1) = \beta_G(x)$ for all $x \in LG$ and $\gamma(1, y) = \beta_H(y)$ for all $y \in LH$, so $\gamma = \beta$. □

Since $LG \times LH$ is L-local, this result says that there is a natural isomorphism

$$L(G \times H) \cong LG \times LH,$$

and the same is true for every product with a finite number of factors. This result (or special cases of it) was found independently in [**9**], [**33**], [**40**], and [**44**].

The following result is a straightforward consequence, using Theorem 3.4.

Theorem 3.6. *If A is any finite abelian group, then LA is also finite abelian, and $\eta_A \colon A \to LA$ is surjective.* □

In fact, if A is finite abelian, then LA is completely determined by the following sequence of integers. For each prime p, let d_p be the largest exponent d such that $\mathbb{Z}/p^d$ is L-local. This will be called the *p-transitional dimension* of L. (It is infinite if $\mathbb{Z}/p^d$ is L-local for all d, and it is zero if $L\mathbb{Z}/p = 0$.) The key observation is that, if $n \geq d_p$, then $L\mathbb{Z}/p^n \cong \mathbb{Z}/p^{d_p}$, since $L\mathbb{Z}/p^n \cong \mathbb{Z}/p^k$ for some $k \leq n$ by Theorem 3.4

(in fact, $k \leq d_p$, since $\mathbb{Z}/p^k$ is L-local), and k cannot be smaller than d_p, since the projection $\mathbb{Z}/p^n \twoheadrightarrow \mathbb{Z}/p^k$ is an L-equivalence and thus it is orthogonal to $\mathbb{Z}/p^{d_p}$.

After this achievement, in order to describe the effect of idempotent functors on finitely generated abelian groups, only $L\mathbb{Z}$ remains to be discussed. Which groups can be of the form $L\mathbb{Z}$ for some L, besides $\mathbb{Q}$, its subrings, and $\mathbb{Z}/n$ for all n? It is surprising to discover that there is a proper class (i.e., not a set) of nonisomorphic abelian groups of the form $L\mathbb{Z}$ for some L, and that all admit a commutative ring structure. Here we shall only mention the fundamental facts about $L\mathbb{Z}$ and refer to [**16**] for further details.

Theorem 3.7. *If L is any idempotent functor on abelian groups, then $L\mathbb{Z}$ is the underlying abelian group of a commutative ring R with* 1 *such that evaluation at* 1 *yields a ring isomorphism* $\operatorname{Hom}(R,R) \cong R$.

Proof. The coaugmentation $\eta\colon \mathbb{Z} \to L\mathbb{Z}$ induces a bijection

$$\operatorname{Hom}(L\mathbb{Z}, L\mathbb{Z}) \cong \operatorname{Hom}(\mathbb{Z}, L\mathbb{Z}),$$

which is in fact an isomorphism of abelian groups. But $\operatorname{Hom}(\mathbb{Z}, L\mathbb{Z}) \cong L\mathbb{Z}$, by an isomorphism sending each φ to $\varphi(1)$. Hence, $L\mathbb{Z}$ acquires a ring structure, with the multiplication borrowed from composition in the endomorphism ring $\operatorname{Hom}(L\mathbb{Z}, L\mathbb{Z})$. The unity of this multiplication is the element $\eta(1) \in L\mathbb{Z}$, which we keep denoting by 1. Let us denote by R the abelian group $L\mathbb{Z}$ with this ring structure. In order to see that R is a commutative ring, let r be any element of R and consider the two group homomorphisms $R \to R$ given by

$$\varphi_r(x) = rx, \qquad \psi_r(x) = xr.$$

Then $\varphi_r(1) = \psi_r(1)$ and this implies that $\varphi_r = \psi_r$, for any $r \in R$, so R is commutative. Next, we aim to show that the isomorphism $\operatorname{Hom}(R,R) \cong R$ is not only a group isomorphism, but a ring isomorphism. Let $\varphi\colon R \to R$ be any group homomorphism. Fix $r \in R$ and define two group homomorphisms $R \to R$ by

$$\alpha_r(x) = \varphi(r)x, \qquad \beta_r(x) = \varphi(rx).$$

Then $\alpha_r(1) = \beta_r(1)$ and hence $\alpha_r = \beta_r$. This shows that every $\varphi \in \operatorname{Hom}(R,R)$ is automatically an R-module map. From this fact we infer that

$$(\varphi \circ \psi)(1) = \varphi(\psi(1)) = \varphi(1)\psi(1),$$

so evaluation at 1 is a ring homomorphism, as claimed. □

Rings R with 1 for which the map $\operatorname{Hom}(R,R) \to R$ sending each φ to $\varphi(1)$ is bijective (hence R is necessarily commutative) are called *rigid rings* in [**16**], since they have as few additive endomorphisms as possible. (They are more frequently called *E-rings* by other authors, as in [**27**].) It was shown in [**27**] that there exist rigid rings of arbitrarily large cardinality. If R is a rigid ring, then Lemma 2.1 applies to the map $u\colon \mathbb{Z} \to R$ sending $1 \in \mathbb{Z}$ to $1 \in R$. It follows that $L_u\mathbb{Z} \cong R$, so all rigid rings occur as images of $\mathbb{Z}$ under idempotent functors. For this reason, the abelian groups of the form $L\mathbb{Z}$ constitute a proper class. Familiar examples of rigid rings include the subrings of the rationals, $\mathbb{Z}/n$ for all n, the p-adics $\hat{\mathbb{Z}}_p$, and each of the products

$$\prod_{p \in P} \mathbb{Z}/p, \qquad \prod_{p \in P} \mathbb{Z}_{(p)}, \qquad \prod_{p \in P} \hat{\mathbb{Z}}_p,$$

for any set of primes P, among many other examples.

In what follows, let L be any idempotent functor on abelian groups, with coaugmentation η. We view rings and modules as abelian groups with additional structure. Rings are associative and have a unity, which is denoted by 1.

THEOREM 3.8. *Let R be any ring and let M be any left R-module. Then LM admits a unique left R-module structure such that $\eta_M\colon M \to LM$ is an R-module homomorphism.*

PROOF. From the universal property of L it follows that, if f and g are any two abelian group homomorphisms $A \to B$, then $L(f+g) = Lf + Lg$. Hence, the natural map $\mathrm{Hom}(A,B) \to \mathrm{Hom}(LA,LB)$ is a ring homomorphism for all A and B. If we set both A and B equal to the underlying abelian group of M, and let $R \to \mathrm{Hom}(M,M)$ be the structure map of M as a left R-module, then the composite

$$R \to \mathrm{Hom}(M,M) \to \mathrm{Hom}(LM,LM)$$

endows LM with a left R-module structure satisfying $\eta_M(ra) = r\eta_M(a)$ for all $r \in R$ and $a \in M$, as claimed. The uniqueness of this R-module structure is again inferred from the universal property of L. □

Of course, the same statement holds for right R-modules.

THEOREM 3.9. *Let R be any ring. Then LR is either zero or it admits a unique ring structure such that $\eta_R\colon R \to LR$ is a ring homomorphism. If R is commutative, then LR is also commutative.*

PROOF. Let us denote η_R by η for simplicity. It induces an isomorphism of abelian groups

$$\mathrm{Hom}(LR,LR) \cong \mathrm{Hom}(R,LR).$$

View LR as a *right* R-module in the same way as in Theorem 3.8. Then this isomorphism restricts to a monomorphism $\mathrm{Hom}_R(LR,LR) \to \mathrm{Hom}_R(R,LR)$, and we have $\mathrm{Hom}_R(R,LR) \cong LR$ via evaluation at 1. Next we show that this monomorphism is in fact an isomorphism. Given any right R-module map $\varphi\colon R \to LR$, take its preimage $\psi \in \mathrm{Hom}(LR,LR)$, that is, the unique group homomorphism such that $\psi \circ \eta = \varphi$. We need to show that ψ is an R-module map. But this follows from the universal property of η, since

$$\psi(\eta(r)s) = \psi(\eta(rs)) = \varphi(rs) = \varphi(r)s = \psi(\eta(r))s$$

for all $r, s \in R$. Hence, LR inherits the ring structure of $\mathrm{Hom}_R(LR,LR)$. The identity map of LR goes to $\eta(1)$, so $\eta(1) = 1$. In order to check that η is a ring homomorphism, pick any two elements r, s in R and let φ, ψ be R-module maps from LR to itself such that $\varphi(1) = \eta(r)$ and $\psi(1) = \eta(s)$. Then $\eta(r)\eta(s) = \varphi(\psi(1)) = \varphi(\eta(s)) = \varphi(1)s = \eta(r)s = \eta(rs)$, as needed.

Let $*$ be any multiplication in LR which is compatible with η. Then $\eta(r) * 1 = \eta(r) * \eta(1) = \eta(r)$ for all $r \in R$, and this implies that $a * 1 = a$ for all $a \in LR$. Then, for any fixed element $a \in LR$, the R-module maps $LR \to LR$ given by

$$\alpha_a(x) = ax, \qquad \beta_a(x) = a * x$$

satisfy $\alpha_a(1) = \beta_a(1)$ and hence coincide.

If R is commutative, then LR is also commutative because $\varphi_a(x) = ax$ and $\psi_a(x) = xa$ satisfy $\varphi_a(1) = \psi_a(1)$ and hence coincide, for all $a \in LR$. (The commutativity of R is needed in order that they be both R-module maps.) In this case, we say that LR is a *rigid R-algebra*, similarly as above. □

The next result is of great value in practice. It tells us, for example, that if $L\mathbb{Z} = \mathbb{Z}_P$ for some set of primes P, then LA will be a $\mathbb{Z}_P$-module for any abelian group A whatsoever.

THEOREM 3.10. *If M is any left R-module, then the left R-module structure of LM can be uniquely extended to a left LR-module structure.*

PROOF. First of all, observe that if B is any L-local abelian group, then $\mathrm{Hom}(A, B)$ is L-local for any abelian group A, since for any L-equivalence $C \to D$ we have $\mathrm{Hom}(D, \mathrm{Hom}(A, B)) \cong \mathrm{Hom}(A, \mathrm{Hom}(D, B)) \cong \mathrm{Hom}(A, \mathrm{Hom}(C, B)) \cong \mathrm{Hom}(C, \mathrm{Hom}(A, B))$. Let $R \to \mathrm{Hom}(M, M)$ be the structure map of M as an R-module. Then the composite

$$R \to \mathrm{Hom}(M, M) \to \mathrm{Hom}(LM, LM)$$

factors uniquely through $\eta_R \colon R \to LR$, since $\mathrm{Hom}(LM, LM)$ is L-local. In order to check that the resulting map $LR \to \mathrm{Hom}(LM, LM)$ is a ring homomorphism, use twice the universal property of η. □

THEOREM 3.11. *If K is a field, then either $LK = 0$ or K is L-local.*

PROOF. If K is a field, then LK is necessarily free as a K-module. Hence, if $LK \neq 0$, then K is a retract of LK, so it is L-local. □

Thus, there is a big contrast between the wide range of possibilities for $L\mathbb{Z}$ and the necessary triviality of $L\mathbb{Q}$ or $L\mathbb{Z}/p$. We can even extend Theorem 3.11 as follows.

THEOREM 3.12. *If V is any vector space over a field K, then either $LV = 0$ or V is L-local.*

PROOF. If $LK = 0$, then $LV = 0$ by (1.2). Otherwise, K is L-local. Then $V \cong \bigoplus_{i \in I} K$ for some set of indices I. But the embedding of V into $\prod_{i \in I} K$ splits as a homomorphism of vector spaces. This shows that V is a retract of an L-local abelian group, so it is L-local. □

4. Homotopy idempotent functors

We shall work in the homotopy category $\mathcal{H}$ of CW-complexes, where a morphism $X \to Y$ is an equivalence class of maps $f \colon X \to Y$ under the homotopy relation; that is, $f \simeq g$ if there is a map $F \colon X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$, where I denotes the closed interval $[0, 1]$.

A *homotopy functor* is a functor L on CW-complexes which carries homotopy equivalences into homotopy equivalences. As we next explain, every homotopy functor L yields a functor in the homotopy category $\mathcal{H}$, which we denote with the same letter L. Thus, we have to show that $f \simeq g$ implies $Lf \simeq Lg$, assuming that L carries homotopy equivalences into homotopy equivalences. The following argument is due to Quillen [**42**]. Let $F \colon X \times I \to Y$ be a homotopy from f to g. Denote by $i_0 \colon X \to X \times I$ and $i_1 \colon X \to X \times I$ the inclusions $i_0(x) = (x, 0)$, $i_1(x) = (x, 1)$. Since the projection $p \colon X \times I \to X$ is a homotopy equivalence, the map $Lp \colon L(X \times I) \to LX$

is also a homotopy equivalence, because L is a homotopy functor. Hence, the equality $Lp \circ Li_0 = Lp \circ Li_1$ implies that $Li_0 \simeq Li_1$, and we infer that

$$Lf = L(F \circ i_0) = LF \circ Li_0 \simeq LF \circ Li_1 = L(F \circ i_1) = Lg,$$

as claimed.

A *homotopy idempotent functor* is a homotopy functor L together with a natural transformation $\eta\colon \mathrm{Id} \to L$, such that L is idempotent when viewed in the homotopy category, with coaugmentation η. Hence, to every map $f\colon X \to Y$ we associate another map $Lf\colon LX \to LY$ such that $\eta_Y \circ f$ is equal to $Lf \circ \eta_X$ (not just homotopic). However, the natural map $\eta_X\colon X \to LX$ is only universal up to homotopy. That is, for every map $\alpha\colon X \to LY$ there is a map $\beta\colon LX \to LY$ such that $\beta \circ \eta_X \simeq \alpha$, and β is unique up to homotopy with this property. In general, LLX and LX will not be homeomorphic, but homotopy equivalent.

A CW-complex X is L-local if it is homotopy equivalent to LY for some Y. A map $f\colon X \to Y$ is an L-equivalence if $Lf\colon LX \to LY$ is a homotopy equivalence. Thus, $\eta_X\colon X \to LX$ is an L-equivalence, and it has the following universal properties:

1. η_X is initial in the homotopy category among all maps from the space X into L-local spaces;
2. η_X is terminal in the homotopy category among all L-equivalences going out of the space X.

It is very often convenient to consider the category $\mathcal{S}$ whose objects are simplicial sets and whose morphisms are simplicial maps. As explained in [**42**], the homotopy category derived from $\mathcal{S}$ by formally inverting the weak equivalences (i.e., maps inducing isomorphisms of homotopy groups) is equivalent to $\mathcal{H}$. In other words, we want to state theorems about $\mathcal{H}$, and in order to do so we may use CW-complexes or simplicial sets as "models", depending on the most suitable toolwork in each circumstance.

A homotopy functor in $\mathcal{S}$ is one that carries weak equivalences into weak equivalences. Homotopy idempotent functors are defined in the same way as above. The only important technical detail is that L-local simplicial sets have to be asked to be fibrant (i.e., satisfying the Kan extension condition; see [**19**]).

The following fundamental fact about homotopy idempotent functors (Theorem 4.1 below) was discovered by Farjoun in [**25**] with a nonessential continuity assumption, and later generalized in [**17**] to the form given here. Given CW-complexes X and Y, one can consider the space $\mathrm{map}(X, Y)$ of maps $X \to Y$ with the compact-open topology. However, once mapping spaces enter into the discussion, it is more convenient to use simplicial sets as models. If X and Y are simplicial sets, we denote by $\mathrm{map}(X, Y)$ the simplicial mapping space from X to Y, whose n-simplices are simplicial maps $\Delta[n] \times X \to Y$, where $\Delta[n]$ denotes the standard n-simplex. If the simplicial set Y is fibrant, then the geometric realization $|\mathrm{map}(X, Y)|$ has the same weak homotopy type as the topological mapping space $\mathrm{map}(|X|, |Y|)$. The techniques used in the following arguments are simplicial in nature and would be cumbersome (or even wrong) if stated for CW-complexes.

Theorem 4.1. *Let L be any homotopy idempotent functor in the category of simplicial sets. For every space X, the natural map $\eta_X\colon X \to LX$ induces a weak equivalence* $\mathrm{map}(LX, Y) \to \mathrm{map}(X, Y)$ *for every L-local space Y.*

Proof. Let $\mathcal{M}(X,Y)$ denote the category whose objects are the sequences $X \to C \leftarrow Y$ in which the map $Y \to C$ is a weak equivalence, and whose morphisms are commutative diagrams. Then, as explained in [**17**] and [**28**], the nerve $N\mathcal{M}(X,Y)$ of this category is naturally weakly equivalent to the mapping space $\mathrm{map}(X,Y)$, if Y is fibrant. Observe that the functor L induces natural maps $N\mathcal{M}(X,Y) \to N\mathcal{M}(LX,LY)$, which we keep denoting by L, as every sequence $X \to C \leftarrow Y$ can be mapped into $LX \to LC \leftarrow LY$. (This fact is one of the reasons why we use such nerves instead of mapping spaces, since a functor L need not induce a map from the space $\mathrm{map}(X,Y)$ to $\mathrm{map}(LX,LY)$ in general; L is called continuous when it does.)

Now, in order to prove the theorem, we check that, for every space X, the map $f\colon N\mathcal{M}(LX,Y) \to N\mathcal{M}(X,Y)$ induced by $\eta_X\colon X \to LX$ is a homotopy equivalence. We also denote f by $(\eta_X)^*$ when it is more convenient. In fact, f is induced by the functor $F\colon \mathcal{M}(LX,Y) \to \mathcal{M}(X,Y)$ sending a sequence $LX \to C \leftarrow Y$ that we denote by (φ_1,φ_2) to the sequence $(\varphi_1 \circ \eta_X, \varphi_2)$. Define another functor $G\colon \mathcal{M}(X,Y) \to \mathcal{M}(LX,Y)$ by sending (ψ_1,ψ_2) to $(L\psi_1, L\psi_2 \circ \eta_Y)$, and let g the map induced by G on the nerves. Then one checks that f and g are homotopy inverse to each other, as follows. The fact that η is a natural transformation yields a natural transformation $\mathrm{Id} \to F \circ G$, which tells us that $f \circ g \simeq \mathrm{id}$; that is $(\eta_X)^* \circ (\eta_Y)^* \circ L \simeq \mathrm{id}$. Exactly the same argument shows that the composite $(\eta_Y)^* \circ (\eta_{LX})^* \circ L$, depicted as

$$N\mathcal{M}(LX,Y) \to N\mathcal{M}(LLX,LY) \to N\mathcal{M}(LX,LY) \to N\mathcal{M}(LX,Y),$$

is homotopic to the identity map. Moreover, we have an equality $(L\eta_X)^* \circ L = L \circ (\eta_X)^*$, since they are induced by the same functor. To conclude, recall that $L\eta_X \simeq \eta_{LX}$, since L is a homotopy idempotent functor. This yields

$$g \circ f = (\eta_Y)^* \circ L \circ (\eta_X)^* = (\eta_Y)^* \circ (L\eta_X)^* \circ L \simeq (\eta_Y)^* \circ (\eta_{LX})^* \circ L \simeq \mathrm{id},$$

as claimed. □

Now assume that X and Y are simplicial sets with distinguished base points x_0 and y_0. Let $\mathrm{map}_*(X,Y)$ denote the pointed mapping space, whose n-simplices are simplicial maps $\Delta[n] \rtimes X \to Y$, where $\Delta[n] \rtimes X$ is the space obtained by collapsing $\Delta[n] \vee \{x_0\}$ inside $\Delta[n] \times X$. (The geometric realization $|\mathrm{map}_*(X,Y)|$ has the same weak homotopy type as the space of maps $f\colon |X| \to |Y|$ such that $f(x_0) = y_0$, with the compact-open topology, if Y is fibrant.) There is a fibration

$$\mathrm{map}_*(X,Y) \to \mathrm{map}(X,Y) \to Y \tag{4.1}$$

where the second arrow is evaluation at the base point; i.e., the map induced by the inclusion $\{x_0\} \hookrightarrow X$ and the isomorphism $\mathrm{map}(\{x_0\},Y) \cong Y$. If Y is connected, this fibration tells us that $\eta_X\colon X \to LX$ induces a weak equivalence $\mathrm{map}(LX,Y) \to \mathrm{map}(X,Y)$ if and only if it induces a weak equivalence of pointed mapping spaces $\mathrm{map}_*(LX,Y) \to \mathrm{map}_*(X,Y)$. The latter is often more useful in practice.

Theorem 4.2. *Let L be any homotopy idempotent functor in the category of simplicial sets. A map $f\colon X \to Y$ is an L-equivalence if and only if the induced map* $\mathrm{map}(Y,Z) \to \mathrm{map}(X,Z)$ *is a weak equivalence for all L-local spaces Z.*

Proof. The assumption $\mathrm{map}(Y,Z) \simeq \mathrm{map}(X,Z)$ implies that $\pi_0 \mathrm{map}(Y,Z) \cong \pi_0 \mathrm{map}(X,Z)$ when Z is L-local, and this tells us that f is an L-equivalence, since it is orthogonal to all L-local spaces in the homotopy category. To prove the converse, consider the commutative diagram

$$\begin{array}{ccc} \mathrm{map}(LY,Z) & \rightarrow & \mathrm{map}(LX,Z) \\ \downarrow & & \downarrow \\ \mathrm{map}(Y,Z) & \rightarrow & \mathrm{map}(X,Z), \end{array}$$

where the vertical arrows are induced by the natural maps η_Y and η_X, and Z is L-local. Thus, the vertical arrows are weak equivalences by Theorem 4.1. The top horizontal arrow is also a weak equivalence, since the map $LX \to LY$ is a weak equivalence by assumption. Therefore, the bottom horizontal arrow is a weak equivalence, as claimed. □

From these results we infer the following.

Theorem 4.3. *For every homotopy idempotent functor L, the class of L-local spaces is closed under homotopy limits and homotopy retracts, and the class of L-equivalences is closed under homotopy colimits and homotopy retracts.*

Proof. Let $\mathcal{I}$ be any small category and $F\colon \mathcal{I} \to \mathcal{S}$ be a diagram in the category of simplicial sets. Suppose that $F(i)$ is L-local for all $i \in \mathcal{I}$. We have to show that the homotopy inverse limit $\mathrm{holim}\, F$ is L-local. For this, let $f\colon X \to Y$ be any given L-equivalence. Then there are weak equivalences $\mathrm{map}(Y, \mathrm{holim}\, F) \simeq \mathrm{holim}\ \mathrm{map}(Y,F) \simeq \mathrm{holim}\ \mathrm{map}(X,F) \simeq \mathrm{map}(X, \mathrm{holim}\, F)$, as desired. The argument to show that every homotopy retract of an L-local space is L-local is the same as in Proposition 1.3. The claim about L-equivalences follows similarly. □

Also the analogue of (1.2) holds for homotopy colimits.

Theorem 4.4. *Given any diagram $F\colon \mathcal{I} \to \mathcal{S}$ in the category of simplicial sets, the natural map* $\mathrm{hocolim}\, F \to \mathrm{hocolim}\, LF$ *is an L-equivalence.*

Proof. Similarly as above, we have weak equivalences $\mathrm{map}(\mathrm{hocolim}\, LF, X) \simeq \mathrm{holim}\ \mathrm{map}(LF,X) \simeq \mathrm{holim}\ \mathrm{map}(F,X) \simeq \mathrm{map}(\mathrm{hocolim}\, F, X)$ if X is L-local. □

Theorem 4.5. *Every homotopy idempotent functor sends connected spaces to connected spaces.*

Proof. This argument is due to Tai [**45**]. Suppose that X is connected and LX is not. Then the inclusion $j\colon \{*\} \hookrightarrow S^0$ is a retract of $\eta\colon X \to LX$ and hence j is an L-equivalence. It follows that j induces a weak equivalence $\mathrm{map}(S^0, LX) \simeq \mathrm{map}(\{*\}, LX)$. But this implies that LX is contractible, which is inconsistent with the fact that LX is not connected. □

The argument used in this proof tells us in fact that, if there is an L-equivalence which does not induce a bijection of connected components, then LX is contractible for all X.

It is an open problem to decide if every homotopy idempotent functor sends 1-connected spaces to 1-connected spaces. Several results in this direction were obtained by Tai in [**45**].

5. Localizing with respect to a map

As in the case of groups, every map $f\colon X \to Y$ between CW-complexes yields a homotopy idempotent functor L_f as follows. A CW-complex Z is called *f-local* if the map

$$\operatorname{map}(Y, Z) \to \operatorname{map}(X, Z) \tag{5.1}$$

induced by f is a weak homotopy equivalence. (If one works with simplicial sets, then (5.1) is asked to be a weak equivalence and Z is asked to be fibrant.) Equivalently, Z is f-local if and only if each of its connected components is f-local, and a connected space Z is f-local if and only if the map of pointed mapping spaces

$$\operatorname{map}_*(Y, Z) \to \operatorname{map}_*(X, Z)$$

induced by f is a weak homotopy equivalence; in order to prove this claim, use the fibration (4.1). The existence of a homotopy functor L_f yielding a reflection (called *f-localization*) from the homotopy category $\mathcal{H}$ onto the full subcategory of f-local spaces was proved by Farjoun in [**24**].

Farjoun asked in [**24**] if every homotopy idempotent functor is homotopy equivalent to L_f for some map f, since all the known examples supported this belief. In [**17**] it was shown that it is impossible to answer this question affirmatively using the ZFC axioms, since an affirmative answer to this question would imply the existence of measurable cardinals, which cannot be proved in ZFC. On the other hand, it was also shown in [**17**] that Vopěnka's principle (which is explained in [**1**] or [**32**]) implies that every homotopy idempotent functor is homotopy equivalent to L_f for some map f. Hence, a negative answer to Farjoun's question in ZFC would imply the inconsistency of Vopěnka's principle, so it is not to be expected either.

A fundamental example of an f-localization is localization with respect to a wedge of power maps of the circle $\rho_n\colon S^1 \to S^1$, $z \mapsto z^n$, where $n \in P'$ for a set of primes P. In this case, connected f-local spaces are those X such that the self-map of their loop space $\Omega X \to \Omega X$ raising every loop ω to ω^n is a weak homotopy equivalence for every $n \in P'$. Such spaces are called *P-local*.

As explained in [**14**], since ΩX is not connected in general, for $k \geq 1$ the homomorphism

$$\pi_k((\Omega X)_\omega) \to \pi_k((\Omega X)_{\omega^n}) \tag{5.2}$$

induced by ρ_n on the connected component of a loop ω can be described as follows. We use the homotopy equivalence $(\Omega X)_\omega \simeq (\Omega X)_0$ given by multiplication by ω^{-1}, and the standard isomorphism $\pi_k((\Omega X)_0) \cong \pi_{k+1}(X)$. Under these transformations, the homomorphism (5.2) becomes an endomorphism of $\pi_{k+1}(X)$, and as such it takes the form

$$a \mapsto a + [\omega] \cdot a + [\omega]^2 \cdot a + \cdots + [\omega]^{n-1} \cdot a$$

for $a \in \pi_{k+1}(X)$, $k \geq 1$, where $[\omega]$ denotes the class of ω in the fundamental group $\pi_1(X)$. (The action is the ordinary conjugation action of the fundamental group on the higher homotopy groups.) Since the self-map of $\pi_1(X)$ induced by ρ_n is $x \mapsto x^n$, we infer that a connected space X is P-local if and only if $\pi_1(X)$ is a P-local group in the sense of Section 2, and the endomorphisms $a \mapsto (1 + x + x^2 + \cdots + x^{n-1})a$ of $\pi_{k+1}(X)$ are bijective for $k \geq 1$, $n \in P'$, and every $x \in \pi_1(X)$.

Inspired by the latter condition, if G is any group and A is any $\mathbb{Z}G$-module, we call A a *P-local module* if the endomorphism $a \mapsto (1 + x + x^2 + \cdots + x^{n-1})a$ is an automorphism of A for $n \in P'$ and every $x \in G$. Notice that, if one considers the semidirect product $A \rtimes G$, then the nth power map in $A \rtimes G$ takes the form

$$(a, x)^n = ((1 + x + x^2 + \cdots + x^{n-1})a, x^n).$$

Hence, if G is a P-local group, then A is a P-local module if and only if the semidirect product $A \rtimes G$ is P-local.

Using this terminology, a connected space is P-local if and only if its fundamental group is a P-local group and its higher homotopy groups are P-local modules over the integral group ring of the fundamental group. Equivalently, X is P-local if and only if the semidirect products $\pi_k(X) \rtimes \pi_1(X)$ are P-local for all k (note that the semidirect product of $\pi_1(X)$ with itself under the conjugation action is isomorphic to the direct product).

Various features of P-local modules were described in [**14**], as well as many properties of P-localization of spaces. If a space X is 1-connected, then the condition of being P-local reduces to the classical condition that its homotopy groups $\pi_k(X)$ be $\mathbb{Z}_P$-modules for $k \geq 2$. Good motivation and a thorough discussion of P-localization of 1-connected (and nilpotent) spaces can be found in [**31**].

We close this section by recalling a purely algebraic description of P-equivalences between connected spaces.

Theorem 5.1. *Let P be any set of primes. A map $f: X \to Y$ of connected spaces is a P-equivalence if and only if the induced homomorphism $\pi_1(X) \to \pi_1(Y)$ is a P-equivalence of groups, and f induces isomorphisms $H^n(Y; A) \cong H^n(X; A)$ for all n and every P-local module A over the integral group ring of $\pi_1(Y)_P$.*

Proof. Suppose first that the stated algebraic condition is satisfied. Let Z be any (connected) P-local space and assume given a map $g: X \to Z$. Replace f by a cofibration if it is not already one. Then the existence and uniqueness (up to homotopy) of a map $h: Y \to Z$ such that $h \circ f \simeq g$ is guaranteed by obstruction theory through the skeleta of Y. Indeed, the homotopy groups of Z become P-local modules over $\pi_1(Y)_P$ via the homomorphism $\pi_1(Y)_P \to \pi_1(Z)$ determined by $g_*: \pi_1(X) \to \pi_1(Z)$ and the isomorphism $\pi_1(X)_P \cong \pi_1(Y)_P$.

To prove the converse, first use the fact that if G is any P-local group then a $K(G, 1)$ is a P-local space, in order to infer by means of (7.1) that $f_*: \pi_1(X) \to \pi_1(Y)$ is a P-equivalence of groups. Second, if G is any P-local group and A is any P-local module over $\mathbb{Z}G$, then the classifying spaces for cohomology with twisted A coefficients with fundamental group G are P-local spaces, and this yields the desired cohomology isomorphisms. □

Further details can be found in [**14**], where it is also proved that cohomology can be replaced by homology in the statement of Theorem 5.1.

6. Structures preserved by homotopy idempotent functors

Recall that an *H-space* is a space X with base point x_0, together with a map $\mu: X \times X \to X$ such that $\mu \circ i_1 \simeq \mathrm{id}_X$ and $\mu \circ i_2 \simeq \mathrm{id}_X$, where i_1 and i_2 denote the inclusions of X into $X \times X$ sending x to (x, x_0) and (x_0, x), respectively. An *H-map* $f: X \to Y$ between H-spaces is one satisfying $f \circ \mu_X \simeq \mu_Y \circ (f \times f)$. As we next show, every localization of an H-space is an H-space. This fact is based on the following result, which is analogous to Lemma 3.5 for groups.

Lemma 6.1. *Let L be any homotopy idempotent functor. For all X and Y, the product map $\eta_X \times \eta_Y \colon X \times Y \to LX \times LY$ is an L-equivalence.*

Proof. Let Z be any L-local space. Then we have natural weak equivalences $\mathrm{map}(LX \times LY, Z) \simeq \mathrm{map}(LX, \mathrm{map}(LY, Z)) \simeq \mathrm{map}(LX, \mathrm{map}(Y, Z))$, and back to $\mathrm{map}(X \times Y, Z)$ by symmetry. □

Since $LX \times LY$ is an L-local space (by Theorem 4.3), we have obtained a natural homotopy equivalence

$$h_{X,Y} \colon LX \times LY \to L(X \times Y)$$

such that $h_{X,Y} \circ (\eta_X \times \eta_Y) \simeq \eta_{X \times Y}$. Of course, the same is true for every finite product of spaces.

Theorem 6.2. *Let L be any homotopy idempotent functor. If X is an H-space, then LX is also an H-space, and $\eta_X \colon X \to LX$ is an H-map.*

Proof. Let $\mu_X \colon X \times X \to X$ be the multiplication map. Define another map $\mu_{LX} \colon LX \times LX \to LX$ as the composite $L\mu_X \circ h_{X,X}$, and abbreviate $h_{X,X}$ to h for simplicity of notation. Then

$$\mu_{LX} \circ i_1 = L\mu_X \circ h \circ i_1 \simeq L\mu_X \circ Li_1 = L(\mu_X \circ i_1) \simeq L(\mathrm{id}_X) = \mathrm{id}_{LX},$$

and similarly with i_2. (In order to check that $h \circ i_1 \simeq Li_1$, argue as follows: $h \circ i_1 \circ \eta_X = h \circ (\eta_X \times \eta_X) \circ i_1 \simeq \eta_{X \times X} \circ i_1 = Li_1 \circ \eta_X$.)

Finally, we check that η_X is an H-map. Since η is a natural transformation, we have $\eta_X \circ \mu_X = L\mu_X \circ \eta_{X \times X} \simeq L\mu_X \circ h \circ (\eta_X \times \eta_X) = \mu_{LX} \circ (\eta_X \times \eta_X)$. □

An H-space is called homotopy associative if $\mu \circ (\mu \times \mathrm{id}) \simeq \mu \circ (\mathrm{id} \times \mu)$. It is called homotopy commutative if the twist map $T \colon X \times X \to X \times X$ given by $T(x, y) = (y, x)$ satisfies $\mu \circ T \simeq \mu$.

Theorem 6.3. *Let L be any homotopy idempotent functor. If X is a homotopy associative H-space, then so is LX. If X is a homotopy commutative H-space, then so is LX.*

Proof. Since $\eta_X \times \eta_X \times \eta_X$ is an L-equivalence and η_X is an H-map, we argue as follows: $\mu_{LX} \circ (\mu_{LX} \times \mathrm{id}_{LX}) \circ (\eta_X \times \eta_X \times \eta_X) \simeq \mu_{LX} \circ (\eta_X \times \eta_X) \circ (\mu_X \times \mathrm{id}_X) \simeq \eta_X \circ \mu_X \circ (\mu_X \times \mathrm{id}_X) \simeq \eta_X \circ \mu_X \circ (\mathrm{id}_X \times \mu_X)$, and continue by symmetry. The argument for homotopy commutativity is the same. □

Farjoun proved in [**24**] that, if f is any map and X is any space, then

$$L_f(\Omega X) \simeq \Omega L_{\Sigma f} X,$$

where Σ denotes suspension. Therefore, f-localization sends loop spaces to loop spaces. Moreover, $\eta \colon \Omega X \to L_f(\Omega X)$ is a loop map. From these facts we immediately infer the following.

Theorem 6.4. *If L is any homotopy idempotent functor and Y is a loop space, then LY is also a loop space, and $\eta \colon Y \to LY$ is a loop map.*

PROOF. If $Y = \Omega X$, then $LY \simeq L_\eta Y \simeq \Omega L_{\Sigma\eta} X$, since η is an η-equivalence and LY is η-local. □

A space X is called a generalized Eilenberg–Mac Lane space (shortly a GEM) if it is a weak product of Eilenberg–Mac Lane spaces $\prod_{n=1}^{\infty} K(G_n, n)$, where G_1 is also abelian. Such a space is characterized by the fact that its fundamental group is abelian and all its Postnikov invariants vanish; hence its homotopy type is completely determined by its homotopy groups. Every GEM is an infinite loop space, since $K(G_n, n) \simeq \Omega K(G_n, n+1)$ for all n.

Abelian topological groups are GEM spaces. If we consider the category whose objects are GEM spaces and whose morphisms are maps which are compatible with the infinite loop structure, then the corresponding homotopy category is equivalent to the homotopy category of simplicial abelian groups and their homomorphisms. The latter is isomorphic to the homotopy category of abelian chain complexes and chain maps; see [**19**], [**23**]. Under these equivalences of categories, a $K(G, n)$ is represented by a chain complex of free abelian groups, which is just a free abelian presentation of G concentrated in dimensions n and $n + 1$. Hence, a GEM map $K(G, n) \to K(H, m)$ is necessarily nullhomotopic unless $m = n$ or $m = n + 1$. (However, of course, there are essential maps $K(G, n) \to K(H, m)$ for various values of m which are not GEM maps, i.e., which cannot be forever delooped.) The same arguments apply to simplicial R-modules and chain complexes of R-modules, for any commutative ring R.

For every connected space X, the infinite symmetric product (or Dold–Thom construction) $SP^\infty X$ is the direct limit of the quotients $SP^k X = X^k/\Sigma_k$ of the k-fold product of X with itself by the action of the symmetric group Σ_k, acting by permuting the factors. Thus, there is a natural map $j_X\colon X \to SP^\infty X$. The homotopy groups of $SP^\infty X$ turn out to be the integral homology groups of X; in fact, $SP^\infty X \simeq \prod_{n=1}^{\infty} K(H_n(X; \mathbb{Z}), n)$, so it is a GEM. In the category of simplicial abelian groups, it is represented by the free simplicial abelian group on a simplicial set representing X. A fundamental property of this construction is that a space X is a GEM if and only if the natural map $j_X\colon X \to SP^\infty X$ admits a left homotopy inverse (see Lemma 4.B.2.1 in [**24**]).

Using this fact, Farjoun proves in 4.B of [**24**] that if X is a GEM then $L_f X$ is also a GEM, for every map f, and moreover the localization map $X \to L_f X$ is a GEM map. The argument holds in fact for every homotopy idempotent functor L. We indicate it for completeness. Pick a map $r\colon SP^\infty X \to X$ such that $r \circ j_X \simeq \mathrm{id}_X$ and find a left homotopy inverse of $j_{LX}\colon LX \to SP^\infty LX$ as follows. There is a diagram

$$\begin{array}{ccccc} SP^\infty LX & \rightarrow & LSP^\infty LX & \leftarrow & LSP^\infty X \\ \uparrow & & \uparrow & & \uparrow \\ LX & \rightarrow & LLX & \leftarrow & LX \end{array}$$

where the first vertical arrow is j_{LX}, the second vertical arrow is Lj_{LX} and the third vertical arrow is Lj_X. The upper horizontal arrows are $\eta_{SP^\infty LX}$ and $LSP^\infty \eta_X$. The lower horizontal arrows are η_{LX} and $L\eta_X$. Hence, the left square commutes since η is a natural transformation, and the right square commutes since j is a natural transformation. Of course, the lower arrows are homotopy equivalences, and the upper right arrow is also a homotopy equivalence, since SP^∞ may be viewed as a homotopy colimit (see Proposition 4.B.6 in [**24**]), so we use Theorem 4.4 above.

Now the composite $Lr \circ (LSP^\infty \eta_X)^{-1} \circ \eta_{SP^\infty LX}$ is a left homotopy inverse of j_{LX}, as desired.

THEOREM 6.5. *Let L be any homotopy idempotent functor. For every abelian group G and every integer n, we have $LK(G,n) \simeq K(A,n) \times K(B,n+1)$ for some abelian groups A and B. Moreover, if G is an R-module for a commutative ring R, then A and B are also R-modules.*

PROOF. As said above, the space $LK(G,n)$ is a GEM and the localization map $\eta\colon K(G,n) \to LK(G,n)$ is a GEM map. Hence, the composites of η with the projections of $LK(G,n)$ onto each of its factors are nullhomotopic, except in dimensions n and $n+1$. The universal property of η implies that all the projections are nullhomotopic except those corresponding to the dimensions n and $n+1$. The same argument holds for R-modules. □

From this result, the following facts were derived in [**16**].

THEOREM 6.6. *Let L be any homotopy idempotent functor. If G is a finitely generated abelian group and $n \geq 1$, then $LK(G,n) \simeq K(A,n)$ for some abelian group A. In fact A is a direct sum of copies of a rigid ring and finite cyclic groups.*

PROOF. Write $LK(G,n) \simeq K(A,n) \times K(B,n+1)$ for some abelian groups A and B. Then $K(A,n)$ is L-local, as it is a retract of an L-local space. Hence it is orthogonal to the localization map $\eta\colon K(G,n) \to LK(G,n)$, and this yields an isomorphism $\mathrm{Hom}(A,A) \cong \mathrm{Hom}(G,A)$. Similarly, using the fact that $K(B,n+1)$ is L-local we obtain

$$\mathrm{Hom}(B,B) \oplus \mathrm{Ext}(A,B) \cong \mathrm{Ext}(G,B) \tag{6.1}$$

if $n \geq 2$, or else

$$\mathrm{Hom}(H_2(A;\mathbb{Z}),B) \oplus \mathrm{Hom}(B,B) \oplus \mathrm{Ext}(A,B) \cong \mathrm{Hom}(H_2(G;\mathbb{Z}),B) \oplus \mathrm{Ext}(G,B)$$

in the case $n = 1$. Now if $G = \mathbb{Z}$ then $\mathrm{Hom}(B,B) = 0$, so $B = 0$. This says that if L is any homotopy idempotent functor, then $LK(\mathbb{Z},n) \simeq K(A,n)$ where $\mathrm{Hom}(A,A) \cong \mathrm{Hom}(\mathbb{Z},A)$, that is, A is a rigid ring in the sense of Section 3. Moreover, A is the localization of $\mathbb{Z}$ in the category of groups with respect to the homomorphism $\mathbb{Z} \to A$ induced by $\eta\colon K(\mathbb{Z},n) \to K(A,n)$ on the nth homotopy group. This determines all possible localizations of a $K(\mathbb{Z},n)$.

If $G = \mathbb{Z}/p^r$, for some prime p and $r \geq 1$, then B is a $\mathbb{Z}/p^r$-module and hence it is either zero or a direct sum of cyclic groups $\mathbb{Z}/p^j$ with $1 \leq j \leq r$. For convenience, let us replace L by L_η, where $\eta\colon K(G,n) \to LK(G,n)$ is the localization map, without changing the notation. As in Section 3, let n_p be the largest integer k such that $K(\mathbb{Z}/p,k)$ is L-local. If $n > n_p$, then $K(\mathbb{Z}/p,n)$ is killed by L and it follows inductively that $LK(G,n)$ is contractible, similarly as in the beginning of Section 3. (Here, instead of Lemma 3.2 above, we apply Theorem 1.H.1 in [**24**] to $L = L_\eta$, stating that if $F \to E \to X$ is a fibration of connected spaces and LF is contractible, then the map $E \to X$ is an L-equivalence.) If $n < n_p$ then $K(G,n)$ is L-local, since both $K(\mathbb{Z}/p,n)$ and $K(\mathbb{Z}/p,n+1)$ are L-local, and we may argue by induction using the fibrations

$$K(\mathbb{Z}/p^j,n) \to K(\mathbb{Z}/p^{j-1},n) \to K(\mathbb{Z}/p,n+1),$$

since, by Theorem 4.3, the homotopy fibre of any map between L-local spaces is L-local. Thus only the case $n = n_p$ remains to be discussed. Let i_p be the largest integer j such that $K(\mathbb{Z}/p^j, n_p)$ is L-local. If $r \le i_p$ then $K(G, n)$ is again L-local. Otherwise, we claim that $LK(G, n) \simeq K(\mathbb{Z}/p^{i_p}, n)$ when $n = n_p$. Let A and B be as above. If $B \ne 0$, then $K(\mathbb{Z}/p^j, n+1)$ is a retract of $K(B, n+1)$ for some $1 \le j \le r$, and hence $K(\mathbb{Z}/p, n+1)$ is L-local, contradicting the assumption that $n = n_p$. This shows that $B = 0$. If $A = 0$, then $K(\mathbb{Z}/p, n)$ would be killed by L, which is not the case. Thus A is a nonzero $\mathbb{Z}/p^r$-module. Since $\mathrm{Hom}(A, A) \cong \mathrm{Hom}(G, A)$, we have $A = \mathbb{Z}/p^j$ for some $j \le r$. If $K(\mathbb{Z}/p^{j+1}, n)$ were L-local, then we would have $\mathrm{Hom}(\mathbb{Z}/p^j, \mathbb{Z}/p^{j+1}) \cong \mathrm{Hom}(\mathbb{Z}/p^r, \mathbb{Z}/p^{j+1})$, which is absurd. Hence, $j = i_p$, as claimed. Since localization commutes with finite direct products (by Lemma 6.1), the proof of the theorem is complete. □

If G is not finitely generated, then the abelian groups A and B appearing in Theorem 6.5 can both be nonzero, as in the following example. Let L be localization with respect to ordinary homology with mod p coefficients. Then

$$LK(\mathbb{Z} \oplus \mathbb{Z}(p^\infty), n) \simeq K(\hat{\mathbb{Z}}_p, n) \times K(\hat{\mathbb{Z}}_p, n+1).$$

If G is not abelian, then the effect of a homotopy idempotent functor L on a $K(G, 1)$ can be very complicated. Indeed, $LK(G, 1)$ can have infinitely many nontrivial homotopy groups. This phenomenon is discussed in the next sections.

7. Compatibility with the fundamental group

Localizations in the category of groups are intimately related with homotopical localizations. For each group G, choose a connected CW-complex of $K(G, 1)$ type in a functorial way. Thus, every homomorphism $G \to H$ is induced by a natural map $K(G, 1) \to K(H, 1)$. Now, given any homotopy idempotent functor L on connected CW-complexes, define a group G to be *L-local* if the space $K(G, 1)$ is L-local. A group homomorphism will be called an *L-equivalence* if it is orthogonal to all L-local groups. In order to show that these definitions are consistent, we need to check that a group which is orthogonal to all L-equivalences is necessarily L-local. With this aim, we prove the following result.

Proposition 7.1. *Let L be any homotopy idempotent functor. If $f\colon X \to Y$ is an arbitrary L-equivalence of connected spaces, then the induced homomorphism $f_*\colon \pi_1(X) \to \pi_1(Y)$ is an L-equivalence of groups.*

Proof. Let G be any L-local group. Then the space $K(G, 1)$ is L-local and hence orthogonal to f in the homotopy category. Since it is connected, we may use pointed mapping spaces by (4.1), in order to obtain a bijection $[Y, K(G, 1)] \cong [X, K(G, 1)]$ induced by f. (We denote by $[X, Z]$ the set of pointed homotopy classes of maps from X to Z.) Recall that for every connected space X we have a natural bijection

$$[X, K(G, 1)] \cong \mathrm{Hom}(\pi_1(X), G). \tag{7.1}$$

Hence, f induces a bijection $\mathrm{Hom}(\pi_1(Y), G) \cong \mathrm{Hom}(\pi_1(X), G)$, so G and the homomorphism $f_*\colon \pi_1(X) \to \pi_1(Y)$ are orthogonal, as claimed. □

Now let G be a group which is orthogonal to all L-equivalences. Let $f\colon X \to Y$ be any L-equivalence of connected spaces. Then, as we just proved, the induced homomorphism $f_*: \pi_1(X) \to \pi_1(Y)$ is an L-equivalence of groups. Hence, it is orthogonal to G and this implies, using again (7.1), that $K(G,1)$ is orthogonal to f in the homotopy category. This proves that $K(G,1)$ is L-local, as needed.

A priori, this orthogonal pair formed by L-local groups and L-equivalences need not be associated with any idempotent functor L on groups. However, as shown in Theorem 7.2 below, this difficulty disappears if L is localization with respect to some map f (hence, as explained in [**17**], the difficulty is related with large-cardinal principles from set theory). For a group G, we say that an L-localization of G exists if there is an L-equivalence from G to an L-local group, which we then denote by LG. For example, if G is abelian, then LG exists for any homotopy idempotent functor L on spaces, since by Theorem 6.5 we have $LK(G,1) \simeq K(A,1) \times K(B,2)$ for some abelian groups A and B; then the induced homomorphism $G \to A$ is an L-equivalence of groups by Proposition 7.1 (since the localization map $K(G,1) \to LK(G,1)$ is an L-equivalence of spaces) and the group A is L-local, since $K(A,1)$ is a retract of $LK(G,1)$.

A well-known example of an idempotent functor on groups which is induced by an idempotent functor on spaces is localization with respect to ordinary homology. Indeed, Bousfield's notion of homological localization of groups [**7**] is derived from homological localization of spaces as we described, since a group G is HR-local (where R is any commutative ring with 1) if and only if $K(G,1)$ is local with respect to ordinary homology with coefficients in R.

This example illustrates the fact that, although L-equivalences of connected spaces induce L-equivalences of fundamental groups, it is not true in general that every L-equivalence of groups is induced by some L-equivalence of spaces. For instance, if G is a group such that $H_1(G;\mathbb{Z}) = 0$ and $H_2(G;\mathbb{Z}) \neq 0$, then the trivial homomorphism $1 \to G$ is an $H\mathbb{Z}$-equivalence, although there is no homology equivalence of spaces inducing $1 \to G$ on fundamental groups (however, there is one inducing $G \to 1$). This is based on the fact that homomorphisms induced by homology equivalences are surjective on the second homology group; see [**7**].

Localizations at primes follow the same pattern. Recall from Section 5 that, for a set of primes P, the space $K(G,1)$ is P-local if and only if the group G is P-local (i.e., uniquely P'-divisible). More generally, we have the following result from [**10**].

THEOREM 7.2. *Let L_f be localization with respect to some map $f: X \to Y$ between connected CW-complexes. Then localization with respect to the induced homomorphism $f_*: \pi_1(X) \to \pi_1(Y)$ is an idempotent functor on groups whose image class consists precisely of groups G such that $K(G,1)$ is f-local.*

PROOF. If $K(G,1)$ is f-local, then G is f_*-local, by (7.1). Conversely, if a group G is f_*-local, then by (7.1) the map f induces a bijection $[Y, K(G,1)] \cong [X, K(G,1)]$. In other words, the induced map of pointed mapping spaces

$$\mathrm{map}_*(Y, K(G,1)) \to \mathrm{map}_*(X, K(G,1)) \tag{7.2}$$

is bijective on connected components. But, since mapping spaces whose target is a $K(G,1)$ are homotopically discrete, we conclude that (7.2) is a weak homotopy equivalence, so $K(G,1)$ is f-local. □

A homotopy idempotent functor L is called π_1-*compatible* if, for all connected spaces X, the group $\pi_1(LX)$ is L-local. Since the localization map $\eta_X: X \to LX$ is always an L-equivalence, the induced homomorphism $\pi_1(X) \to \pi_1(LX)$ is necessarily an L-equivalence of groups, by Proposition 7.1. Therefore, if L is π_1-compatible then for every space X there exists an L-equivalence from $\pi_1(X)$ to an L-local group; that is, $\pi_1(LX) \cong L\pi_1(X)$ for all spaces X. Thus, if L is π_1-compatible, then an associated localization functor for groups exists and it is just given by $LG = \pi_1(LK(G,1))$.

As shown in [**10**], localization with respect to the trivial map from $K(\mathbb{Z}/p,1)$ onto a point is not π_1-compatible, where p is any prime. Important examples of π_1-compatible functors arise from the next result.

Theorem 7.3. *Let Y be a point or a connected CW-complex of dimension one, and let Z be a connected CW-complex of dimension less than or equal to two. Then localization with respect to any map $f\colon Y \to Z$ is π_1-compatible.*

Proof. Let L_f denote localization with respect to $f\colon Y \to Z$. Let X be any connected space, and let $G = \pi_1(L_f X)$. By Theorem 7.2, we have to prove that G is orthogonal to $f_*\colon \pi_1(Y) \to \pi_1(Z)$. Given any homomorphism $\varphi\colon \pi_1(Y) \to G$, we use the fact that $\dim Y \le 1$ in order to find a map $g\colon Y \to L_f X$ inducing φ on fundamental groups. Since $L_f X$ is orthogonal to f in the homotopy category, there exists a map $h\colon Z \to L_f X$ such that $h \circ f \simeq g$, and h is unique up to homotopy with this property. Then $h_*\colon \pi_1(Z) \to G$ is a homomorphism such that $h_* \circ f_* = \varphi$. If $\psi\colon \pi_1(Z) \to G$ is another homomorphism such that $\psi \circ f_* = \varphi$, then we use the fact that $\dim Z \le 2$ in order to find a map $j\colon Z \to L_f X$ inducing ψ on fundamental groups. Then $j \circ f$ induces φ on fundamental groups and, since $\dim Y \le 1$, we may infer that $j \circ f \simeq g$. Now the homotopy uniqueness of h implies that $h \simeq j$, so $h_* = \psi$, as needed to complete the argument. □

For instance, localization with respect to any map between wedges of circles is π_1-compatible. This is the case, of course, with localizations at sets of primes. Ordinary homological localizations are also special cases, as shown in [**26**].

8. Preservation of asphericity

Spaces of $K(G,1)$ type are also called *aspherical*. Thus, a CW-complex is aspherical if and only if it is homotopy equivalent to the orbit space X/G of a contractible space X under a free, properly discontinuous action of a discrete group G. It is not true in general that LX is aspherical if X is aspherical and L is homotopy idempotent. However, the following facts are to be mentioned.

1. Let $X = K(F,1)$ where F is a free group. It has been conjectured by Farjoun that LX is aspherical for every homotopy idempotent functor L. Some advances in this direction have been made in [**5**].
2. If $X = K(G,1)$ where G is abelian, then $LX \simeq K(A,1) \times K(B,2)$ for some abelian groups A and B, as shown in [**24**]. Moreover, if G is finitely generated then $B = 0$, by Theorem 6.6 above.
3. If $X = K(N,1)$ where N is nilpotent, then the P-localization X_P is aspherical for every set of primes P; see [**31**].

In fact, if one considers localization at a set of primes P, and G is any group, then a necessary and sufficient condition for $K(G,1)_P$ to be a $K(G_P,1)$ was given in [**14**] as follows. The P-localization homomorphism $\eta\colon G \to G_P$ lifts to a map

$K(G,1) \to K(G_P,1)$ whose target is a P-local space. Hence, we seek precisely a condition under which this map $K(G,1) \to K(G_P,1)$ is a P-equivalence.

THEOREM 8.1. *For a group G and a set of primes P, the space $K(G,1)_P$ is a $K(G_P,1)$ if and only if the P-localization homomorphism $\eta\colon G \to G_P$ induces isomorphisms $H_n(G;A) \cong H_n(G_P;A)$ for all P-local modules A over the integral group ring of G_P.*

PROOF. Since $K(G_P,1)$ is P-local, the natural map $K(G,1) \to K(G_P,1)$ is a P-localization if and only if it is a P-equivalence. Thus our claim follows from Theorem 5.1. □

In the next section we analyze whether or not this condition is satisfied for certain classes of groups. We close this section by displaying an illuminating example where it fails. This example is based on arguments due to Bousfield–Kan (see VII.4.4 in [**8**]) and Cohen [**18**].

Let G be the symmetric group Σ_3 on three elements, and choose $P = \{3\}$. Since G is generated by elements of order 2 and the localization homomorphism $\eta\colon G \to G_P$ kills all the elements of order prime to 3, we see that $\eta(G) = 1$ and the universal property of η tells us then that $G_P = 1$. Hence, the space $K(G,1)_P$ is 1-connected. Since it is P-local and 1-connected, its homotopy groups and integral homology groups are P-local. The first integral homology groups of G are $H_1(G;\mathbb{Z}) \cong \mathbb{Z}/2$, $H_2(G;\mathbb{Z}) = 0$, and $H_3(G;\mathbb{Z}) \cong \mathbb{Z}/6$. Therefore, $\pi_2\, K(G,1)_P \cong H_2(K(G,1)_P;\mathbb{Z}) \cong H_2(K(G,1)_P;\mathbb{Z}_P) \cong H_2(K(G,1);\mathbb{Z}_P) = 0$, and similarly $\pi_3\, K(G,1)_P \cong \mathbb{Z}/3$. Pick a map $f\colon S^3 \to K(G,1)_P$ inducing the projection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/3$ on the third homotopy group. Then the composite of f with a map $\rho_3\colon S^3 \to S^3$ of degree 3 is nullhomotopic. Let F be the homotopy fibre of ρ_3, so we have a commutative diagram of fibre sequences

$$\begin{array}{ccccc} F & \to & S^3 & \to & S^3 \\ \downarrow & & \downarrow & & \downarrow \\ \Omega K(G,1)_P & \to & \{*\} & \to & K(G,1)_P, \end{array}$$

where a scrutiny of the associated Serre spectral sequences shows that the map $F \to \Omega K(G,1)_P$ induces an isomorphism on integral homology. Hence it is a homotopy equivalence. In other words, there is a homotopy fibre sequence

$$\Omega K(G,1)_P \;\to\; S^3 \;\to\; S^3,$$

where the second arrow is a map of degree 3. Since $\pi_n\, \Omega K(G,1)_P \cong \pi_{n+1}\, K(G,1)_P$, the space $K(G,1)_P$ is a 3-local space with finite homotopy groups, and it has nonzero homotopy in dimension $n+1$ whenever S^3 has 3-torsion in homotopy in dimension n. According to a theorem of McGibbon and Neisendorfer [**37**], this happens in arbitrarily high dimensions.

9. Localizing infra-nilmanifolds

We recall the following concepts e.g. from [**20**]. Let V be a contractible nilpotent Lie group and let C be a maximal compact subgroup of the automorphism group $\mathrm{Aut}(V)$. A discrete subgroup G of the semidirect product $V \rtimes C$ is called an *almost-crystallographic group* if the quotient of V by the action of G via affine diffeomorphisms is compact. If G is torsion-free then it is called an *almost-Bieberbach group* and the quotient of V by the action of G is a manifold. Such manifolds are called *infra-nilmanifolds* (they are nilmanifolds if $G \subset V$). Infra-nilmanifolds are thus aspherical. In the special case when $V = \mathbb{R}^n$ and C is the orthogonal group $O(n)$, the corresponding concepts are crystallographic groups, Bieberbach groups, and flat Riemannian manifolds.

If G is an almost-crystallographic group, then $N = G \cap V$ is the unique maximal nilpotent normal subgroup of G (called the *Fitting subgroup* of G), and the quotient G/N is finite. Thus, almost-crystallographic groups are virtually nilpotent and finitely generated. A classification of almost-crystallographic groups is given up to dimension four in [**20**].

In this section we describe the first steps of a study of the effect of P-localization on the homotopy type of infra-nilmanifolds. Let us first examine the two-dimensional compact flat Riemannian manifolds, namely the torus and the Klein bottle. The torus $S^1 \times S^1$ is a $K(\mathbb{Z} \oplus \mathbb{Z}, 1)$, so its localization at each set of primes P is just $K(\mathbb{Z}_P \oplus \mathbb{Z}_P, 1)$. The fundamental group of the Klein bottle has a presentation

$$G = \langle x, y \mid yxy^{-1} = x^{-1} \rangle. \tag{9.1}$$

Thus, G fits into a short exact sequence

$$\mathbb{Z} \oplus \mathbb{Z} \rightarrowtail G \twoheadrightarrow \mathbb{Z}/2$$

where the kernel is generated by x and y^2. In this extension, the action of $\mathbb{Z}/2$ on $\mathbb{Z} \oplus \mathbb{Z}$ is given by the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If P is any set of primes such that $2 \in P$, then $\mathbb{Z}/2$ is a finite P-local group and hence, according to Theorem 2.1 in [**12**], the P-localization of G fits into a commutative diagram

$$\begin{array}{ccccc} \mathbb{Z} \oplus \mathbb{Z} & \rightarrowtail & G & \twoheadrightarrow & \mathbb{Z}/2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}_P \oplus \mathbb{Z}_P & \rightarrowtail & G_P & \twoheadrightarrow & \mathbb{Z}/2 \end{array}$$

where the vertical arrows are P-localizations. Moreover, by considering the map of Lyndon–Hochschild–Serre spectral sequences associated with this diagram, one sees that the homomorphism $G \to G_P$ induces isomorphisms in homology with arbitrary P-local coefficients. Indeed, if A is any P-local coefficient module over G_P, then the induced homomorphism

$$H_n(\mathbb{Z} \oplus \mathbb{Z}; A) \to H_n(\mathbb{Z}_P \oplus \mathbb{Z}_P; A)$$

is an isomorphism for all n, since the groups involved are abelian; see Theorem 4.3 in [**14**]. Therefore, as stated in Theorem 8.1 above, the map $K(G, 1) \to K(G_P, 1)$ induced by the P-localization homomorphism $G \to G_P$ is a P-equivalence of spaces and hence a P-localization.

Now let us suppose that $2 \notin P$. In this case, the relation $(yx)^2 = (x^{-1}yx)^2$ in G tells us that x is in the kernel of the P-localization homomorphism $G \to G_P$. Therefore, the projection of G onto the (infinite cyclic) quotient Q by the normal subgroup N generated by x is a P-equivalence. Hence, the P-localization of G is the composite of the projection onto Q followed by the P-localization of Q. Next we show that this homomorphism also induces isomorphisms in homology with arbitrary P-local coefficients, so the following theorem will be proved.

Theorem 9.1. *For every set of primes P, the P-localization of the Klein bottle is aspherical.*

Proof. As above, we denote by G the fundamental group of the Klein bottle and we use the presentation (9.1). Thus, we may also display G into an extension of infinite cyclic groups

$$N \rightarrowtail G \twoheadrightarrow Q \tag{9.2}$$

where $N = \langle x \rangle$ and $Q = \langle \bar{y} \rangle$. By our previous remarks, we may suppose that $2 \notin P$. Since Q is abelian, in order to verify the condition stated in Theorem 8.1, we only have to show that the projection $G \twoheadrightarrow Q$ induces isomorphisms in homology with any P-local coefficient module A over Q_P. Let us denote $\omega(a) = \bar{y} \cdot a$ for each $a \in A$, where Q acts on A via Q_P. The spectral sequence associated with (9.2) with coefficients in A has $E^2_{r,s} = H_r(Q; H_s(N; A))$ and converges to $H_*(G; A)$. Since both N and Q are infinite cyclic, the spectral sequence collapses at the E^2-term. Since the action of G on A factors through Q, the induced action of N on A is trivial. Therefore, $H_0(N; A) \cong A$ with the given action of Q, while $H_1(N; A) \cong A$ with an action of Q where $\bar{y}$ acts by sending $a \mapsto -\omega(a)$, for every $a \in A$. Since A is a P-local module and $2 \notin P$, the endomorphism $1 + \omega$ is an automorphism of A. Hence,

$$H_2(G; A) \cong H_1(Q; H_1(N; A)) \cong H^0(Q; H_1(N; A)) \cong \mathrm{Ker}\,(1 + \omega) = 0$$

and $H_0(Q; H_1(N; A)) \cong \mathrm{Coker}\,(1 + \omega) = 0$. Therefore,

$$H_1(G; A) \cong H_1(Q; H_0(N; A)) \cong H_1(Q; A),$$

as needed. □

The study of P-localizations of infra-nilmanifolds of higher dimensions and, more generally, P-localizations of aspherical spaces with finitely generated, virtually nilpotent fundamental group has been undertaken by Descheemaeker in [**22**]. A basic question to answer is in which cases the P-localization of such a space is again aspherical. (One should not really expect that other properties of a more geometric nature be preserved under localization. Keep in mind that the fundamental group and the integral homology groups of P-local spaces are almost never finitely generated, so they cannot belong to compact manifolds.)

The following result is relevant to the proposed analysis.

Theorem 9.2. *Let $Y \to X \to K(Q, 1)$ be a homotopy fibre sequence where Q is a P-torsion group, for a set of primes P. Then $Y_P \to X_P \to K(Q, 1)$ is a homotopy fibre sequence.*

Proof. By localizing fibrewise at P we obtain a commutative diagram of homotopy fibre sequences

$$\begin{array}{ccccc} Y & \to & X & \to & K(Q,1) \\ \downarrow & & \downarrow & & \downarrow \\ Y_P & \to & E & \to & K(Q,1) \end{array}$$

in which the middle map $X \to E$ is a P-equivalence; see e.g. [**4**]. The fundamental group of E fits into a group extension

$$\pi_1(Y_P) \rightarrowtail \pi_1(E) \twoheadrightarrow Q$$

so the group $\pi_1(E)$ is P-local, since, by Theorem 11.5 in [**6**], every extension of a P-local group by a P-torsion group is P-local. Similarly, the semidirect products $\pi_k(E) \rtimes \pi_1(E)$ are P-local for $k \geq 2$, since they fit into group extensions

$$\pi_k(Y_P) \rtimes \pi_1(Y_P) \rightarrowtail \pi_k(E) \rtimes \pi_1(E) \twoheadrightarrow Q.$$

Therefore, the space E is P-local, so $E \simeq X_P$. □

Now let X be any infra-nilmanifold. Let N be the Fitting subgroup of the fundamental group $\pi_1(X)$, and let $Q = \pi_1(X)/N$. Then X is covered by a nilmanifold Z with fundamental group N. Suppose given a set of primes P. Let S be the subgroup of Q generated by all the P'-torsion elements of Q. Let Y be the manifold covering X with $\pi_1(Y)$ equal to the preimage of S in $\pi_1(X)$. Thus, there are group extensions

$$N \rightarrowtail \pi_1(Y) \twoheadrightarrow S; \qquad \pi_1(Y) \rightarrowtail \pi_1(X) \twoheadrightarrow Q_P,$$

which are induced on fundamental groups by homotopy fibre sequences

$$Z \to Y \to K(S, 1); \qquad Y \to X \to K(Q_P, 1).$$

The fact that $S_P = 1$ tells us, by Theorem 2.1 in [**12**], that $\pi_1(Y_P)$ is nilpotent. However, it seems difficult to analyze the higher homotopy groups of Y_P in general. We do not know whether or not Y_P is necessarily a nilpotent space.

By Theorem 9.2, the study of the homotopy type of X_P essentially reduces to the study of Y_P, since there is a homotopy fibre sequence

$$Y_P \to X_P \to K(Q_P, 1). \tag{9.3}$$

For instance, if Q is P-torsion, then X_P is aspherical, since Y is then nilpotent. This is a special case of Proposition 12.3.1 in [**9**]. Descheemaeker has proved in [**22**] that the converse is true if X is orientable; that is, X_P is aspherical if and only if Q is P-torsion, when X is orientable. It is also shown in [**22**] that every P-localization of any nonorientable infra-nilmanifold of dimension three is aspherical.

Motivated by results in [**11**], we expect that when X_P is not aspherical it will have an infinite number of nonzero homotopy groups.

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