The definability of almost disjoint families and long well-orders at higher cardinals

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XVI International Luminy Workshop in Set Theory, 14. September 2021

Joint work in progress with Sandra Müller (Vienna).

Introduction

Mathematical objects whose existence relies on the *Axiom of Choice* are often referred to as *pathological sets*.

For many types of pathological sets of real numbers, results from descriptive set theory show that these objects cannot be defined by simple formulas in second-order arithmetic.

Moreover, many canonical extensions of the axioms of $\mathbf{ZFC}$, like large cardinal assumptions or forcing axioms, imply that these objects are not definable in second-order arithmetic at all.

In the following, we want to study the *set-theoretic* definability of pathological sets of higher cardinalities.

More specifically, we aim to generalize classical non-definability results for sets of reals numbers to subsets of the power set $\mathcal{P}(\kappa)$ of an uncountable cardinal κ that are definable by Σ_1-formulas with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$.

Definition

- A formula in the language $\mathcal{L}_\in$ of set theory is a Δ_0-*formula* if it is contained in the smallest collection of $\mathcal{L}_\in$-formulas that contains all atomic $\mathcal{L}_\in$-formulas and is closed under negation, disjunction and bounded quantification.
- An $\mathcal{L}_\in$-formula is a Σ_1-*formula* if it is of the form $\exists x \ \varphi(x)$ for some Δ_0-formula φ.

The starting point of our work is a *perfect set theorem* for Σ_1-definable sets at limits of measurable cardinals.

Given cardinals $\mu \geq \omega$ and $\nu > 1$, we equip the set ${}^{\mu}\nu$ of all functions from μ to ν with the topology whose basic open sets consists of all functions that extend a given function $s : \xi \longrightarrow \nu$ with $\xi < \mu$.

Moreover, we equip $\mathcal{P}(\kappa)$ with the topology induced by ${}^{\kappa}2$.

An injection $\iota : {}^{\mu}\nu \longrightarrow \mathcal{P}(\kappa)$ is a *perfect embedding* if it induces a homeomorphism between ${}^{\mu}\nu$ and the subspace $\operatorname{ran}(\iota)$ of $\mathcal{P}(\kappa)$.

Theorem

Let κ be a limit of measurable cardinals and let D be a subset of $\mathcal{P}(\kappa)$ that is definable by a Σ_1-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$.

If D has cardinality greater than κ, then there is a perfect embedding $\iota : {}^{\mathrm{cof}(\kappa)}\kappa \longrightarrow \mathcal{P}(\kappa)$ with $\mathrm{ran}(\iota) \subseteq D$.

In the case of singular limits of measurable cardinals, it is possible to show that the consistency strength of the assumption of the above theorem is optimal for its conclusion.

Theorem

Let κ be a singular strong limit cardinal with the property that for every subset D of $\mathcal{P}(\kappa)$ of cardinality greater than κ that is definable by a Σ_1-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$, there is a perfect embedding $\iota : {}^{\mathrm{cof}(\kappa)}\kappa \longrightarrow \mathcal{P}(\kappa)$ with $\mathrm{ran}(\iota) \subseteq D$.

Then there is an inner model with a sequence of measurable cardinals of length $\mathrm{cof}(\kappa)$.

A classical result of Mathias shows that there are no analytic maximal almost disjoint families in $\mathcal{P}(\omega)$.

The techniques developed in the proof of the above perfect set theorem allow us to prove an analog of this result for large cardinals.

Theorem

Let κ be a Ramsey cardinal that is a limit of measurable cardinals and let A be a subset of $\mathcal{P}(\kappa)$ that is definable by a Σ_1-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$.

If A has cardinality greater than κ, then there exist distinct $x, y \in A$ with the property that $x \cap y$ is unbounded in κ.

Finally, we consider singular cardinals κ and definable *long well-orders* in $\mathcal{P}(\kappa)$, i.e. well-orderings of subsets of $\mathcal{P}(\kappa)$ of order-type at least κ^+.

Theorem

Let κ be a cardinal of countable cofinality that is a limit of measurable cardinals.

If there exists a well-ordering of a subset of $\mathcal{P}(\kappa)$ of cardinality greater than κ that is definable by a Σ_1-formula with parameter κ, then there is a $\mathbf{\Sigma}^1_3$-well-ordering of the reals.

Our techniques allow us to determine the exact consistency strength of the non-existence of Σ_1-definable long well-orderings of subsets of a singular strong limit cardinal of countable cofinality.

Theorem

The following statements are equiconsistent over $\mathbf{ZFC}$*:*

- *There exist infinitely many measurable cardinals.*
- *There exists a singular cardinal* κ *with the property that no well-ordering of a subset of* $\mathcal{P}(\kappa)$ *of cardinality greater than* κ *is definable by a* Σ_1*-formula with parameters in* $\mathrm{H}(\kappa) \cup \{\kappa\}$.

Thin subsets

We now discuss the central ideas of the proof of the following result:

Theorem

Let κ be a limit of measurable cardinals and let D be a subset of $\mathcal{P}(\kappa)$ that is definable by a Σ_1-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$.

If D has cardinality greater than κ, then there is a perfect embedding $\iota : {}^{\mathrm{cof}(\kappa)}\kappa \longrightarrow \mathcal{P}(\kappa)$ with $\mathrm{ran}(\iota) \subseteq D$.

Let κ be a limit cardinal and let $\langle \kappa_\alpha \mid \alpha < \operatorname{cof}(\kappa) \rangle$ be a strictly increasing sequence of measurable cardinals with supremum κ. Moreover, if $\operatorname{cof}(\kappa) < \kappa$, then $\kappa_0 > \operatorname{cof}(\kappa)$.

Given $\alpha < \operatorname{cof}(\kappa)$, let U_α be a normal ultrafilter on κ_α and let

$$\langle \langle N^\alpha_\beta \mid \beta \in \mathrm{On} \rangle, \ \langle j^\alpha_{\beta,\gamma} : N^\alpha_\beta \longrightarrow N^\alpha_\gamma \mid \beta \leq \gamma \in \mathrm{On} \rangle \rangle$$

denote the linear iteration of $\langle \mathrm{V}, U_\alpha \rangle$.

Then $|\mathcal{P}(\kappa)|^{N^\alpha_\kappa} = \kappa$ for all $\alpha < \operatorname{cof}(\kappa)$.

Given $D \subseteq \mathcal{P}(\kappa)$ with $|D| > \kappa$, pick $x_* \in D$ with $x_* \notin N^\alpha_\kappa$ for all $\alpha < \operatorname{cof}(\kappa)$.

For each $\alpha < \operatorname{cof}(\kappa)$, we have $j^\alpha_{0,\kappa}(x_*) \cap \kappa \neq x_*$ and hence there is $\bar{\alpha} < \kappa$ with

$$j_{0,\bar{\alpha}}(x_*) \cap \bar{\alpha} \ \neq \ x_* \cap \bar{\alpha}.$$

We can now inductively construct

- a system $\langle \kappa_s \mid s \in {}^{<\mathrm{cof}(\kappa)}\kappa \rangle$ of inaccessible cardinals less than κ, and
- a system $\langle I_s \mid s \in {}^{\leq\mathrm{cof}(\kappa)}\kappa \rangle$ of linear iterations of $\langle \mathrm{V}, \{U_\alpha \mid \alpha < \mathrm{cof}(\kappa)\} \rangle$ of length at most κ with well-founded limit.

with the property that the following statements hold:

- $i^{I_s}_{0,\infty}(\kappa) = \kappa$ and $i^{I_s}_{0,\infty} \restriction \kappa_0 = \mathrm{id}_{\kappa_0}$.
- If $s \subsetneq t$, then $\kappa_s < \kappa_t$, I_t extends I_s and $i^{I_s}_{0,\infty}(x_*) \restriction \kappa_s = i^{I_t}_{0,\infty}(x_*) \restriction \kappa_s$.
- If $s \in {}^{<\mathrm{cof}(\kappa)}\kappa$ and $\beta < \gamma < \kappa$, then $\kappa_{s^\frown\langle\beta\rangle} < \kappa_{s^\frown\langle\gamma\rangle}$ and

$$i^{I_{s^\frown\langle\beta\rangle}}_{0,\infty}(x_*) \restriction \kappa_{s^\frown\langle\beta\rangle} \;\neq\; i^{I_{s^\frown\langle\gamma\rangle}}_{0,\infty}(x_*) \restriction \kappa_{s^\frown\langle\beta\rangle}.$$

If D is definable by a Σ_1-formula with parameters in $\mathrm{H}(\kappa_0) \cup \{\kappa\}$, then Σ_1-upwards absoluteness ensures that $i^{I_s}_{0,\infty}(x_*) \in D$ for all $s \in {}^{\mathrm{cof}(\kappa)}\kappa$.

This yields a perfect embedding

$$\iota : {}^{\mathrm{cof}(\kappa)}\kappa \longrightarrow D;\ s \longmapsto i^{I_s}_{0,\infty}(x_*).$$

This construction can be extended to sequences $s \in {}^{<\mathrm{cof}(\kappa)}\kappa$ of limit length such that

- s is an element of a forcing extension $\mathrm{V}[G]$ of V in which $\langle \mathrm{V}, \{U_\alpha \mid \alpha < \mathrm{cof}(\kappa)\}\rangle$ is still iterable.
- Every proper initial segment of s is an element of V.

Long well-orders

We now discuss the proof of this equiconsistency result:

Theorem

The following statements are equiconsistent over $\mathbf{ZFC}$*:*

- *There exist infinitely many measurable cardinals.*
- *There exists a singular cardinal* κ *with the property that no well-ordering of a subset of* $\mathcal{P}(\kappa)$ *of cardinality greater than* κ *is definable by a* Σ_1*-formula with parameters in* $\mathrm{H}(\kappa) \cup \{\kappa\}$.

One direction of the proof uses the theory of *short core models* developed by Peter Koepke.

The other direction relies on a combination of the above constructions with *diagonal Prikry forcing*.

Assume that $\vec{\kappa} = \langle \kappa_n \mid n < \omega \rangle$ is a strictly increasing sequence of measurable cardinals with limit κ. Pick a sequence $\vec{U} = \langle U_n \mid n < \omega \rangle$ with the property that U_n is a normal ultrafilter on κ_n for each $n < \omega$.

Let G be generic for the corresponding *diagonal Prikry forcing* $\mathbb{P}_{\vec{U}}$ and assume that, in $\mathrm{V}[G]$, there exists a well-ordering $\trianglelefteq$ of a subset D of $\mathcal{P}(\kappa)$ of cardinality greater than κ that can be defined by a Σ_1-formula $\varphi(v_0, \ldots, v_3)$, a parameter $z \in \mathrm{H}(\kappa)$ and the parameter κ.

Since forcing with $\mathbb{P}_{\vec{U}}$ does not add bounded subsets of κ and the Boolean completion of $\mathbb{P}_{\vec{U}}$ is weakly homogeneous, we know that $z \in \mathrm{V}$, $D \subseteq \mathrm{V}$ and

$$D \ = \ \{y \in \mathcal{P}(\kappa)^{\mathrm{V}} \mid \mathbb{1}_{\mathbb{P}_{\vec{U}}} \Vdash \varphi(\check{y}, \check{y}, \check{z}, \check{\kappa})\} \ \in \ \mathrm{V}.$$

Let $c \in {}^{\omega}\kappa$ denote the sequence added by G. Let x_* be the element of D given by the above construction in $\mathrm{V}[G]$ and let

$$i_* = i^{I_c}_{0,\infty} : \mathrm{V} \longrightarrow M_* = M^{I_c}_{\infty}$$

denote the corresponding iteration map.

Then $i_*(\kappa) = \kappa$, $i_*(z) = z$ and, if we set $y_* = i_*(x_*)$, then elementarity implies that

$$\mathbb{1}_{i_*(\mathbb{P}_{\vec{U}})} \Vdash^{M_*} \varphi(\check{y}_*, \check{y}_*, \check{z}, \check{\kappa}).$$

Using a *Mathias criterion* for $\mathbb{P}_{\vec{U}}$ due to Fuchs, we find an M_*-generic filter on $i_*(\mathbb{P}_{\vec{U}})$ in $\mathrm{V}[G]$.

But this shows that $y_* \in D \subseteq \mathrm{V}$ and this allows us to define c in V, a contradiction.

Σ_1-definability at ω_1

Using results of Woodin on the Π_2-maximality of $\mathbb{P}_{max}$-extensions of $\mathrm{L}(\mathbb{R})$ and unpublished work of Chan–Jackson–Trang on the non-existence of maximal almost disjoint families in $\mathrm{L}(\mathbb{R})$, it is possible to prove analogs of the above results for ω_1.

Theorem

Assume that either there is a measurable cardinal above infinitely many Woodin cardinals or Woodin's Axiom $(*)$ holds.

- *No well-ordering of a subset of $\mathcal{P}(\omega_1)$ of cardinality greater than $\aleph_1$ is definable by a Σ_1-formula with parameters in $\mathrm{H}(\aleph_1) \cup \{\omega_1\}$.*
- *If A is a set of cardinality greater than $\aleph_1$ that consists of unbounded subsets of ω_1 and is definable by a Σ_1-formula with parameters in $\mathrm{H}(\aleph_1) \cup \{\omega_1\}$, then there exist distinct $x, y \in A$ with the property that $x \cap y$ is unbounded in ω_1.*

The following observation shows that the above implications cannot be generalized from ω_1 to ω_2.

Proposition

- *If the* $\mathbf{BPFA}$ *holds, then there exists an almost disjoint family of cardinality* $2^{\aleph_2}$ *in* $\mathcal{P}(\omega_2)$ *that is definable by a* Σ_1*-formula with parameters in* $\mathrm{H}(\aleph_2) \cup \{\omega_2\}$*.*
- *If there is a supercompact cardinal, then, in a generic extension of the ground model, there exists exists an almost disjoint family of cardinality* $2^{\aleph_2}$ *in* $\mathcal{P}(\omega_2)$ *that is definable by a* Σ_1*-formula and the parameter* ω_2*.*

Thank you for listening!