Large cardinals, strong logics and reflection principles

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Introduction

The groundbreaking work of Cohen and Gödel revealed that many natural mathematical questions are not answered by the standard axiomatization of mathematics provided by the axioms of $\mathbf{ZFC}$.

This initiated the programme to search for intrinsically justified extensions of these axioms that settle important mathematical questions left open by $\mathbf{ZFC}$.

In this search for the right axiomatization of mathematics, large cardinal axioms play an outstanding role.

These axioms postulate the existence of cardinal numbers having certain properties that make them very large, and whose existence cannot be proved in **ZFC**, because it implies the consistency of **ZFC** itself.

Large cardinal axioms answer many important questions left open by **ZFC** in the desired way and are therefore strong candidate for new axioms of mathematics.

Moreover, these principles also allow us to measure the consistency strength of other axioms and order them into a linear hierarchy based on their consistency strength.

Despite their central role in modern set theory, large cardinals are still surrounded by many open fundamental questions:

- **What is a large cardinal?**
  Even though set theorists have an intuitive understanding of these axioms, there is no widely accepted definition of what a large cardinal actually is, and, without such a definition, it seems impossible to develop a general theory of large cardinals that allows proofs of their observed properties.

- **Are large cardinal axioms true?**
  Although large cardinal axioms provide the desired answers to many questions left open by $\mathbf{ZFC}$, the question whether they are true and should therefore be added to the standard axiomatization of set theory remains open.

A fruitful way of approaching these questions is given by various results proving equivalences between the following three types of statements:

- The existence of large cardinals.
- Compactness properties of strong logics.
- Reflection principles strengthening the downward Löwenheim–Skolem Theorem.

Vopěnka's Principle

Definition (Vopěnka)

Vopěnka's Principle is the scheme of axioms stating that for every proper class of structures of the same signature, there is an elementary embedding between two distinct members of the class.

This reflection principle was recently used to answer long-standing open questions in other areas of mathematics, like category theory, commutative algebra and homotopy theory.

Bagaria showed how Vopěnka's Principle can be characterized through the existence of elementary embeddings.

Definition (Bagaria)

Let n be a natural number.

- $C^{(n)}$ is the class of all ordinals α satisfying $\mathrm{V}_\alpha \prec_{\Sigma_n} \mathrm{V}$.
- A cardinal κ is $C^{(n)}$-*extendible* if for every ordinal $\lambda > \kappa$, there is an ordinal $\mu > \lambda$ and an elementary embedding $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\mu$ with $\mathrm{crit}(j) = \kappa$ and $\lambda < j(\kappa) \in C^{(n)}$.

Theorem (Bagaria)

The following schemes are equivalent over $\mathbf{ZFC}$*:*

- *Vopěnka's Principle.*
- *For every natural number* n*, there is a* $C^{(n)}$*-extendible cardinal.*

Vopěnka's Principle is closely related to compactness properties of strong logics.

An *abstract logic* is a pair $(\mathcal{L}, \models_{\mathcal{L}})$ consisting of

- a class function $\mathcal{L}$ that maps signatures σ to sets $\mathcal{L}(\sigma)$ of $\mathcal{L}$-sentences, and
- a satisfaction relation $\models_{\mathcal{L}}$ that determines which $\mathcal{L}$-sentences $\phi \in \mathcal{L}(\sigma)$ hold in σ-structures

that satisfies certain canonical rules about invariance under isomorphic copies, extensions of signatures, and boundedness of the sizes of signatures generating sentences.

Given an abstract logic $\mathcal{L}$ and a cardinal κ, an $\mathcal{L}$-theory T is *$<\kappa$-satisfiable* if every subtheory of cardinality less than κ is satisfiable.

A cardinal κ is a *strong compactness cardinal* of an abstract logic $\mathcal{L}$ if every $<\kappa$-satisfiable $\mathcal{L}$-theory is satisfiable.

Theorem (Makowsky)

The following schemes are equivalent over $\mathbf{ZFC}$*:*

- *Vopěnka's Principle.*
- *Every abstract logic has a strong compactness cardinal.*

Weak compactness cardinals

A cardinal κ is a *weak compactness cardinal* of an abstract logic $\mathcal{L}$ if every $<\kappa$-satisfiable $\mathcal{L}$-theory of cardinality κ is satisfiable.

Recent work of Boney, Dimopoulos, Gitman and Magidor connects this weaker property to the large cardinal notion of *subtleness*, introduced by Jensen and Kunen in their studies of strong diamond principles.

Definition (Jensen–Kunen)

A cardinal δ is *subtle* if for every sequence $\langle A_\gamma \subseteq \gamma \mid \gamma < \delta \rangle$ and every closed unbounded subset C of δ, there exist $\beta < \gamma$ in C with the property that $A_\beta = A_\gamma \cap \beta$.

Definition

We let "Ord *is subtle*" denote the scheme of axioms stating that for every sequence $\langle A_\gamma \subseteq \gamma \mid \gamma \in \mathrm{Ord} \rangle$ and every closed unbounded class C of ordinals, there exist $\beta < \gamma$ in C with the property that $A_\beta = A_\gamma \cap \beta$.

Theorem (Boney–Dimopoulos–Gitman–Magidor)

The following schemes are equivalent over $\mathbf{ZFC}$ *together with the existence of a definable global well-ordering:*

- Ord *is subtle.*
- *Every abstract logic has a stationary class of weak compactness cardinals.*

This result raises two questions:

- Is it necessary to assume the existence of a global well-ordering?
- Can we characterize the existence of weak compactness cardinals for all abstract logics through large cardinal properties of Ord?

Proposition

The following statements are equivalent for every infinite cardinal δ:

- *The cardinal δ is subtle.*
- *For all closed unbounded subsets C of δ and all sequences $\langle E_\gamma \mid \gamma < \delta \rangle$ with $\emptyset \neq E_\gamma \subseteq \mathcal{P}(\gamma)$ for all $\gamma < \delta$, there are $\beta < \gamma$ in C and $A \in E_\gamma$ with $A \cap \beta \in E_\beta$.*

Definition

We let "Ord *is essentially subtle*" denote the scheme of axioms stating that for every closed unbounded class C of ordinals and every class sequence $\langle E_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ with $\emptyset \neq E_\alpha \subseteq \mathcal{P}(\alpha)$ for all $\alpha \in \mathrm{Ord}$, there exist $\alpha < \beta$ in C and $A \in E_\beta$ with $A \cap \alpha \in E_\alpha$.

Theorem

The following schemes of sentences are equivalent over $\mathbf{ZFC}$*:*

- Ord *is essentially subtle.*
- *Every abstract logic has a stationary class of weak compactness cardinals.*

Theorem (Bagaria–L.)

The following statements are equivalent for every cardinal $\delta \in C^{(1)}$:

- *For all $\xi < \delta$ and every sequence $\langle A_\gamma \subseteq \gamma \mid \gamma < \delta \rangle$, there are cardinals $\xi < \mu < \nu < \delta$ with $A_\mu = A_\nu \cap \mu$.*
- *The cardinal δ is either subtle or a limit of subtle cardinals.*
- *For all $\xi < \delta$ and every sequence $\langle E_\gamma \mid \gamma < \delta \rangle$ such that $\emptyset \neq E_\gamma \subseteq \mathcal{P}(\gamma)$ holds for all $\gamma < \delta$, there exist cardinals $\xi < \mu < \nu < \delta$ and $A \in E_\nu$ with $A \cap \mu \in E_\mu$.*

Definition

We let "Ord *is essentially closure subtle*" denote the scheme of axioms stating that all $\xi \in \mathrm{Ord}$ and every class sequence $\langle E_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ such that $\emptyset \neq E_\alpha \subseteq \mathcal{P}(\alpha)$ holds for all $\alpha \in \mathrm{Ord}$, there exist cardinals $\xi < \mu < \nu$ and $A \in E_\nu$ with $A \cap \mu \in E_\mu$.

Theorem

The following schemes of sentences are equivalent over $\mathbf{ZFC}$*:*

- Ord *is essentially closure subtle.*
- *Every abstract logic has a weak compactness cardinal.*

We now explore the differences between the assumption

$$\text{“}\mathrm{Ord}\ \textit{is essentially subtle}\text{”}$$

and the assumption

$$\text{“}\mathrm{Ord}\ \textit{is essentially closure subtle}\text{”}.$$

Proposition

If Φ is a sentence in the language of set theory with the property that $\mathbf{ZFC} + \Phi$ is consistent, then

$$\mathbf{ZFC} + \Phi \nvdash \text{“}\mathrm{Ord}\ \textit{is essentially subtle}\text{”}.$$

Theorem

The following statements are equivalent:

- *There exists a sentence* Φ *in the language of set theory such that the theory* $\mathbf{ZFC} + \Phi$ *is consistent and*

$$\mathbf{ZFC} + \Phi \vdash \text{“}\mathrm{Ord} \textit{ is essentially closure subtle”}.$$

- $\mathbf{ZFC} + \text{“}\mathrm{Ord} \textit{ is essentially closure subtle”} \nvdash \text{“}\mathrm{Ord} \textit{ is essentially subtle”}.$
- *The theory*

$$\mathbf{ZFC} + \textit{“There is a proper class of subtle cardinals”}$$

is consistent.

The techniques developed in the proofs of the above results also allow us to show that the existence of weak compactness cardinals for all abstract logics does not imply the existence of strongly inaccessible cardinals in V.

Theorem

The following schemes are equiconsistent over $\mathbf{ZFC}$*:*

- *There is a proper class of subtle cardinals.*
- Ord *is essentially closure subtle and there are no inaccessible cardinals.*

Weakly $C^{(n)}$-shrewd cardinals

We now relate the existence of weak compactness cardinals to large cardinal properties.

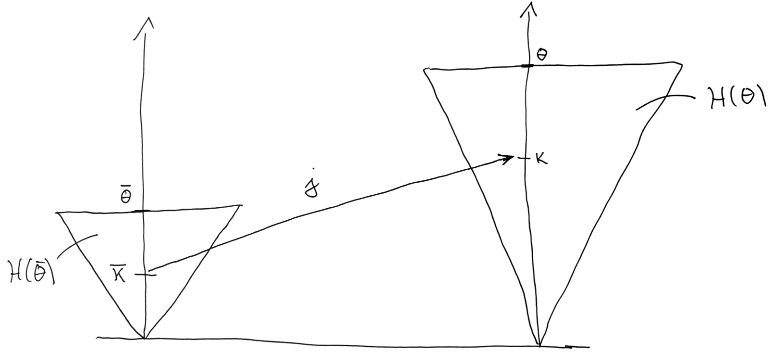
The starting point of these results is the following classical result:

Theorem (Magidor)

The following statements are equivalent for every cardinal κ:

- κ *is supercompact.*
- *For every cardinal $\theta > \kappa$ and all $z \in \mathrm{H}(\theta)$, there exist*
  - *cardinals $\bar{\kappa} < \bar{\theta} < \kappa$, and*
  - *an elementary embedding $j : \mathrm{H}(\bar{\theta}) \longrightarrow \mathrm{H}(\theta)$*

  such that $\mathrm{crit}(j) = \bar{\kappa}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.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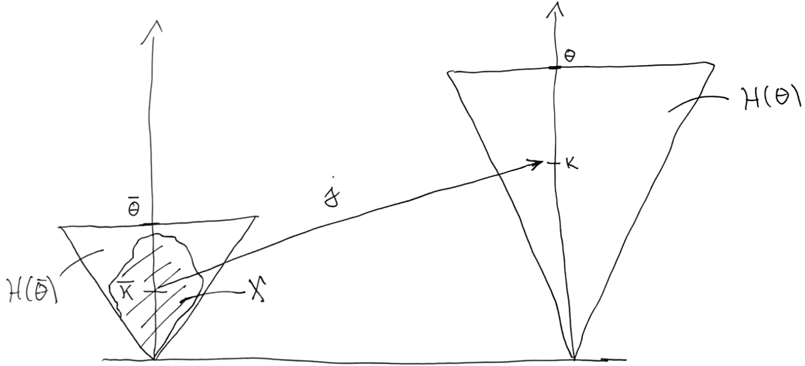
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X

Theorem

The following statements are equivalent for every cardinal κ:

- *For all cardinals $\theta > \kappa$ and all $z \in \mathrm{H}(\theta)$, there exist*
    - *cardinals $\bar{\kappa} < \bar{\theta} < \kappa$,*
    - *an elementary submodel X of $\mathrm{H}(\bar{\theta})$, and*
    - *an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$*

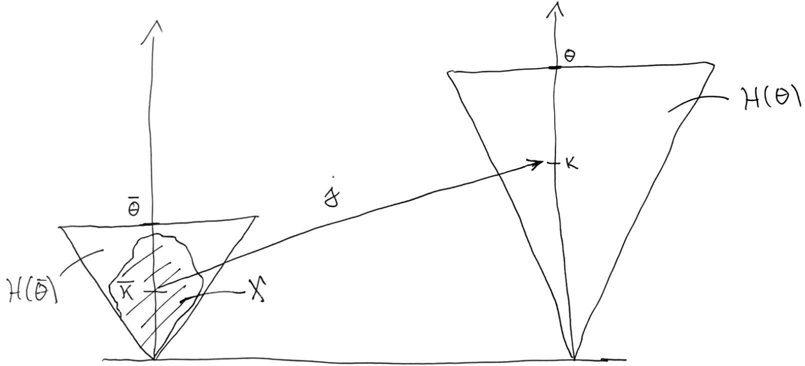
  such that $\bar{\kappa} + 1 \subseteq X$, $j \upharpoonright \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.
- *κ is a shrewd cardinal.*
- *κ is a strongly unfoldable cardinal.*

Definition (Rathjen)

A cardinal κ is *shrewd* if for every $\mathcal{L}_{\in}$-formula $\Phi(v_0, v_1)$, every ordinal $\gamma > \kappa$ and every subset A of V_κ such that $\Phi(A, \kappa)$ holds in V_γ, there exist ordinals $\alpha < \beta < \kappa$ such that $\Phi(A \cap \mathrm{V}_\alpha, \alpha)$ holds in V_β.

Definition (Villaveces)

An inaccessible cardinal κ is *strongly unfoldable* if for every ordinal λ and every transitive ZF^--model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there is a transitive set N with $V_\lambda \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) \geq \lambda$.



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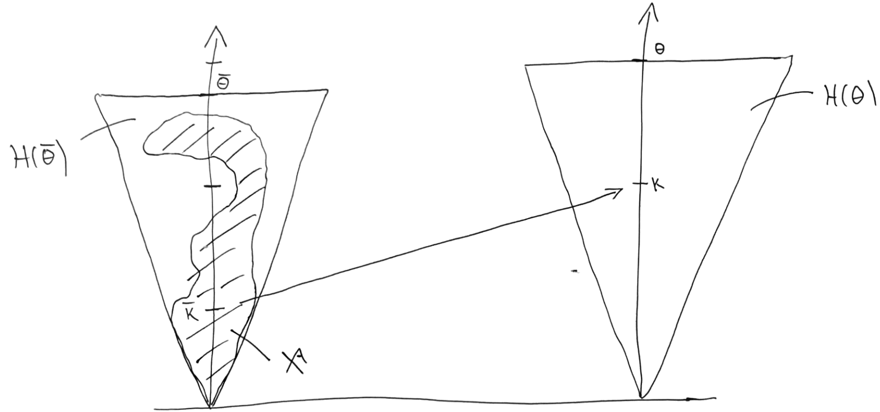

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Definition

An infinite cardinal κ is *weakly shrewd* if for every $\mathcal{L}_\in$-formula $\Phi(v_0, v_1)$, every cardinal $\theta > \kappa$ and every subset A of κ with the property that $\Phi(A, \kappa)$ holds in $\mathrm{H}(\theta)$, there exist cardinals $\bar{\kappa} < \bar{\theta}$ with the property that $\bar{\kappa} < \kappa$ and $\Phi(A \cap \bar{\kappa}, \bar{\kappa})$ holds in $\mathrm{H}(\bar{\theta})$.

Lemma

The following statements are equivalent for every infinite cardinal κ:

- *κ is a weakly shrewd cardinal.*
- *For all cardinals $\theta > \kappa$ and all $z \in \mathrm{H}(\theta)$, there exist*
  - *cardinals $\bar{\kappa} < \bar{\theta}$,*
  - *an elementary submodel X of $\mathrm{H}(\bar{\theta})$, and*
  - *an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$*

  with $\bar{\kappa} + 1 \subseteq X$, $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa > \bar{\kappa}$ and $z \in \mathrm{ran}(j)$.

Definition

Given a natural number $n > 0$, a cardinal κ is *weakly $C^{(n)}$-shrewd* if for every cardinal $\kappa < \theta \in C^{(n)}$ and every $z \in \mathrm{H}(\theta)$, there exists

- a cardinal $\bar{\theta} \in C^{(n)}$,
- a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$,
- an elementary submodel X of $\mathrm{H}(\bar{\theta})$, and
- an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$

such that $\bar{\kappa} + 1 \subseteq X$, $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.

Theorem

The following schemes of sentences are equivalent over $\mathbf{ZFC}$*:*

- Ord *is essentially closure subtle.*
- *For every natural number* $n > 0$*, there is a proper class of weakly* $C^{(n)}$*-shrewd cardinals.*
- *Every logic has a weak compactness cardinal.*

Theorem

The following schemes of sentences are equivalent over $\mathbf{ZFC}$*:*

- Ord *is essentially subtle.*
- *For every natural number* $n > 0$*, there is a proper class of weakly* $C^{(n)}$*-shrewd cardinals that are elements of* $C^{(n+1)}$*.*
- *For every natural number* $n > 0$*, there is a weakly* $C^{(n)}$*-shrewd cardinal that is an element of* $C^{(n+1)}$*.*
- *Every logic has a stationary class of weak compactness cardinals.*

Structural reflection

The above results establish a connection between large cardinal axioms and weak compactness cardinals for abstract logics.

I will now present joint work with Joan Bagaria that shows how these assumptions are related to principles strengthening the downward Löwenheim–Skolem Theorem.

These results will show that certain patterns repeat in all regions of the large cardinal hierarchy. They rely on the following reflection principle:

Definition (Bagaria)

Given a class $\mathcal{C}$ of structures of the same countable signature and an infinite cardinal κ, we let $\mathrm{SR}_{\mathcal{C}}(\kappa)$ denote the statement that for every structure $\mathfrak{B}$ in $\mathcal{C}$, there exists a structure $\mathfrak{A}$ in $\mathcal{C}$ of cardinality less than κ and an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$.

Theorem (Bagaria)

The following statements are equivalent for every cardinal κ:

- *The cardinal κ is the least supercompact cardinal.*
- *The cardinal κ is the least cardinal with the property that $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*

Theorem (Bagaria)

The following statements are equivalent for every natural number $n > 0$ and every cardinal κ:

- *The cardinal κ is the least $C^{(n)}$-extendible cardinal.*
- *The cardinal κ is the least cardinal with the property that $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_{n+2}-formula with parameters in V_κ.*

Theorem (Bagaria)

The following schemes are equivalent over $\mathbf{ZFC}$*:*

- *Vopěnka's Principle.*
- *For every class* $\mathcal{C}$ *of structures of the same countable signature, there is a cardinal* κ *such that* $\mathrm{SR}_{\mathcal{C}}(\kappa)$ *holds.*

Theorem (Bagaria–L.)

The following statements are equivalent for every uncountable cardinal δ*:*

- *The cardinal* δ *is a Vopěnka cardinal, i.e.,* δ *is an inaccessible cardinal with the property that for every set* $\mathcal{C} \in \mathrm{V}_{\delta+1} \setminus \mathrm{V}_{\delta}$ *of graphs, there are two members of the set with a homomorphism between them.*
- *For every set* $\mathcal{C}$ *of structures of the same type with* $\mathcal{C} \subseteq V_{\delta}$*, there is a cardinal* $\kappa < \delta$ *with the property that the principle* $\mathrm{SR}_{\mathcal{C}}(\kappa)$ *holds.*

We now want to find analogs of these characterizations for the large cardinal notions introduced earlier.

These results are based on the following restrictions of the principle SR:

Definition (Bagaria–Väänänen)

Let $\mathcal{C}$ be a non-empty class of structures of the same countable signature and let κ be an infinite cardinal.

- $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ denotes the statement that for every structure $\mathfrak{B}$ in $\mathcal{C}$ of cardinality κ, there exists a structure $\mathfrak{A}$ in $\mathcal{C}$ of cardinality less than κ and an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$.
- $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ denotes the statement that $\mathcal{C}$ contains a structure of cardinality less than κ.

Theorem (Bagaria–L.)

The following statements are equivalent for every natural number $n > 0$ and every cardinal κ:

- *The cardinal κ is the least weakly $C^{(n)}$-shrewd cardinal in $C^{(n+1)}$.*
- *The cardinal κ is the least cardinal with the property that the principles $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ and $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ hold for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_{n+1}-formula with parameters in V_{κ}.*

Theorem (Bagaria–L.)

The following schemes of sentences are equivalent over $\mathbf{ZFC}$*:*

- Ord *is essentially subtle.*
- *For every class* $\mathcal{C}$ *of structures of the same countable signature, there exists a cardinal* κ *such that* $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ *and* $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ *hold.*

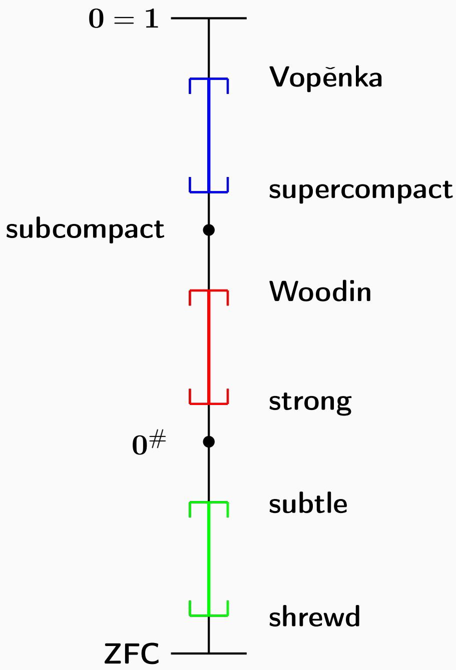
Theorem (Bagaria–L.)

The following statements are equivalent for every uncountable cardinal δ*:*

- *The cardinal* δ *is the least subtle cardinal.*
- *The cardinal* δ *is the least cardinal with the property that for every set* $\mathcal{C}$ *of structures of the same countable signature with* $\mathcal{C} \subseteq V_\delta$*, there exists a cardinal* $\kappa < \delta$ *with the property that the principles* $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ *and* $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ *hold.*

The above results show that the interval in the large cardinal hierarchy between supercompactness and Vopěnkaness has the same structure as the interval between shrewdness and subtleness.

Using results of Bagaria and Wilson on the concept of *product structural reflection* introduced by Wilson, it is possible to show that the same pattern repeats in the interval between strongness and Woodinness.



$0 = 1$
Vopěnka
supercompact
subcompact
Woodin
strong
$0^{\#}$
subtle
shrewd
ZFC

Thank you for listening!