Fresh subsets of measurable ultrapowers

Philipp Moritz Lücke
Institut de Matemàtica, Universitat de Barcelona.

Joint work with Sandra Müller (Vienna).

Hebrew University – Bar Ilan University Set Theory Seminar, 27. January 2021.

Introduction

Given a normal ultrafilter U on a measurable cardinal, we let $\mathrm{Ult}(\mathrm{V}, U)$ denote the induced ultrapower and we let $j_U : \mathrm{V} \longrightarrow \mathrm{Ult}(\mathrm{V}, U)$ denote the corresponding elementary embedding.

We identify $\mathrm{Ult}(\mathrm{V}, U)$ with its transitive collapse and its elements $[f]_U$ with their images under the given transitive collapse.

The work presented in this talk studies the closure properties of measurable ultrapowers.

The following proposition lists the basic closure and non-closure properties of models of the form $\mathrm{Ult}(\mathrm{V}, U)$.

Proposition

Let U be a normal ultrafilter on a measurable cardinal δ.

- ${}^{\delta}\mathrm{Ult}(\mathrm{V}, U) \subseteq \mathrm{Ult}(\mathrm{V}, U)$.
- $U \notin \mathrm{Ult}(\mathrm{V}, U)$.
- $j_U[\delta^+] \notin \mathrm{Ult}(\mathrm{V}, U)$.

The following theorem of Cummings shows that additional closure properties of $\mathrm{Ult}(\mathrm{V}, U)$ can highly depent on the underlying model of set theory.

Theorem (Cummings)

The following statements are equiconsistent over the theory $\mathbf{ZFC}$*:*

- *There exists a* $\mathcal{P}_2(\delta)$*-hypermeasure.*
- *There exists a normal ultrafilter* U *on a measurable cardinal* δ *with the property that* $\mathcal{P}(\delta^+) \subseteq \mathrm{Ult}(\mathrm{V}, U)$*.*

In the following, we want to study the closure and non-closure properties of measurable ultrapowers through the following notion introduced by Hamkins:

Definition (Hamkins)

Given a class M, a set A of ordinals is *fresh over* M if $A \notin M$ and $A \cap \alpha \in M$ for all $\alpha < \mathrm{lub}(A)$.

The above theorem can now be rephrased in the following way:

Theorem (Cummings)

The following statements are equiconsistent over the theory $\mathbf{ZFC}$*:*

- *There exists a* $\mathcal{P}_2(\delta)$*-hypermeasure.*
- *There exists a normal ultrafilter* U *on a measurable cardinal* δ *with the property that no unbounded subset of* δ^+ *is fresh over* $\mathrm{Ult}(\mathrm{V}, U)$*.*

For a given normal ultrafilter U, we now to determine the class of limit ordinals containing an unbounded subset that is fresh over $\mathrm{Ult}(\mathrm{V}, U)$.

For the images of regular cardinals under the embedding j_U, this question was already studied by Assaf Shani.

Moreover, Hiroshi Sakai investigated stronger closure properties of measurable ultrapowers that imply the non-existence of unbounded fresh subsets at many ordinals.

We continue by listing the obvious closure properties of measurable ultrapowers with respect to the non-existence of fresh subsets.

We will later show that this list covers all provable closure and non-closure properties of measurable ultrapowers, in the sense that there are ...

- ... models of set theory in which fresh subsets exist at all limit ordinals that are not ruled out by these properties, and
- ... models in which fresh subsets only exist at limit ordinals, where their existence is guaranteed by them.

Basic closure and non-closure properties

Proposition

Let U be a normal ultrafilter on a measurable cardinal δ.

If $\kappa > \delta$ is the minimal cardinal with $\mathcal{P}(\kappa) \nsubseteq \mathrm{Ult}(\mathrm{V}, U)$, then there is an unbounded subset of κ that is fresh over $\mathrm{Ult}(\mathrm{V}, U)$.

Proof.

Every element of $\mathcal{P}(\kappa) \setminus \mathrm{Ult}(\mathrm{V}, U)$ is unbounded in κ and fresh over $\mathrm{Ult}(\mathrm{V}, U)$. $\square$

Proposition

Let U be a normal ultrafilter on a measurable cardinal δ.

If λ is a limit ordinal with $\mathrm{cof}(\lambda)^{\mathrm{Ult}(\mathrm{V},U)} = j_U(\delta^+)$, then there is an unbounded subset of λ that is fresh over $\mathrm{Ult}(\mathrm{V},U)$.

Proof.

Fix a cofinal function $c : j_U(\delta^+) \longrightarrow \lambda$ in $\mathrm{Ult}(\mathrm{V},U)$ and set

$$A \;=\; (c \circ j_U)[\delta^+].$$

Since $j_U[\delta^+]$ is a cofinal subset of $j_U(\delta^+)$ of order-type δ^+, we know that A is a cofinal subset of λ of order-type δ^+.

Then the closure of $\mathrm{Ult}(\mathrm{V},U)$ under δ-sequences implies that every proper initial segment of A is an element of $\mathrm{Ult}(\mathrm{V},U)$.

Finally, since $j_U[\delta^+] \notin \mathrm{Ult}(\mathrm{V},U)$, we can conclude that A is fresh over $\mathrm{Ult}(\mathrm{V},U)$. $\square$

Proposition

Let U be a normal ultrafilter on a measurable cardinal δ.

If $2^\delta = \delta^+$ holds and λ is a limit ordinal with $\mathrm{cof}(\lambda) = \delta^+$, then there is an unbounded subset of λ that is fresh over $\mathrm{Ult}(\mathrm{V}, U)$.

Proof.

First, assume that $\mathrm{cof}(\lambda)^{\mathrm{Ult}(\mathrm{V},U)} = \delta^+$.

Since $2^\delta = \delta^+$, there is an unbounded subset A of δ^+ that is fresh over $\mathrm{Ult}(\mathrm{V}, U)$ and we can use a cofinal function from δ^+ to λ in $\mathrm{Ult}(\mathrm{V}, U)$ to find an unbounded subset of λ that is fresh over $\mathrm{Ult}(\mathrm{V}, U)$.

Now, assume that $\mathrm{cof}(\lambda)^{\mathrm{Ult}(\mathrm{V},U)} > \delta^+$ and fix an unbounded subset A of λ of order-type δ^+.

Then the closure of $\mathrm{Ult}(\mathrm{V}, U)$ under δ-sequences implies that A is fresh over $\mathrm{Ult}(\mathrm{V}, U)$. $\square$

Proposition

Let U be a normal ultrafilter on a measurable cardinal δ and let λ be a limit ordinal.

If $\mathrm{cof}(\lambda) \leqslant \delta$, then no unbounded subset of λ is fresh over $\mathrm{Ult}(\mathrm{V}, U)$.

Proof.

This statement follows directly from the closure of $\mathrm{Ult}(\mathrm{V}, U)$ under δ-sequences. $\square$

Proposition

Let U be a normal ultrafilter on a measurable cardinal δ and let λ be a limit ordinal.

If the cardinal $\mathrm{cof}(\lambda)$ is weakly compact, then no unbounded subset of λ is fresh over $\mathrm{Ult}(\mathrm{V}, U)$.

Assume that $\kappa = \mathrm{cof}(\lambda)$ is a weakly compact cardinal greater than δ and fix an unbounded subset A of λ with $A \cap \gamma \in \mathrm{Ult}(\mathrm{V}, U)$ for all $\gamma < \lambda$.

Fix sequences $\langle f_\alpha \mid \alpha < \kappa \rangle$ and $\langle g_\alpha \mid \alpha < \kappa \rangle$ of functions with domain δ such that the sequence $\langle [f_\alpha]_U \mid \alpha < \kappa \rangle$ is cofinal in λ and $[g_\alpha]_U = A \cap [f_\alpha]_U$ for all $\alpha < \kappa$.

Let $c : [\kappa]^2 \longrightarrow U$ denote the unique function with the property that

$$c(\{\alpha, \beta\}) \ = \ \{\xi < \delta \mid f_\alpha(\xi) < f_\beta(\xi), \ g_\alpha(\xi) = f_\alpha(\xi) \cap g_\beta(\xi)\}$$

holds for all $\alpha < \beta < \kappa$.

Since $\kappa > 2^\delta = |U|$ is weakly compact, there is an unbounded subset H of κ and $X \in U$ with $c[H]^2 = \{X\}$.

Pick a function g with domain δ and $g(\xi) = \bigcup \{g_\alpha(\xi) \mid \alpha \in H\}$ for $\xi \in X$.

Then $[g]_U \cap [f_\alpha]_U = [g_\alpha]_U$ for $\alpha \in H$ and $[g]_U = A \in \mathrm{Ult}(\mathrm{V}, U)$. $\square$

Proposition

Let U be a normal ultrafilter on a measurable cardinal δ and let λ be a limit ordinal.

If there exists a ${<}(2^\delta)^+$-closed ultrafilter F on λ that contains all cobounded subsets of λ, then no unbounded subset of λ is fresh over $\mathrm{Ult}(\mathrm{V}, U)$.

Assume that A is an unbounded subset of λ that is fresh over $\mathrm{Ult}(\mathrm{V}, U)$.

Given $\eta < \lambda$, fix functions f_η and g_η with domain δ satisfying $[f_\eta]_U = \eta$ and $[g_\eta]_U = A \cap \eta$.

Given $\eta < \lambda$ and $X \in U$, define $A_{\eta,X}$ to be the set of all $\zeta \in (\eta, \lambda)$ with

$$X = \{\xi < \delta \mid f_\eta(\xi) < f_\zeta(\xi),\ g_\eta(\xi) = g_\zeta(\xi) \cap f_\eta(\xi)\}.$$

If $\eta < \lambda$, then

$$\bigcup\{A_{\eta,X} \mid X \in U\} \ = \ (\eta, \lambda) \ \in \ F.$$

Then there is a sequence $\langle X_\eta \in U \mid \eta < \lambda\rangle$ with $A_{\eta,X_\eta} \in F$ for all $\eta < \lambda$.

Given $X \in U$, set $E_X = \{\eta < \lambda \mid X_\eta = X\}$.

Since $\bigcup\{E_X \mid X \in U\} = \lambda$, there is $X_* \in U$ with $E_{X_*} \in F$.

We found $X_* \in U$ and $E_{X_*} \in F$ with $A_{\eta,X_*} \in F$ for all $\eta \in E_{X_*}$, where

$$A_{\eta,X_*} = \{\zeta \in (\eta,\lambda) \mid X_* = \{\xi < \delta \mid f_\eta(\xi) < f_\zeta(\xi),\ g_\eta(\xi) = g_\zeta(\xi) \cap f_\eta(\xi)\}\}.$$

Claim

If $\eta, \zeta \in E_{X_*}$, $\xi \in X_*$ and $\alpha = \min(f_\eta(\xi), f_\zeta(\xi))$, then

$$g_\eta(\xi) \cap \alpha \ = \ g_\zeta(\xi) \cap \alpha.$$

Proof of the Claim.

Pick $\rho \in A_{\eta,X_*} \cap A_{\zeta,X_*} \in F$. Then $\alpha < f_\rho(\xi)$ and

$$g_\eta(\xi) \cap \alpha \ = \ g_\rho(\xi) \cap \alpha \ = \ g_\zeta(\xi) \cap \alpha. \qquad \square$$

We found $X_* \in U$ and $E_{X_*} \in F$ with

$$g_\eta(\xi) \cap \alpha \ = \ g_\zeta(\xi) \cap \alpha$$

whenever $\eta, \zeta \in E_{X_*}$, $\xi \in X_*$ and $\alpha = \min(f_\eta(\xi), f_\zeta(\xi)$.

Pick a function g with domain δ and

$$g(\xi) \ = \ \bigcup\{g_\eta(\xi) \mid \eta \in E_{X_*}\}$$

for all $\xi \in X_*$. By the above claim, we now know that $g(\xi) \cap f_\eta(\xi) = g_\eta(\xi)$ holds for all $\eta \in E_{X_*}$ and all $\xi \in X_* \in U$. But this directly implies that

$$[g]_U \cap \eta \ = \ [g_\eta]_U \ = \ A \cap \eta$$

for all $\eta \in E_{X_*}$. Hence, we can conclude that $[g]_U = A \in \mathrm{Ult}(\mathrm{V}, U)$, a contradiction. $\square$

By the *filter extension property* of strongly compact cardinals, the above result directly implies the following corollary.

This corollary also follows directly from results of Hiroshi Sakai.

Corollary

Let U be a normal ultrafilter on a measurable cardinal δ and let λ be a limit ordinal.

If the interval $(\delta, \mathrm{cof}(\lambda))$ contains a strongly compact cardinal, then no unbounded subset of λ is fresh over $\mathrm{Ult}(\mathrm{V}, U)$. $\square$

Sakai's result

We now use a theorem of Hiroshi Sakai to show that the above results include all provable non-closure properties of measurable ultrapowers.

Sakai's result relies on another concept isolated by Hamkins:

Definition (Hamkins)

Given an infinite cardinal κ, an inner model M has the κ-*approximation property* if $X \in M$ holds for every set X of ordinals with the property that $X \cap a \in M$ for all $a \in M$ with $|a|^M < \kappa$.

Proposition

Let κ be an uncountable regular cardinal and let M be an inner model with the κ-approximation property.

If λ is a limit ordinal with $\mathrm{cof}(\lambda) \geqslant \kappa$, then no unbounded subset of λ is fresh over M.

Theorem (Sakai)

Let δ be a measurable cardinal and let W be an inner model such that the GCH holds in W, V is a $\mathrm{Col}((\delta^+)^{\mathrm{V}}, {<}(\delta^{++})^{\mathrm{V}})^W$-generic extension of W and $(\delta^{++})^{\mathrm{V}}$ is strongly compact in W.

If U is a normal ultrafilter on δ, then the inner model $\mathrm{Ult}(\mathrm{V}, U)$ has the δ^{++}-approximation property.

Corollary

In the situation of the above theorem, the following statements are equivalent for every normal ultrafilter U on δ and every limit ordinal λ:

- *There is an unbounded subset of λ that is fresh over $\mathrm{Ult}(\mathrm{V}, U)$.*
- $\mathrm{cof}(\lambda) = \delta^+$.

Measurable ultrapowers of canonical inner models

We now want to show that the above results also include all provable closure properties of measurable ultrapowers.

More specifically, we show that measurable ultrapowers of canonical inner models possess the minimal amount of closure properties with respect to freshness.

Our results apply to a large class of canonical inner models, so called *Jensen-style extender models*, that can contain various large cardinals below supercompact cardinals.

Theorem (L.–Müller)

Assume that V *is a Jensen-style extender model that does not have a subcompact cardinal.*

Then the following statements are equivalent for every normal ultrafilter U *on a measurable cardinal* δ *and every limit ordinal* λ*:*

- *There is an unbounded subset of* λ *that is fresh over* $\mathrm{Ult}(\mathrm{V}, U)$*.*
- *The cardinal* $\mathrm{cof}(\lambda)$ *is greater than* δ *and not weakly compact.*

The proof of this result makes use of the existence of various *square sequences* in these models.

Definition

- (Todorčević) Given an uncountable regular cardinal κ, a sequence $\langle C_\gamma \mid \gamma \in \mathrm{Lim} \cap \kappa \rangle$ is a $\square(\kappa)$-*sequence* if the following statements hold:
  - C_γ is a closed unbounded subset of γ for all $\gamma \in \mathrm{Lim} \cap \kappa$.
  - If $\gamma \in \mathrm{Lim} \cap \kappa$ and $\beta \in \mathrm{Lim}(C_\gamma)$, then $C_\beta = C_\gamma \cap \beta$.
  - There is no closed unbounded subset C of κ with $C \cap \gamma = C_\gamma$ for all $\gamma \in \mathrm{Lim}(C)$.
- (Jensen) Given an infinite cardinal κ, a $\square(\kappa^+)$-sequence $\langle C_\gamma \mid \gamma \in \mathrm{Lim} \cap \kappa^+ \rangle$ is a $\square_\kappa$-*sequence* if $\mathrm{otp}(C_\gamma) \leqslant \kappa$ holds for all $\gamma \in \mathrm{Lim} \cap \kappa^+$.

Seminal results of Schimmerling and Zeman show that, in Jensen-style extender models, a $\square_\nu$-sequence exists if and only if ν is not a subcompact cardinal.

In addition, Zeman extended results of Jensen by showing that $\square(\kappa)$-sequences exist in these models for all inaccessible, non-weakly compact cardinals.

Finally, it should be noted that results of Kypriotakis and Zeman show that, in Jensen-style extender models, a $\square(\kappa^+)$-sequence exists if κ is the least subcompact cardinal.

Theorem

Let U be a normal ultrafilter on a measurable cardinal δ. Assume that the following statements hold:

- *The GCH holds at all cardinals greater than or equal to δ.*
- *If $\kappa > \delta^+$ is a regular cardinal that is not weakly compact, then there exists a $\square(\kappa)$-sequence.*
- *If $\kappa > \delta$ is a singular cardinal, then there exists a $\square_\kappa$-sequence.*

Then the following statements are equivalent for every limit ordinal λ:

- *There is an unbounded subset of λ that is fresh over $\mathrm{Ult}(\mathrm{V}, U)$.*
- *The cardinal $\mathrm{cof}(\lambda)$ is greater than δ and not weakly compact.*

In the following, we discuss the central parts of the proof of this result.

We first observe that it suffices to concentrate on the regular cardinals of the ultrapower.

Proposition

Let U be a normal ultrafilter on a measurable cardinal δ and let λ be a limit ordinal. If there is an unbounded subset of $\mathrm{cof}(\lambda)^{\mathrm{Ult}(\mathrm{V},U)}$ that is fresh over $\mathrm{Ult}(\mathrm{V},U)$, then there is an unbounded subset of λ that is fresh over $\mathrm{Ult}(\mathrm{V},U)$.

Indexed squares

The following result allows us to handle the image points of regular cardinals in the proof of the above theorem.

It strengthens results of Sakai and Shani.

Theorem

Let U be a normal ultrafilter on a measurable cardinal δ and let $\kappa > \delta$ be a regular cardinal.

If there exists a $\square(\kappa)$-sequence, then there is a closed unbounded subset of $j_U(\kappa)$ that is fresh over $\mathrm{Ult}(\mathrm{V}, U)$.

This result can already be used to show that the conclusion of Sakai's theorem has high consistency strength.

First, if U is a normal ultrafilter on a measurable cardinal δ with the property that for every limit ordinal λ with $\mathrm{cof}(\lambda) \in \{\delta^{++}, \delta^{+++}\}$, no unbounded subset of λ is fresh over $\mathrm{Ult}(\mathrm{V}, U)$, then $\kappa = \delta^{++}$ is a countably closed regular cardinal greater than $\max\{2^{\aleph_0}, \aleph_3\}$ with the property that there are no $\square(\kappa)$- and no $\square_\kappa$-sequences.

By a result of Schimmerling, the existence of such a cardinal implies *Projective Determinacy*.

In addition, work of Jensen, Schimmerling, Schindler and Steel shows that the existence of such a cardinal implies the existence of a sharp for a proper class model with a proper class of strong cardinals and a proper class of Woodin cardinals.

Now, assume that U is a normal ultrafilter on a measurable cardinal δ and $\kappa > \delta$ is a singular strong limit cardinal with the property that unbounded subsets of limit ordinals of cofinality κ^+ are not fresh over $\mathrm{Ult}(\mathrm{V}, U)$.

Then the above theorem shows that there are no $\square_\kappa$-sequences and therefore the work of Steel shows that AD holds in $\mathrm{L}(\mathbb{R})$.

Even stronger consequences of this conclusion can be derived with the help of results of Sargsyan.

Definition (Lambie-Hanson)

Let $\delta < \kappa$ be infinite regular cardinals. A $\square^{\mathrm{ind}}(\kappa, \delta)$-sequence is a matrix

$$\langle C_{\gamma,\xi} \mid \gamma < \kappa,\ i(\gamma) \leqslant \xi < \delta \rangle$$

satisfying the following statements:

- If $\gamma \in \mathrm{Lim} \cap \kappa$, then $i(\gamma) < \delta$.
- If $\gamma \in \mathrm{Lim} \cap \kappa$ and $i(\gamma) \leqslant \xi < \delta$, then $C_{\gamma,\xi}$ is a club in γ.
- If $\gamma \in \mathrm{Lim} \cap \kappa$ and $i(\gamma) \leqslant \xi_0 < \xi_1 < \delta$, then $C_{\gamma,\xi_0} \subseteq C_{\gamma,\xi_1}$.
- If $\beta, \gamma \in \mathrm{Lim} \cap \kappa$ and $i(\gamma) \leqslant \xi < \delta$ with $\beta \in \mathrm{Lim}(C_{\gamma,\xi})$, then $\xi \geqslant i(\beta)$ and $C_{\beta,\xi} = C_{\gamma,\xi} \cap \beta$.
- If $\beta, \gamma \in \mathrm{Lim} \cap \kappa$ with $\beta < \gamma$, then there is an $i(\gamma) \leqslant \xi < \delta$ such that $\beta \in \mathrm{Lim}(C_{\gamma,\xi})$.
- There is no closed unbounded subset C of κ and $\xi < \delta$ such that $\xi \geqslant i(\gamma)$ and $C_{\gamma,\xi} = C \cap \gamma$ hold for all $\gamma \in \mathrm{Lim}(C)$.

Theorem (Lambie-Hanson–L.)

Let $\delta < \kappa$ be infinite regular cardinals. If there exists a $\Box(\kappa)$-sequence, then there exists a $\Box^{\mathrm{ind}}(\kappa, \delta)$-sequence.

Theorem

Let U be a normal ultrafilter on a measurable cardinal δ and let $\kappa > \delta$ be a regular cardinal.

If there exists a $\Box(\kappa)$-sequence, then there is a closed unbounded subset of $j_U(\kappa)$ that is fresh over $\mathrm{Ult}(\mathrm{V}, U)$.

Fix a $\square^{\mathrm{ind}}(\kappa,\delta)$-sequence

$$\langle C_{\gamma,\xi} \mid \gamma < \kappa,\ i(\gamma) \leqslant \xi < \delta\rangle.$$

Given $\gamma \in \mathrm{Lim} \cap \kappa$, let $f_\gamma : \delta \longrightarrow \mathcal{P}(\gamma)$ denote the unique function with $f_\gamma(\xi) = \emptyset$ for all $\xi < i(\gamma)$ and $f_\gamma(\xi) = C_{\gamma,\xi}$ for all $i(\gamma) \leqslant \xi < \delta$.

By Łos' Theorem, if $\beta, \gamma \in \mathrm{Lim} \cap \kappa$ with $\beta \leqslant \gamma$, then $[f_\gamma]_U$ is closed unbounded in $j_U(\gamma)$ and

$$[f_\beta]_U \ = \ [f_\gamma]_U \cap j_U(\beta)$$

holds. Define

$$A \ = \ \bigcup\{[f_\gamma]_U \mid \gamma \in \mathrm{Lim} \cap \kappa\}.$$

Since $j_U(\kappa) = \sup(j_U[\kappa])$, the set A is closed unbounded in $j_U(\kappa)$ with $A \cap j_U(\gamma) = [f_\gamma]_U$ for all $\gamma \in \mathrm{Lim} \cap \kappa$.

Assume that $A \in \mathrm{Ult}(\mathrm{V}, U)$ and pick $f : \delta \longrightarrow \mathcal{P}(\kappa)$ with $[f]_U = A$ and the property that $f(\xi)$ is a closed unbounded subset of κ for all $\xi < \delta$.

Given $\gamma \in \mathrm{Lim} \cap \kappa$, we have

$$\{\xi \in [i(\gamma), \delta) \mid f_\gamma(\xi) = f(\xi) \cap \gamma\} \ \in \ U.$$

Hence, we can find $\xi < \delta$ with the property that $\xi \geqslant i(\gamma)$ and $f(\xi) \cap \gamma = C_{\gamma,\xi}$ holds for unboundedly many γ below κ.

Pick $\beta \in \mathrm{Lim}(f(\xi))$ and $\beta < \gamma < \kappa$ with $\xi \geqslant i(\gamma)$ and $f(\xi) \cap \gamma = C_{\gamma,\xi}$.

Then $\beta \in \mathrm{Lim}(C_{\gamma,\xi})$, $\xi \geqslant i(\beta)$ and $C_{\beta,\xi} = C_{\gamma,\xi} \cap \beta = f(\xi) \cap \beta$.

Hence, there is a club C in κ and $\xi < \delta$ with $\xi \geqslant i(\beta)$ and $C \cap \beta = C_{\beta,\xi}$ for all $\beta \in \mathrm{Lim}(C)$, a contradiction. $\square$

Successors of singular cardinals

The other important case in the proof of our main theorem are limit ordinals whose cofinality is the successor of a singular cardinal of cofinality equal to the given measurable cardinal.

These ordinals will be handled by the next result:

Theorem

Let U be a normal ultrafilter on a measurable cardinal δ and let κ be a singular cardinal of cofinality δ.

If the GCH holds above δ and there exists a $\square_\kappa$-sequence, then there is a closed unbounded subset of κ^+ that is fresh over $\mathrm{Ult}(\mathrm{V}, U)$.

The proof of the above result makes use of the following standard lemmas:

Lemma

Let U be a normal ultrafilter on a measurable cardinal δ.

If $\nu > \delta$ is a cardinal with $\mathrm{cof}(\nu) \neq \delta$ and $\lambda^\delta < \nu$ for all $\lambda < \nu$, then $j_U(\nu) = \nu$ and $j_U(\nu^+) = \nu^+$.

Lemma

Let κ be a singular cardinal and let S be a stationary subset of κ^+.

If there exists a $\square_\kappa$-sequence, then there exists a $\square_\kappa$-sequence $\langle C_\gamma \mid \gamma \in \mathrm{Lim} \cap \kappa^+ \rangle$ and a stationary subset E of S such that $\mathrm{otp}(C_\gamma) < \kappa$ and $\mathrm{Lim}(C_\gamma) \cap E = \emptyset$ for all $\gamma \in \mathrm{Lim} \cap \kappa^+$.

Fix a $\square_\kappa$-sequence $\langle C_\gamma \mid \gamma \in \mathrm{Lim} \cap \kappa^+ \rangle$ and a stationary subset E of $S^{\kappa^+}_\delta$ such that $\mathrm{otp}(C_\gamma) < \kappa$ and $\mathrm{Lim}(C_\gamma) \cap E = \emptyset$ for all $\gamma \in \mathrm{Lim} \cap \kappa^+$.

By our assumptions, there is a continuous, strictly increasing sequence $\langle \kappa_\xi \mid \xi < \delta \rangle$ of j_U-fixed points that is cofinal in κ.

The normality of U then implies that $[\xi \mapsto \kappa_\xi]_U = \kappa$ and $[\xi \mapsto \kappa_\xi^+]_U \leqslant \kappa^+$.

Given $\gamma \in \mathrm{Lim} \cap \kappa^+$, let ξ_γ denote the minimal element ξ of δ with $\kappa_\xi^+ > \mathrm{otp}(C_\gamma)$.

Note that, if $\gamma \in \mathrm{Lim} \cap \kappa^+$ and $\beta \in \mathrm{Lim}(C_\gamma)$, then $\xi_\beta \leqslant \xi_\gamma$.

We inductively construct a sequence

$$\langle f_\gamma \in \prod_{\xi<\delta} \kappa_\xi^+ \mid \gamma < \kappa^+ \rangle$$

with the following properties:

- If $\beta < \gamma < \kappa^+$, then $f_\beta(\xi) < f_\gamma(\xi)$ for coboundedly many $\xi < \delta$.
- If $\gamma \in \mathrm{Lim} \cap \kappa^+$ and $\beta \in \mathrm{Lim}(C_\gamma)$, then $f_\beta(\xi) < f_\gamma(\xi)$ for all $\xi_\gamma \leqslant \xi < \delta$.
- If $\gamma \in \mathrm{Lim} \cap \kappa^+$, then $f_\gamma(\xi) \in \mathrm{Lim}$ for all $\xi < \delta$.

Given $\beta < \gamma < \kappa^+$, we then have

$$[f_\beta]_U \;<\; [f_\gamma]_U \;<\; [\xi \mapsto \kappa_\xi^+]_U.$$

In particular, this shows that $[\xi \mapsto \kappa_\xi^+]_U = \kappa^+$.

The construction of this sequence is straightforward:

- $f_0(\xi) = 0$ for all $\xi < \delta$.
- $f_{\gamma+1}(\xi) = f_\gamma(\xi) + 1$ for all $\gamma < \kappa^+$ and $\xi < \delta$.
- If $\gamma \in \mathrm{Lim} \cap \kappa^+$ with $\mathrm{Lim}(C_\gamma)$ bounded in γ and $\xi < \delta$, then

$$f_\gamma(\xi) \;=\; \min\{\rho \in \mathrm{Lim} \mid \rho > f_\beta(\xi) \textit{ for all } \beta \in C_\gamma \setminus \max(\mathrm{Lim}(C_\gamma))\}.$$

- If $\gamma \in \mathrm{Lim} \cap \kappa^+$ with $\mathrm{Lim}(C_\gamma)$ unbounded in γ, then

$$f_\gamma(\xi) \;=\; \begin{cases} \omega, & \textit{if } \xi < \xi_\gamma. \\ \sup\{f_\beta(\xi) \mid \beta \in \mathrm{Lim}(C_\gamma)\}, & \textit{if } \xi_\gamma \leqslant \xi < \delta. \end{cases}$$

Since $2^\kappa = \kappa^+$ holds, there is an enumeration $\langle h_\alpha \mid \alpha < \kappa^+ \rangle$ of $\prod_{\xi<\delta} \mathcal{P}(\kappa_\xi^+)$ of order-type κ^+.

Let $\langle \gamma_\alpha \mid \alpha < \kappa^+ \rangle$ denote the monotone enumeration of the stationary set E that is avoided by our $\Box_\kappa$-sequence.

We inductively define a sequence $\langle c_\gamma \mid \gamma \in \mathrm{Lim} \cap \kappa^+ \rangle$ of functions with domain δ satisfying the following statements for all $\gamma \in \mathrm{Lim} \cap \kappa^+$:

(a) $c_\gamma(\xi)$ is a closed unbounded subset of $f_\gamma(\xi)$ for all $\xi < \delta$.

(b) If $\beta \in \mathrm{Lim} \cap \gamma$, then $c_\beta(\xi) = c_\gamma(\xi) \cap f_\beta(\xi)$ for coboundedly many $\xi < \delta$.

(c) If $\gamma \notin E$, $\beta \in \mathrm{Lim}(C_\gamma)$ and $\xi_\gamma \leqslant \xi < \delta$, then $c_\beta(\xi) = c_\gamma(\xi) \cap f_\beta(\xi)$.

(d) If $\alpha < \kappa^+$ and $\xi_{\gamma_\alpha} \leqslant \xi < \delta$, then $c_{\gamma_\alpha}(\xi) \neq h_\alpha(\xi) \cap f_{\gamma_\alpha}(\xi)$.

Then $[c_\gamma]_U$ is a closed unbounded subset of $[f_\gamma]_U$ for all $\gamma \in \mathrm{Lim} \cap \kappa^+$.

Moreover, $[c_\beta]_U = [c_\gamma]_U \cap [f_\beta]_U$ for all $\beta, \gamma \in \mathrm{Lim} \cap \kappa^+$ with $\beta < \gamma$.

In particular, there is a club C in κ^+ with $C \cap [f_\gamma]_U = [c_\gamma]_U$ for all $\gamma \in \mathrm{Lim} \cap \kappa^+$.

Finally, we have

$$C \cap [f_{\gamma_\alpha}]_U \ = \ [c_{\gamma_\alpha}]_U \ \neq \ [h_\alpha]_U \cap [f_{\gamma_\alpha}]_U$$

for all $\alpha < \kappa^+$ and hence $C \notin \mathrm{Ult}(\mathrm{V}, U)$.

Case

$\gamma = \gamma_\alpha \in E$ for some $\alpha < \kappa^+$.

Let $\langle \beta_\xi \mid \xi < \delta \rangle$ be the monotone enumeration of a subset of $\mathrm{Lim}(C_\gamma)$ of order-type δ that is closed unbounded in γ.

Given $\xi_\gamma \leqslant \xi < \delta$, we have $f_{\beta_\xi}(\xi) < f_\gamma(\xi)$ and we can pick a club subset C^γ_ξ of $f_\gamma(\xi)$ with $\min(C^\gamma_\xi) = f_{\beta_\xi}(\xi)$ and $C^\gamma_\xi \neq h_\alpha(\xi) \cap [f_{\beta_\xi}(\xi), f_\gamma(\xi))$.

Define

$$c_\gamma(\xi) \; = \; \begin{cases} \omega, & \textit{for all } \xi < \xi_\gamma. \\ c_{\beta_\xi}(\xi) \; \cup \; C^\gamma_\xi, & \textit{for all } \xi_\gamma \leqslant \xi < \delta. \end{cases}$$

Then $c_\gamma(\xi)$ is a club subset of $f_\gamma(\xi)$ for all $\xi < \delta$ and

$$c_\gamma(\xi) \neq h_\alpha(\xi) \cap f_\gamma(\xi)$$

holds for all $\xi_\gamma \leqslant \xi < \delta$. Moreover, it is easy to check that c_γ also satisfies the other desired properties. $\square$

Thank you for listening!