Small Models, Large Cardinals, and Induced Ideals

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Introduction

The theory of large cardinals plays central role in modern set theory.

Originating from the work of Hausdorff on cardinal arithmetics and the work of Ulam on the *measure problem*, large cardinal axioms postulate the existence of cardinal numbers having certain properties that make them very large, and whose existence cannot be proved in $\mathbf{ZFC}$, because it implies the consistency of $\mathbf{ZFC}$ itself.

We present some examples of classical large cardinal notions.

Definition (Hausdorff, Sierpiński–Tarski, Zermelo)

An uncountable cardinal is *inaccessible* if it is a regular strong limit cardinal.

Definition (Erdős–Tarski)

An uncountable cardinal κ is *weakly compact* if for every function $c : [\kappa]^2 \longrightarrow 2$, there is an unbounded subset H of κ with $|c[H]^2| = 1$.

Definition (Ulam)

An uncountable cardinal κ is *measurable* if there exists a $<\kappa$-complete, non-principal ultrafilter on κ.

The special role of large cardinals arises from two empirical facts:

- First, there is strong evidence that for every extension of $\mathbf{ZFC}$, the consistency of the given theory is either equivalent to the consistency of $\mathbf{ZFC}$, or to the consistency of some extension of $\mathbf{ZFC}$ by large cardinal axioms.
- Second, all large cardinal notions studied so far are linearly ordered by their consistency strength.

In combination, these two phenomena allow for an ordering of all extensions of $\mathbf{ZFC}$ (and therefore of all mathematical statements!) in a linear hierarchy based on their consistency strength.

Despite their central role in modern set theory, large cardinals are still surrounded by many open conceptual questions:

- There is no widely accepted formal definition of the intuitive concept of large cardinals. Instead there are several common ways to formulate such principles, such as elementary embeddings and partition properties, and for many axioms equivalent formulations of different types can be found.
- Moreover, although the linearity of the ordering of mathematical theories by their consistency strength seems to be a fundamental fact of mathematics, it has not been possible to prove the general validity of this principle and, without a formal definition for the concept of large cardinals, it is not even clear how such an argument should look like.

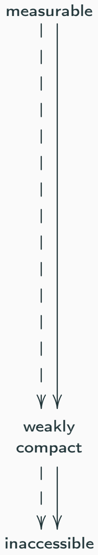
We re-examine our earlier examples in the light of the above discussion.

Lemma

- *If κ is an inaccessible cardinal, then V_κ is a model of $\mathbf{ZFC}$.*
- *Weakly compact cardinals are inaccessible limits of inaccessibles.*
- *Measurable cardinals are weakly compact limits of weakly compacts.*

In the following diagram, we write $\Phi_0 \longrightarrow \Phi_1$ to denote that a large cardinal property Φ_0 implies a large cardinal property Φ_1.

Moreover, we write $\Phi_0 \dashrightarrow \Phi_1$ to denote that, over $\mathbf{ZFC}$, the existence of a cardinal with property Φ_0 has strictly larger consistency strength than the existence of a cardinal with property Φ_1.


measurable
weakly
compact
inaccessible

In order to show that the ordering of large cardinal properties under direct implication is not linear, we discuss two more classical notions:

Definition (Jensen–Kunen)

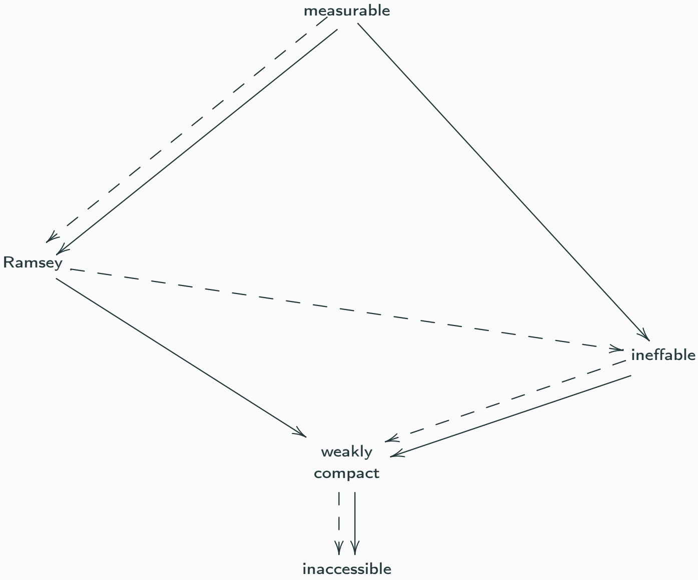
A cardinal κ of uncountable cofinality is *ineffable* if for every function $c : [\kappa]^2 \longrightarrow 2$, there is a stationary subset H of κ with $|c[H]^2| = 1$.

Definition (Erdős–Hajnal)

An infinite cardinal κ is *Ramsey* if for every function $c : [\kappa]^{<\omega} \longrightarrow 2$, there is an unbounded subset H of κ with $|c[H]^n| = 1$ for all $n < \omega$.

Lemma

- *Ineffable cardinals are weakly compact limits of weakly compacts.*
- *Ramsey cardinals are weakly compact limits of ineffables.*
- *Measurable cardinals are ineffable, Ramsey and limits of cardinals that are both ineffable and Ramsey.*
- *Ineffable cardinals are* $\mathbf{\Pi}^1_2$*-indescribable.*
- *The least Ramsey cardinal is* $\mathbf{\Pi}^1_2$*-describable.*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Small models and filters

Starting from measurability upwards, many important large cardinal notions are defined through the existence of certain ultrafilters.

These filters can then be used in ultrapower constructions to produce elementary embeddings $j : \mathrm{V} \longrightarrow M$ of the set-theoretic universe V into some transitive class M with the large cardinal in question as their critical point.

In addition, a great variety of results shows that many prominent large cardinal properties below measurability can be characterized through the existence of filters that only measure sets contained in set-sized models M of set theory.

Definition

Let κ be a cardinal.

- A *weak κ-model* is a model M of $\mathbf{ZFC}^-$ with $|M| = \kappa$ and $\kappa + 1 \subseteq M$.
- A *κ-model* is a transitive weak κ-model M with ${}^{<\kappa}M \subseteq M$.
- If M is a model of $\mathbf{ZFC}^-$ with $\kappa \in M$, then a set $U \subseteq M \cap \mathcal{P}(\kappa)$ is an *M-ultrafilter* on κ if

$$\langle M, \in, U \rangle \models \text{“}U \textit{ is a uniform ultrafilter on } \kappa\text{”}.$$

Classical results now provide examples of characterizations of large cardinals through the existence of ultrafilters for weak κ-models possessing different degrees of amenability and completeness.

Proposition (Folklore)

An uncountable cardinal κ is weakly compact if and only if for every transitive weak κ-model M, there exists an M-ultrafilter U on κ that is $<\kappa$-complete in V.

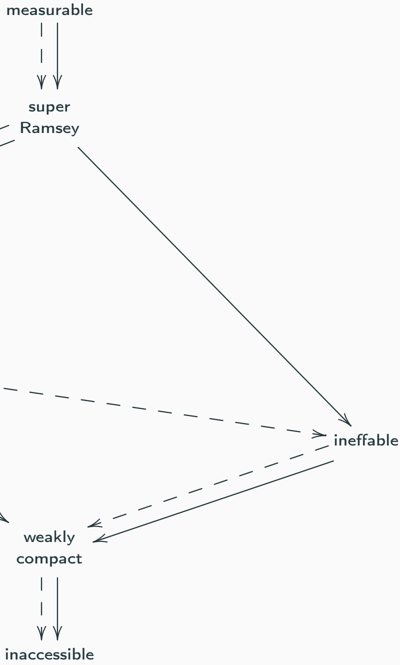
Theorem (Dodd, Mitchell)

An infinite cardinal κ is Ramsey if and only if every subset of κ is contained in a transitive weak κ-model M with the property that there exists an M-ultrafilter U on κ that is M-normal, M-amenable and countably complete.

The above results motivated the definition of new *Ramsey-like* cardinals through the existence of certain ultrafilters for small models.

Definition (Gitman)

- A cardinal κ is *strongly Ramsey* if every subset of κ is contained in a κ-model M with the property that there exists an M-ultrafilter U on κ that is M-normal and M-amenable.
- A cardinal κ is *super Ramsey* if every subset of κ is contained in a κ-model $M \prec \mathrm{H}(\kappa^+)$ with the property that there exists an M-ultrafilter U on κ that is M-normal and M-amenable.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Large cardinal characterizations

The work presented in this talk aims to provide a coherent framework for large cardinal properties up to measurability and their ordering under direct implication and consistency strength.

Our characterization make use of the following schemes that connect large cardinal properties $\Phi(\kappa)$ with properties $\Psi(M,U)$ of set-sized models M of $\mathbf{ZFC}^-$ and M-ultrafilters U.

Scheme A

$\Phi(\kappa)$ holds if and only if for all sufficiently large regular cardinals θ and all $x \in \mathrm{H}(\theta)$, there is a countable model $M \prec \mathrm{H}(\theta)$ with $x \in M$ and an M-ultrafilter U on κ with $\Psi(M, U)$.

Scheme B

$\Phi(\kappa)$ holds if and only if for every $x \subseteq \kappa$, there is a transitive weak κ-model M with $x \in M$ and an M-ultrafilter U on κ with $\Psi(M, U)$.

Scheme C

$\Phi(\kappa)$ if and only if for all sufficiently large regular cardinals θ and all $x \in \mathrm{H}(\theta)$, there is a weak κ-model $M \prec \mathrm{H}(\theta)$ with $x \in M$ and an M-ultrafilter U on κ with $\Psi(M, U)$.

Scheme A

$\Phi(\kappa)$ holds if and only if for all sufficiently large regular cardinals θ and all $x \in \mathrm{H}(\theta)$, there is a countable model $M \prec \mathrm{H}(\theta)$ with $x \in M$ and an M-ultrafilter U on κ with $\Psi(M, U)$.

Scheme C

$\Phi(\kappa)$ if and only if for all sufficiently large regular cardinals θ and all $x \in \mathrm{H}(\theta)$, there is a weak κ-model $M \prec \mathrm{H}(\theta)$ with $x \in M$ and an M-ultrafilter U on κ with $\Psi(M, U)$.

Trivial examples of instances of the Schemes A and C are given by

$$\Phi_{ms}(\kappa) \equiv \text{“}\kappa \textit{ is measurable”}$$

and

$$\Psi_{ms}(M, U) \equiv \text{“}U \textit{ is } M\textit{-normal and } U = F \cap M \textit{ for some } F \in M\text{”}.$$

Scheme B

$\Phi(\kappa)$ holds if and only if for every $x \subseteq \kappa$, there is a transitive weak κ-model M with $x \in M$ and an M-ultrafilter U on κ with $\Psi(M, U)$.

It can be shown that there is no reasonable characterization of measurability through Scheme B. In order to obtain a trivial example for a valid instance of Scheme B, we make the following definition:

Definition

An uncountable cardinal κ is *locally measurable* if and only if for many transitive weak κ-models M there exists an M-normal M-ultrafilter U on κ with $U \in M$.

Scheme B

$\Phi(\kappa)$ holds if and only if for every $x \subseteq \kappa$, there is a transitive weak κ-model M with $x \in M$ and an M-ultrafilter U on κ with $\Psi(M, U)$.

By earlier results and definitions, we obtain the following instances of Scheme B:

- $\Phi_R(\kappa) \equiv$ "κ *is Ramsey*" and

  $$\Psi_R(M, U) \equiv \text{“}U \textit{ is } M\textit{-amenable, } M\textit{-normal and countably complete”}.$$

- $\Phi_{stR}(\kappa) \equiv$ "κ *is strongly Ramsey*" and

  $$\Psi_{stR}(M, U) \equiv \text{“}M \textit{ is a } \kappa\textit{-model, } U \textit{ is } M\textit{-amenable and } M\textit{-normal”}.$$

- $\Phi_{suR}(\kappa) \equiv$ "κ *is super Ramsey*" and

  $$\Psi_{suR}(M, U) \equiv \text{“}M \prec \mathrm{H}(\kappa^+) \textit{ is a } \kappa\textit{-model, } U \textit{ is } M\textit{-amenable and } M\textit{-normal”}.$$

It turns out that all of the large cardinal notions discussed earlier can be characterized through one of the above schemes.

Scheme A

$\Phi(\kappa)$ holds if and only if for all sufficiently large regular cardinals θ and all $x \in \mathrm{H}(\theta)$, there is a countable model $M \prec \mathrm{H}(\theta)$ with $x \in M$ and an M-ultrafilter U on κ with $\Psi(M, U)$.

Theorem

Scheme A holds true for $\Phi_{ia}(\kappa) \equiv$ "κ is inaccessible" and

$$\Psi_{ia}(M, U) \equiv \text{"}U \textit{ is } {<}\kappa\textit{-amenable and } {<}\kappa\textit{-complete for } M\text{"}.$$

Scheme B

$\Phi(\kappa)$ holds if and only if for every $x \subseteq \kappa$, there is a transitive weak κ-model M with $x \in M$ and an M-ultrafilter U on κ with $\Psi(M, U)$.

Theorem

Scheme B holds true in the following cases:

- $\Phi_{wc}(\kappa) \equiv$ *"κ is weakly compact" and*

$$\Psi_{wc}(M, U) \equiv \text{“}U \text{ is } {<}\kappa\text{-amenable for } M \text{ and } M\text{-normal”}.$$

- $\Phi_{ie}(\kappa) \equiv$ *"κ is ineffable" and*

$$\Psi_{ie}(M, U) \equiv \text{“}U \text{ is normal”}.$$

- $\Phi_{iR}(\kappa) \equiv$ *"κ is ineffably Ramsey" and*

$$\Psi_{iR}(M, U) \equiv \text{“}U \text{ is } M\text{-amenable, } M\text{-normal and stationary-complete”}.$$

Scheme C

$\Phi(\kappa)$ if and only if for all sufficiently large regular cardinals θ and all $x \in \mathrm{H}(\theta)$, there is a weak κ-model $M \prec \mathrm{H}(\theta)$ with $x \in M$ and an M-ultrafilter U on κ with $\Psi(M, U)$.

Theorem

Scheme C holds true for $\Phi_{cie}(\kappa) \equiv$ "κ is completely ineffable" and

$$\Psi_{cie}(M, U) \equiv \text{“}U \text{ is } \kappa\text{-amenable for } M \text{ and } M\text{-normal”}.$$

Induced ideals

The above characterizations now allow us to canonically assign ideals to the given large cardinal properties.

The idea behind the definition of these ideals is to consider the collection of all subsets of the given cardinal that are never contained in the ultrafilters witnessing the corresponding large cardinal property.

It then turns out that, in the cases where ideals corresponding to large cardinal properties were already defined, these assignments coincide with the known notions.

In the other cases, these assignments provide the first examples of ideals canonically induced by the given large cardinal property.

Definition

Let $\Psi(M, U)$ be a property of models M and filters U, and let κ be an uncountable cardinal.

- $\mathrm{I}_{\Psi}^{<\kappa}$ is the collection of all $A \subseteq \kappa$ with the property that for all sufficiently large regular cardinals θ, there exists $x \in \mathrm{H}(\theta)$ such that $A \notin U$ holds for all countable $M \prec \mathrm{H}(\theta)$ with $x \in M$ and all M-ultrafilters U on κ with $\Psi(M, U)$.
- $\mathrm{I}_{\Psi}^{\kappa}$ is the collection of all $A \subseteq \kappa$ with the property that there exists $x \subseteq \kappa$ such that $A \notin U$ holds for all transitive weak κ-models M with $x \in M$ and all M-ultrafilters U on κ with $\Psi(M, U)$.
- $\mathrm{I}_{\prec\Psi}^{\kappa}$ is the collection of all $A \subseteq \kappa$ with the property that for all sufficiently large regular cardinals θ, there exists $x \in \mathrm{H}(\theta)$ such that $A \notin U$ holds for all weak κ-models $M \prec \mathrm{H}(\theta)$ with $x \in M$ and all M-ultrafilters U on κ with $\Psi(M, U)$.

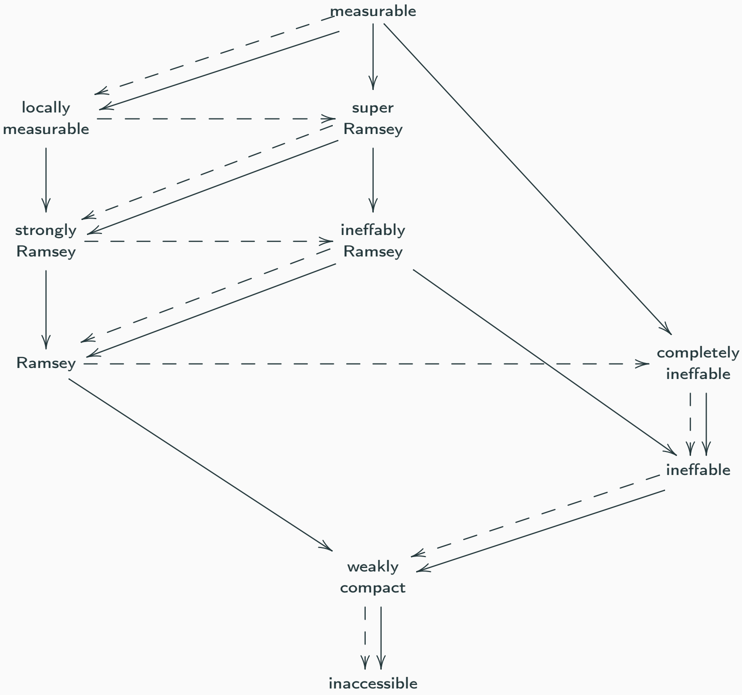
Theorem

- *If* κ *is inaccessible, then* $\mathrm{I}_{ia}^{<\kappa}$ *is the bounded ideal on* κ.
- *If* κ *is a weakly compact cardinal, then* I_{wc}^{κ} *is the weakly compact ideal on* κ.
- *If* κ *is an ineffable cardinal, then* I_{ie}^{κ} *is the ineffable ideal on* κ.
- *If* κ *is a completely ineffable cardinal, then* $\mathrm{I}_{\prec cie}^{\kappa}$ *is the completely ineffable ideal on* κ.
- *If* κ *is a Ramsey cardinal, then* I_{R}^{κ} *is the Ramsey ideal on* κ.
- *If* κ *is an ineffably Ramsey cardinal, then* I_{iR}^{κ} *is the ineffably Ramsey ideal on* κ.
- *If* κ *is a measurable cardinal, then* $\mathrm{I}_{\prec ms}^{\kappa}$ *is the intersection of all complements of normal ultrafilters on* κ.

We now want to use the ideals defined above to study the ordering of large cardinal notions under direct implication and consistency strength.

The following diagram summarizes the ordering of the large cardinal notions discussed above.

Remember that we write $\Phi_0 \longrightarrow \Phi_1$ to denote that a large cardinal property Φ_0 implies a large cardinal property Φ_1, and we write $\Phi_0 \dashrightarrow \Phi_1$ to denote that, over $\mathbf{ZFC}$, the existence of a cardinal with property Φ_0 has strictly larger consistency strength than the existence of a cardinal with property Φ_1.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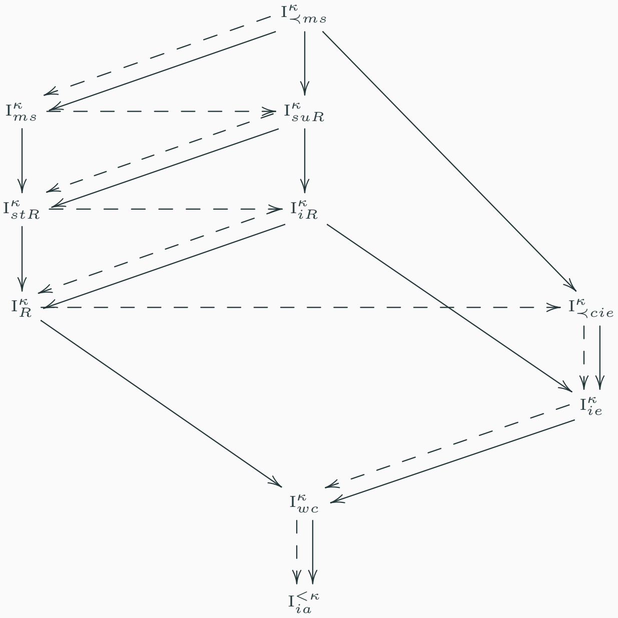
In the next diagram, we compare the provable relations between the ideals introduced above.

In the diagram, we use a solid arrow $I_0 \longrightarrow I_1$ to denote a provable inclusion $I_1 \subseteq I_0$ of large cardinal ideals.

Moreover, if I_1 is an ideal induced by a large cardinal property Φ, then a dashed arrow $I_0 \dashrightarrow I_1$ represents the statement that

$$\{\alpha < \kappa \mid \neg\Phi(\alpha)\} \in I_0$$

provably holds.


$I^\kappa_{\prec ms}$
I^κ_{ms}
I^κ_{suR}
I^κ_{stR}
I^κ_{iR}
I^κ_{R}
$I^\kappa_{\prec cie}$
I^κ_{ie}
I^κ_{wc}
$I^{<\kappa}_{ia}$

Thank you for listening!