SPDEs with fractional noise in space with index $H < 1/2$

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Outline

- Stochastic wave and heat equations with spatially homogeneous noise
- Motivation, objective and strategy
- Stochastic integration
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Stochastic heat and wave equations

We consider the stochastic wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \sigma(u(t,x))\dot{X}(t,x), \quad t \in [0,T],\ x \in \mathbb{R}^d \\ u(0,x) = u_0(x), \qquad \dfrac{\partial u}{\partial t}(0,x) = v_0(x), \end{cases} \tag{SWE}$$

and the stochastic heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) + \sigma(u(t,x))\dot{X}(t,x), \quad t \in [0,T],\ x \in \mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{SHE}$$

- Δ denotes the Laplacian operator on $\mathbb{R}^d$.
- $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function,
- $u_0, v_0 : \mathbb{R} \to \mathbb{R}$ are bounded and Hölder continuous,
- $\dot{X}(t,x)$ is a spatially homogeneous Gaussian noise.

Spatially homogeneous Gaussian noise

On some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $X = \{X(\varphi),\ \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)\}$ be a zero-mean Gaussian process with

$$\mathbb{E}[X(\varphi)X(\psi)] = \int_0^\infty \Gamma\big(\varphi(t,\cdot) * \tilde{\psi}(t,\cdot)\big)\, dt$$

- $\mathcal{D}(\mathcal{O})$: functions in $\mathcal{C}^\infty(\mathcal{O})$ with compact support.
- Γ is a non-negative-definite tempered distribution on $\mathbb{R}^d$.
- $\tilde{\psi}(t,x) = \psi(t,-x)$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$.
- The Lebesgue integral in t means that the process is *white* in time.
- There exists a tempered measure μ on $\mathbb{R}^d$ such that $\mathcal{F}\mu = \Gamma$ in the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions on $\mathbb{R}^d$.

$$\mathbb{E}[X(\varphi)X(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(t,\cdot)(\xi)}\, \mu(d\xi) dt$$

The spectral measure μ satisfies

$$\int_{\mathbb{R}^d} \left(\prod_{j=1}^{d} \frac{1}{1+\xi_j^2} \right) \mu(d\xi) < \infty \qquad \left[\Longleftarrow \int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} \, \mu(d\xi) < \infty \right]$$

In order to solve SPDEs, one aims to construct stochastic integrals with respect to X.

Remark: the process $X = \{X(\varphi),\, \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)\}$ defines a stationary random distribution (Itô 1954, Yaglom 1957). That is,

$$\mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d) \ni \varphi \longmapsto X(\varphi) \in L^2(\Omega)$$

is linear and continuous, and the covariance is invariant under translations:

$$\mathbb{E}[X(\tau_h \varphi) X(\tau_h \psi)] = \mathbb{E}[X(\varphi) X(\psi)] \quad \text{for any} \quad h \in \mathbb{R}_+ \times \mathbb{R}^d.$$

Mild solutions

Fix $T > 0$. A random field $\{u(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^d\}$ is a solution of (SWE) (resp. (SHE)), if it is predictable and, for any $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\,\sigma(u(s,y))\,X(ds,dy) \quad \text{a.s.}$$

where $G_t(x)$ denotes the corresponding fundamental solution: e.g., for $d = 1$ we have

$$G_t(x) = \frac{1}{2} 1_{\{|x|<t\}} \quad \text{(wave)}, \qquad G_t(x) = \frac{1}{(2\pi t)^{1/2}} \exp\left(-\frac{|x|^2}{2t}\right) \quad \text{(heat)}$$

and $w(t,x)$ is the contribution of the initial data:

$$w(t,x) = \frac{1}{2}\int_{x-t}^{x+t} v_0(y)dy + \frac{1}{2}\Big(u_0(x+t) + u_0(x-t)\Big) \quad \text{(wave)},$$

$$w(t,x) = \int_{\mathbb{R}} G_t(x-y)u_0(y)dy \quad \text{(heat)}$$

Motivation and objective

Recall: for any $\varphi, \psi \in \mathcal{D}\big(\mathbb{R}_+ \times \mathbb{R}^d\big)$,

$$\mathbb{E}[X(\varphi)X(\psi)] = \int_0^\infty \Gamma\big(\varphi(t,\cdot) * \tilde{\psi}(t,\cdot)\big)\, dt = \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(t,\cdot)(\xi)}\,\mu(d\xi)dt$$

Most results in the literature assume the following:

(A) Γ is a non-negative-definite tempered measure (or in particular, Γ is a non-negative locally integrable function f).

In this case,

$$\begin{aligned}\mathbb{E}[X(\varphi)X(\psi)] &= \int_0^\infty \int_{\mathbb{R}^d} \big(\varphi(t,\cdot) * \tilde{\psi}(t,\cdot)\big)(x)\,\Gamma(dx)dt \\ &= \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(t,x)f(x-y)\psi(t,y)\,dydxdt.\end{aligned}$$

Under assumptions **(A)** and

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}\,\mu(d\xi) < \infty, \tag{1}$$

Dalang 1999 (also Dalang and Q-S 2011) proved existence of a unique solution to a general class of SPDEs in $\mathbb{R}^d$ including:

(i) (SWE) with $d \in \{1,2,3\}$,

(ii) (SHE) for any $d \geq 1$.

On the other hand, Peszat and Zabczyk 2007 obtained existence and uniqueness of a function-space valued solution to (i) and (ii) under condition (1) and

(B) There exists a constant $C > 0$ such that $\Gamma + C\lambda_d$ is a non-negative measure, where λ_d is the Lebesgue measure on $\mathbb{R}^d$.

As far as (SWE) in any $d \geq 3$ is concerned:

- Peszat 2002 (function-space valued solution): assumption **(B)** and

$$\sup_{\eta \in \mathbb{R}} \int_{\mathbb{R}^d} \frac{1}{1 + |\xi - \eta|^2} \mu(d\xi) < \infty. \tag{2}$$

  He proved that, under **(B)**, (2) is equivalent to (1).

- Dalang and Mueller 2003 (*hybrid* approach): assumption **(A)** and condition (1).
- Conus and Dalang 2008 (random field solution): assumption **(A)** and condition (2).

In all these results, the involved stochastic integral can be interpreted as a stochastic integral with respect to a martingale measure (or cylindrical Wiener process): e.g. Walsh 1986, Da Prato and Zabczyk 1992, Dalang 1999, Dalang and Q-S 2011.

From now on, assume $d = 1$, and consider the following important example:

- Assume that the space correlation behaves like a fractional Brownian motion with $H \in (0,1)$.
- This corresponds to take a spectral measure μ of the form

$$\mu(d\xi) = c_H |\xi|^{1-2H} d\xi, \qquad \text{with} \quad c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}$$

- The measure μ satisfies (1) for all $H \in (0,1)$.
- But condition (2) does not hold for $H < 1/2$.
- In fact, if $H > 1/2$, $\Gamma = \mathcal{F}\mu$ is the locally integrable function $f(x) = H(2H-1)|x|^{2H-2}$, which satisfies **(A)**.
- But if $H < 1/2$, $\Gamma = \mathcal{F}\mu$ is a genuine distribution (Jolis 2010):

$$\Gamma(\varphi) = H(2H-1) \int_{\mathbb{R}} \big(\varphi(x) - \varphi(0)\big) |x|^{2H-2} dx, \quad \varphi \in \mathcal{D}(\mathbb{R})$$

Objective: consider the stochastic wave and heat equations

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + \sigma(u(t,x))\dot{X}(t,x), & t \in [0,T],\ x \in \mathbb{R} \\ u(0,x) = u_0(x), \qquad \dfrac{\partial u}{\partial t}(0,x) = v_0(x), \end{cases} \qquad \text{(SWE)}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + \sigma(u(t,x))\dot{X}(t,x), & t \in [0,T],\ x \in \mathbb{R} \\ u(0,x) = u_0(x) \end{cases} \qquad \text{(SHE)}$$

where we assume that

- $\sigma(z) = az + b$ is an affine function,
- $\dot{X}(t,x)$ is a spatially homogeneous Gaussian noise with spectral measure $\mu(d\xi) = c_H|\xi|^{1-2H} d\xi$ with $H \in (\frac{1}{4}, \frac{1}{2})$.
- $u_0, v_0 : \mathbb{R} \to \mathbb{R}$ are bounded and H-Hölder continuous,

Under the above hypotheses, we aim to prove the following. Assume that $H \in (\frac{1}{4}, \frac{1}{2})$.

Theorem

Equation (SWE) (respectively (SHE)) has a unique solution $u = \{u(t,x), (t,x) \in [0,T] \times \mathbb{R}\}$, *which is* $L^2(\Omega)$*-continuous and satisfies, for any* $p \geq 2$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|u(t,x)|^p\right] < \infty$$

and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \frac{\left(\mathbb{E}\left[|u(s,y)-u(s,z)|^p\right]\right)^{2/p}}{|y-z|^{2-2H}} dydzds < \infty.$$

The latter condition appears in a *natural* way, as we apply techniques from the theory of fractional Sobolev spaces.

Strategy

In order to attain our objective, we have developed the following steps:

1. Properly *interpret* the stochastic integral with respect to our spatially homogeneous noise:

$$\int_0^t \int_{\mathbb{R}} S(s,y) X(ds,dy) \qquad \text{(Basse-O'Connor } et\ al. \text{ 2012)}$$

2. Obtain a new criterion for integrability, based on tools from the theory of fractional Sobolev spaces (Di Nezza *et al.* 2012).

3. Set a Picard iteration scheme, show that it is well-defined and converges, in a convenient topology, to a process which solves our SPDEs.

Related results

Our main result covers the cases

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\,\dot{X}(t,x), & x \in \mathbb{R} \\ u(0,x) = c, \qquad \dfrac{\partial u}{\partial t}(0,x) = 0, & \end{cases}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\,\dot{X}(t,x), & x \in \mathbb{R} \\ u(0,x) = c & \end{cases}$$

- These are the Hyperbolic Anderson Model and Parabolic Anderson Model, resp.
- Study of the series of multiple stochastic integrals with respect to X.
- This method has been applied in
  - Heat equations: Hu 2001, Hu and Nualart 2009, Balan and Tudor 2010, Hu *et al.* 2011.
  - Wave equations: Dalang *et al.* 2008, Dalang and Mueller 2009, Balan 2012.

Stochastic integral

Wiener integral: Let $\mathcal{H}$ be the completion of $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ with respect to

$$\langle \cdot, \cdot \rangle_{\mathcal{H}} := \mathbb{E}[X(\varphi)X(\psi)] = c_H \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(t,\cdot)(\xi)}\, |\xi|^{1-2H} d\xi dt$$

Then, $\varphi \mapsto X(\varphi) \in L^2(\Omega)$ is an isometry which can be extended to $\mathcal{H}$:

$$X(h) = \int_0^\infty \int_{\mathbb{R}} h(t,x) X(dt,dx), \quad h \in \mathcal{H}.$$

For $t \geq 0$ any interval $(x,y] \subset \mathbb{R}$, one proves that $1_{(0,t]\times(x,y]} \in \mathcal{H}$, so we can define the random variable

$$X_t\big((x,y]\big) := X\big(1_{(0,t]\times(x,y]}\big)$$

Problem: we cannot define $X_t(A)$ for all $A \in \mathcal{B}_b(\mathbb{R})$, since in general the function $1_{(0,t]\times A}$ may not be in $\mathcal{H}$ ($H < 1/2$).

But recall that our noise $X = \{X(\varphi),\ \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})\}$ can be viewed as a stationary random distribution (Itô 1954, Yaglom 1957).

Hence, X admits a suitable spectral representation which can be applied to show that

$$X_t((x,y]) := X\left(1_{(0,t]\times(x,y]}\right) = \int_{\mathbb{R}} \mathcal{F}1_{(x,y]}(\xi) M_t(d\xi),$$

where $\{M_t(A), \mathcal{F}_t,\ t \geq 0,\ A \in \mathcal{B}_b(\mathbb{R})\}$ is a (complex valued) martingale measure with zero mean and covariation

$$\langle M(A), M(B)\rangle_t = t\,\mu(A\cap B) = t\,c_H \int_{A\cap B} |\xi|^{1-2H} d\xi, \qquad A, B \in \mathcal{B}_b(\mathbb{R}).$$

$(\mathcal{F}_t)_{t\geq 0}$ denotes the filtration generated by X:

$$\mathcal{F}_t = \sigma\left\{X\left(1_{[0,s]}\phi\right),\ s \in [0,t],\ \phi \in \mathcal{D}(\mathbb{R})\right\}$$

Sketch of the construction of the stochastic integral:

1. $\mathcal{E}$: linear combinations of processes of the form $g(\omega, t, x) = Y(\omega)1_{(a,b]}(t)1_{(c,d]}(x)$. Define

$$\int_0^t \int_{\mathbb{R}} g(s,y)X(ds,dy) := Y(X_{t\wedge b}((u,v]) - X_{t\wedge a}((u,v]))$$

   and extend to $\mathcal{E}$ by linearity.

2. For any $g \in \mathcal{E}$, it holds

$$\int_0^t \int_{\mathbb{R}} g(s,y)X(ds,dy) = \int_0^t \int_{\mathbb{R}} \mathcal{F}g(s,\cdot)(\xi)M(ds,d\xi)$$

3. Let $\mathcal{P}_0$ be the completion of $\mathcal{E}$ with respect to

$$\|g\|_0^2 = \mathbb{E}\int_0^T \int_{\mathbb{R}} |\mathcal{F}g(t,\cdot)(\xi)|^2 c_H|\xi|^{1-2H} d\xi dt.$$

4. By the isometry property of Walsh's stochastic integral, the map $\mathcal{E} \ni g \mapsto \{\int_0^t \int_{\mathbb{R}} g(s,y)X(ds,dy)\}_{t\in[0,T]} \in \mathcal{M}$ is an isometry, where $\mathcal{M}$ is a subspace of the space of continuous square-integrable martingales with $\|N\| = \{E(N_T^2)\}^{1/2}$. This map can be extended to $\mathcal{P}_0$.

Identification of integrands:

Theorem (Basse-O'Connor *et al.* 2012)

The elements of $\mathcal{P}_0$ are predictable functions of the form

$$S : \Omega \times [0, T] \to \mathcal{S}'(\mathbb{R})$$

such that $\mathcal{F}S(\omega, t, \cdot)$ is a locally integrable function for any (ω, t) and

$$\mathbb{E} \int_0^T \int_{\mathbb{R}} |\mathcal{F}S(t,\cdot)(\xi)|^2 \, c_H |\xi|^{1-2H} d\xi dt < \infty.$$

In particular, we have the isometry

$$\mathbb{E} \left| \int_0^t \int_{\mathbb{R}} S(s,x) X(ds, dx) \right|^2 = \mathbb{E} \int_0^t \int_{\mathbb{R}} |\mathcal{F}S(s,\cdot)(\xi)|^2 \, c_H |\xi|^{1-2H} d\xi ds,$$

Remark: all that we have done is valid for any $H \in (0, 1)$.

Criterion for integrability:

A measurable function $g : \mathbb{R} \to \mathbb{R}$ is *tempered* if there exists a tempered distribution $T_g \in \mathcal{S}'(\mathbb{R})$ such that $T_g\varphi = \int_{\mathbb{R}} g(x)\varphi(x)dx$, for all $\varphi \in \mathcal{S}(\mathbb{R})$.

Theorem

Let $S : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be a predictable function, such that

(a) *for almost all $(\omega, t) \in \Omega \times [0, T]$, $S(\omega, t, \cdot)$ is a tempered function,*

(b) *the Fourier transform $\mathcal{F}S(\omega, t, \cdot)$ in $\mathcal{S}'(\mathbb{R})$ is a locally integrable function.*

If

$$I(T) := C_H \mathbb{E} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|S(t,x) - S(t,y)|^2}{|x-y|^{2-2H}} dxdydt < \infty,$$

then $S \in \mathcal{P}_0$ and

$$\mathbb{E} \left| \int_0^T \int_{\mathbb{R}} S(s,x) X(ds,dx) \right|^2 = I(T).$$

The proof of the above criterion is based on the following result, related to the theory of fractional Sobolev spaces (Di Nezza *et al.* 2012):

Proposition

Let $g : \mathbb{R} \to \mathbb{R}$ be a tempered function whose Fourier transform in $\mathcal{S}'(\mathbb{R})$ is a locally integrable function. For any $0 < H < 1/2$,

$$c_H \int_{\mathbb{R}} |\mathcal{F}g(\xi)|^2 |\xi|^{1-2H} d\xi = C_H \int_{\mathbb{R}^2} \frac{|g(x) - g(y)|^2}{|x - y|^{2H-2}} \, dxdy,$$

when either one of the two integrals above is finite.

Picard iteration scheme

For any $(t,x) \in [0,T] \times \mathbb{R}$, set $u^0(t,x) = w(t,x)$ and, for $n \geq 0$,

$$u^{n+1}(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(u^n(s,y))X(ds,dy)$$

Theorem

Let $p \geq 2$ and σ be Lipschitz. Then, $u^n(t,x)$ is well-defined and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|u^n(t,x)|^p\right] < \infty,$$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \frac{\left(\mathbb{E}\left[|u^n(s,y) - u^n(s,z)|^p\right]\right)^{2/p}}{|y-z|^{2-2H}} dydzds < \infty$$

and, for any $h \in \mathbb{R}$ with $|h| < 1$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|u^n(t,x+h) - u^n(t,x)|^2\right] \leq C_n |h|^{2H}$$
$$\sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} \mathbb{E}\left[|u^n(t+h,x) - u^n(t,x)|^2\right] \leq C_n |h|^{\beta},$$

where $\beta = 2H$ for the wave equation, and $\beta = H$ for the heat equation.

Remarks on the proof

We start with $n = 0$. Recall that, for the wave equation,

$$w(t,x) = \int_{\mathbb{R}} G_t(x-y)v_0(y)dy + \frac{\partial}{\partial t}\left(\int_{\mathbb{R}} G_t(x-y)u_0(y)dy\right)$$
$$= \frac{1}{2}\int_{x-t}^{x+t} v_0(y)dy + \frac{1}{2}\big(u_0(x+t) + u_0(x-t)\big),$$

$$G_t(x) = \frac{1}{2}1_{\{|x|<t\}}$$

and, for the heat equation,

$$w(t,x) = \int_{\mathbb{R}} G_t(x-y)u_0(y)dy$$

$$G_t(x) = \frac{1}{(2\pi t)^{1/2}}\exp\left(-\frac{|x|^2}{2t}\right)$$

Using the explicit expression of $G_t(x)$ and that u_0, v_0 are bounded and H-Hölder continuous, one proves

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |w(t,x)| < \infty \qquad \begin{array}{c} \sup_{(t,x)\in[0,T]\times\mathbb{R}} |w(t,x+h) - w(t,x)|^2 \leq C|h|^{2H} \\ \sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} |w(t+h,x) - w(t,x)|^2 \leq C|h|^{\beta} \end{array}$$

It remains to study the expression

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \frac{|w(s,y) - w(s,z)|^2}{|y-z|^{2-2H}}\, dydzds \\ &= \int_0^t \int_{\mathbb{R}} G_{t-s}^2(x-y) \left(\int_{\mathbb{R}} \frac{|w(s,y+z) - w(s,y)|^2}{|z|^{2-2H}} dz \right) dyds \end{aligned}$$

Decomposing the domain of the dz integral, the latter term is bounded by

$$\int_0^t \int_{\mathbb{R}} G_{t-s}^2(x-y) \left(\int_{|z|\leq 1} |z|^{4H-2} dz + \int_{|z|>1} |z|^{2H-2} dz \right) dyds,$$

which is uniformly bounded thanks to condition $H \in (\frac{1}{4}, \frac{1}{2})$.

In order to show that $u^{n+1}(t,x)$ is well-defined, one needs to prove that the following stochastic integral is well-defined:

$$\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(u^n(s,y))X(ds,dy)$$

Hence, setting

$$S_n(s,y) := G_{t-s}(x-y)\sigma(u^n(s,y))1_{[0,t]}(s),$$

one proves

(i) u^n has a predictable modification,

(ii) $S_n(\omega,s,\cdot) \in L^1(\mathbb{R})$ for almost all $(\omega,s) \in \Omega \times [0,T]$,

(iii) S_n satisfies

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|S_n(s,y)-S_n(s,z)|^2}{|y-z|^{2H-2}} \, dydzds < \infty.$$

These conditions, together with all remaining estimates for u^{n+1} in the induction hypothesis, can be proved using the following type of techniques:

- σ Lipschitz, Burkholder-Davis-Gundy inequality, Jensen inequality, Minkowski inequality (integrals), Fubini theorem, Plancherel theorem, and many changes of variables.
- For all $\alpha \in (-1, 1)$,

$$\int_0^T \int_{\mathbb{R}} |\mathcal{F}G_t(\xi)|^2 \, |\xi|^\alpha \, d\xi dt = \begin{cases} C_1 \, T^{2-\alpha} & \text{wave} \\ C_2 \, T^{(1-\alpha)/2} & \text{heat} \end{cases}$$

- For any $\alpha \in (-1, 1)$ and $h \in \mathbb{R}$,

$$\int_0^T \int_{\mathbb{R}} |\mathcal{F}G_{t+h}(y) - \mathcal{F}G_t(y)|^2 \, |\xi|^\alpha \, d\xi dt \leq \begin{cases} CT|h|^{1-\alpha} & \text{wave} \\ C|h|^{(1-\alpha)/2} & \text{heat} \end{cases}$$

- For any $\alpha \in (-1, 1)$ and $h \in \mathbb{R}$,

$$\int_0^T \int_{\mathbb{R}} (1 - \cos(\xi h)) \, |\mathcal{F}G_t(\xi)|^2 \, |\xi|^\alpha \, d\xi dt \leq \begin{cases} CT|h|^{1-\alpha} & \text{wave} \\ C|h|^{1-\alpha} & \text{heat,} \end{cases}$$

In fact, in order to show that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \frac{\left(\mathbb{E}\left[|u^{n+1}(s,y) - u^{n+1}(s,z)|^p\right]\right)^{2/p}}{|y-z|^{2-2H}} dydzds < \infty,$$

we are forced to estimate the term

$$\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{G_{t-s}^2(x-y)}{|z|^{2-2H}} \left(\int_0^s \int_{\mathbb{R}} |1 - e^{-i\xi z}|^2 \, |\mathcal{F}G_{s-r}(\xi)|^2 \, |\xi|^{1-2H} d\xi dr \right) dzdyds$$
$$\leq \left(\int_0^t \int_{\mathbb{R}} G_{t-s}^2(x-y) dyds \right) \left(\int_0^T \int_{\mathbb{R}} |\mathcal{F}G_{s-r}(\xi)|^2 \, |\xi|^{2(1-2H)} d\xi dr \right).$$

The latter integral is finite if and only if

$$-1 < 2(1-2H) < 1 \qquad \Longleftrightarrow \qquad \frac{1}{4} < H < \frac{3}{4}$$

Convergence of Picard iterations

Now we aim to prove that the sequence $\{u^n(t,x),\ n \geq 0\}$ converges in $L^p(\Omega)$.

Here we assume that

$$|\sigma(x) - \sigma(y) - \sigma(u) + \sigma(v)| \leq C|x - y - u + v| \qquad \Longleftrightarrow \qquad \sigma \text{ affine}$$

In fact, we prove convergence in the Banach space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$: space of $L^2(\Omega)$-continuous and adapted processes $Y = \{Y(t,x),\ (t,x) \in [0,T] \times \mathbb{R}\}$ such that $\|Y\|_{\mathcal{X}_1} < \infty$ and $\|Y\|_{\mathcal{X}_2} < \infty$, where

$$\|Y\|_{\mathcal{X}_1} = \sup_{(t,x)\in[0,T]\times\mathbb{R}} \left(\mathbb{E}\left[|Y(t,x)|^p\right]\right)^{1/p}$$

and

$$\|Y\|_{\mathcal{X}_2} = \sup_{(t,x)\in[0,T]\times\mathbb{R}} \left(\int_0^t \int_{\mathbb{R}^2} G^2_{t-s}(x-y) \frac{\left(\mathbb{E}\left[|Y(s,y) - Y(s,z)|^p\right]\right)^{2/p}}{|y-z|^{2-2H}} dydzds\right)^{1/2}$$

For any $Y \in \mathcal{X}$, we define $\|Y\|_{\mathcal{X}} := \|Y\|_{\mathcal{X}_1} + \|Y\|_{\mathcal{X}_2}$.

Theorem

The sequence $(u^n)_{n\geq 0}$ *converges in* $\mathcal{X}$ *to a process u, which is* $L^2(\Omega)$*-continuous, and is the unique solution to equation (SWE) (or (SHE)).*

Proof: We have

$$M_{n+1}(t) \leq \int_0^t \big(M_n(s) + M_{n-1}(s)\big) J(t-s)ds$$

where, setting $m_n := u^n - u^{n-1}$,

$$M_n(t) = \sup_{x\in\mathbb{R}} \Big(\mathbb{E}\big[|m_n(t,x)|^p\big]\Big)^{2/p}$$
$$+ \sup_{x\in\mathbb{R}} \int_0^t \int_{\mathbb{R}^2} G^2_{t-s}(x-y) \frac{\Big(\mathbb{E}\big[|m_n(s,y) - m_n(s,z)|^p\big]\Big)^{2/p}}{|y-z|^{2-2H}} dydzds,$$

$$J(t-s) = \int_{\mathbb{R}} |\mathcal{F}G_{t-s}(\xi)|^2 \, |\xi|^{1-2H} \, d\xi + \int_s^t \int_{\mathbb{R}} G^2_{t-r}(z) \int_{\mathbb{R}} |\mathcal{F}G_{r-s}(\xi)|^2 |\xi|^{2(1-2H)} \, d\xi dzdr$$

In fact, we have showed that

$$\int_s^t \int_{\mathbb{R}} G_{t-r}^2(z) \int_{\mathbb{R}} |\mathcal{F}G_{r-s}(\xi)|^2 |\xi|^{2(1-2H)}\, d\xi dz dr = \begin{cases} C_1\,(t-s)^{4H-1} & \text{wave} \\ C_2\,(t-s)^{2H-1} & \text{heat} \end{cases}$$

We have proved a version of Dalang's Gronwall lemma in order to treat situations of the form

$$f_n(t) \le \int_0^t \big(f_{n-1}(s) + f_{n-2}(s)\big) g(t-s)\, ds$$

Once we know that there exists $u = \lim_n u^n$ in $\mathcal{X}$, we take limits in

$$u^{n+1}(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(u^n(s,y))X(ds,dy)$$

to deduce that $u = \{u(t,x),\ (t,x) \in [0,T] \times \mathbb{R}\}$ solves (SWE) (resp. (SHE)).

Uniqueness has been proved using similar arguments.

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