Contributions to Stochastic Integration and Stochastic Partial Differential Equations

PhD Defense

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Structure of the thesis

This thesis consists of the following five articles

1. The stochastic wave equation in high dimensions: Malliavin differentiability and absolute continuity, [Electronic Journal of Probability]

2. Absolute continuity for SPDEs with an irregular fundamental solution, [Preprint]

3. Logarithmic asymptotics of the densities of SPDEs driven by spatially correlated noise, [Stochastic Analysis and Applications 2014, Springer]

4. Random-field solutions to linear hyperbolic stochastic partial differential equations with variable coefficients, [Submitted]

5. Integration Theory for infinite dimensional volatility modulated Volterra processes, [Bernoulli]

The first four belong to the field of SPDEs and the last one to the area of **stochastic integration theory**. In the first three articles, we investigate the **probability law of solutions to SPDEs**, and in the forth article, we have a result on **existence and uniqueness of solution**.

Part I: SPDEs - Study of the Probability Law

1. The stochastic wave equation in high dimensions: Malliavin differentiability and absolute continuity
2. Absolute continuity for SPDEs with an irregular fundamental solution
3. Logarithmic asymptotics of the densities of SPDEs driven by spatially correlated noise

SPDEs - Formal definition

The central objects are **SPDEs**, given by

$$Lu(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{F}(t,x).$$

In this **formal** equation

- L is a PDO with constant coefficients, in particular the **wave operator**

$$L = \frac{\partial^2}{\partial t^2} - \Delta_d,$$

- b, σ are real Lipschitz-continuous functions, and
- F is a random Gaussian noise given by the isonormal Wiener process on the Hilbert space $\mathcal{H}_T = L^2([0,T]; \mathcal{H})$, where

$$\mathcal{H} = \overline{(\mathcal{S}, \langle \cdot, \cdot \rangle_{\mathcal{H}})}^{\langle \cdot, \cdot \rangle_{\mathcal{H}}},$$

with

$$\langle \phi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi) = \int_{\mathbb{R}^d} (\phi \star \tilde{\psi})(x)\Gamma(dx).$$

SPDEs - Rigorous formulation

We mostly follow the **random-field approach** (Walsh,Dalang,...) of SPDEs. The mild formulation of SPDEs is given by

$$u(t,x) = I_0(t,x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y)\sigma(u(s,y))M(ds,dy)$$
$$+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y)b(u(s,y))dyds.$$

Here

- I_0 is a term accounting for the initial conditions,
- M is the **martingale measure** derived from the random noise term, $M_t(A) = F(1_{[0,t]}1_A)$, and
- Λ is the **fundamental solution** to the associated PDE, $L\Lambda = \delta_{0,0}$.

For the wave equation, the form of the fundamental solution changes with the spatial dimension, but for all dimensions we have

$$\mathcal{F}\Lambda(t)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}.$$

SPDEs - Stochastic integral

The stochastic integral in the SPDE is defined as $L^2(\Omega)$-limit of the regularizations $\Lambda_n(t-s, x-y)Z(s,y)$ and

$$
\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y)Z(s,y)M(ds,dy) \right)^2 \right]
$$
$$
= \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(t-s)(\xi)|^2 \mu_s^Z(d\xi)ds
$$
$$
\leq \int_0^t \mathbb{E}\left[Z(s,0)^2 \right] \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(t-s)(\xi+\eta)|^2 \mu(d\xi)ds.
$$

The curse of dimensionality (for the wave equation)

$d = 1, 2$: fundamental solution is a function, no Fourier transform necessary, p-moments, nonstationary equations, easier expression

$d = 3$: fundamental solution is a nonnegative distribution, p-moments, nonstationary equations, easier expression

$d \geq 4$: fundamental solution is a distribution, only second moment and stationary equations, therefore $I_0 \equiv 0$

SPDEs - Pathwise integral

Similarly, the pathwise integral in the SPDE is defined as the $L^2(\Omega)$-limit of the regularizations $\Lambda_n(t-s, x-y)Z(s,y)$ w.r.t. the following norm

$$\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y)Z(s,y)dyds\right)^2\right]$$

$$= \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(t-s)(\eta)|^2 \nu_s^Z(d\eta)ds$$

$$\leq \int_0^t \mathbb{E}\left[Z(s,0)^2\right] \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(t-s)(\eta)|^2 ds.$$

Standing hypotheses

stochastic integral: $\displaystyle\int_0^t \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(s)(\xi+\eta)|^2 \mu(d\xi)ds < \infty,$

pathwise integral: $\displaystyle\int_0^t \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(s)(\eta)|^2 ds < \infty,$

and some (technical) regularity conditions

The SWE in high dimensions I - Main theorem

First objective: show the Malliavin differentiability of the solution to an SPDE with Λ a general distribution.

Theorem (Sanz-Solé & S. (2013))

Fix $(t, x) \in [0, T] \times \mathbb{R}^d$. Under the standing hypotheses, $\sigma, b \in \mathcal{C}_b^1(\mathbb{R})$ and $I_0(t, x) = 0$, we have $u(t, x) \in \mathbb{D}^{1,2}$ and

$$
\begin{aligned}
Du(t, x) = &\Lambda(t - \cdot, x - *)\sigma(u(\cdot, *)) \\
&+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - y)\sigma'(u(s, y))Du(s, y)M(ds, dy) \\
&+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - y)b'(u(s, y))Du(s, y)dyds.
\end{aligned}
$$

Note that this is an SPDE in the Hilbert space $\mathcal{H}_T$. We need similar conditions on the integrands, i.e. $D\sigma(u)$, as for the real-valued case to well-define both integrals. These can be shown from the respective properties of u.

The SWE in high dimensions II - Idea of the proof

Main problem: we **cannot** use the classic way to show the pointwise convergence of the Malliavin derivatives Du_n of some approximation u_n

Solution: show first that $u \in \mathbb{D}^{1,2}$

Lemma

Let $(F_n)_{n \in \mathbb{N}} \subseteq \mathbb{D}^{1,2}$ such that $\lim_{n \to \infty} F_n = F$ in $L^2(\Omega)$ and $\sup_{n \in \mathbb{N}} \mathbb{E}[\|DF_n\|_{\mathcal{H}_T}^2] < \infty$. Then $F \in \mathbb{D}^{1,2}$ and $DF_n \rightharpoonup DF$ in $L^2(\Omega; \mathcal{H}_T)$.

and then use **commutation formulas**

$$D\bigg(\int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - y)\sigma(u(s, y))M(ds, dy) \bigg)$$

$$= \Lambda(t - \cdot, x - *)\sigma(u(\cdot, *)) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - y)\sigma'(u(s, y))Du(s, y)M(ds, dy),$$

$$D\bigg(\int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - y)b(u(s, y))dyds \bigg)$$

$$= \int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - y)b'(u(s, y))Du(s, y)dyds.$$

The SWE in high dimensions III - Absolute continuity

Second objective: show the absolute continuity of $P_{u(t,x)}$, for all $(t, x) \in (0, T] \times \mathbb{R}^d$, i.e. $P_{u(t,x)} = p_{t,x}\lambda^1$.

Idea: using the Bouleau-Hirsch criterion and showing that the first term in the SPDE dominates the others

$$Du(t, x) = \Lambda(t - \cdot, x - *)\sigma(u(\cdot, *))$$
$$+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - y)\sigma'(u(s, y))Du(s, y)M(ds, dy)$$
$$+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - y)b'(u(s, y))Du(s, y)dyds.$$

Problem: This does not work because Λ may be negative!

Theorem (Sanz-Solé & S. (2013))

Under the standing hypotheses on Λ, $\sigma \equiv c \neq 0$, $b \in \mathcal{C}_b^1(\mathbb{R})$ and for all $t > 0$, we have $\|\Lambda\|_{\mathcal{H}_t}^2 = \int_0^t \|\Lambda(s, *)\|_{\mathcal{H}}^2 ds > 0$. Then, for all $(t, x) \in (0, T] \times \mathbb{R}^d$, the law of $u(t, x)$ is absolutely continuous.

Absolute continuity for SPDEs I - Main idea

Solution for multiplicative noise: use a different, very recent approach.

Lemma (Debussche & Romito (2013))

Let κ be a finite nonnegative measure. Assume that there exist $0 < \alpha \leq a < 1$, $n \in \mathbb{N}$ and $C_n > 0$ such that for all $\phi \in \mathcal{C}_b^\alpha$, $h \in [-1,1]$,

$$\left| \int_{\mathbb{R}} \Delta_h^n \phi(y) \kappa(dy) \right| \leq C_n \|\phi\|_{\mathcal{C}_b^\alpha} |h|^a.$$

Then $\kappa = p\lambda^1$, and $p \in B_{1,\infty}^{a-\alpha}(\mathbb{R})$.

Here $B_{1,\infty}^s(\mathbb{R})$, with $s \in (0,1)$ is the Besov space with norm

$$\|f\|_{B_{1,\infty}^s(\mathbb{R})} = \|f\|_{L^1(\mathbb{R})} + \sup_{|h| \leq 1} |h|^{-s} \|\Delta_h^n f\|_{L^1(\mathbb{R})}.$$

We have used difference operators $(\Delta_h^1 f)(x) = f(x+h) - f(x)$, and

$$(\Delta_h^n f)(x) = (\Delta_h^1(\Delta_h^{n-1} f))(x) = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} f(x+jh).$$

Absolute continuity for SPDEs II - Assumptions and result

Now we assume the standing hypotheses and

- there exists $C, \delta > 0$ such that $\mathbb{E}[(u(t,0) - u(s,0))^2] \leq C|t-s|^\delta$,
- $\inf_{x \in \mathbb{R}} |\sigma(x)| = \sigma_0 > 0$,
- for some $C, \gamma, \gamma_1, \gamma_2 > 0$ and $t_0 \in (0, T]$

$$\int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(s)(\xi)|^2 \mu(d\xi) ds \geq Ct^\gamma, \text{ for all } t \in [0, t_0],$$

$$\int_0^t \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(s)(\xi + \eta)|^2 \mu(d\xi) ds \leq Ct^{\gamma_1},$$

$$\int_0^t \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(s)(\eta)|^2 ds \leq Ct^{\gamma_2}.$$

Theorem (Sanz-Solé & S. (2014))

Under these assumptions and $\bar{\gamma} := \frac{\min\{\gamma_1, \gamma_2\} + \delta}{\gamma} > 1$. Then, for all $(t, x) \in (0, T] \times \mathbb{R}^d$, the probability law of $u(t, x)$ is absolutely continuous, $P_{u(t,x)} = p_{t,x}\lambda^1$, and $p_{t,x} \in B_{1,\infty}^s(\mathbb{R})$ with $0 < s < 1 - \bar{\gamma}^{-1}$.

Absolute continuity for SPDEs III - Idea of the proof

Approximation of $u(t, x)$: for $\varepsilon > 0$ (here: $b \equiv 0$)

$$u^\varepsilon(t, x) = \int_0^{t-\varepsilon} \int_{\mathbb{R}^d} \Lambda(t - s, x - y)\sigma(u(s, y))M(ds, dy)$$
$$+ \sigma(u(t - \varepsilon, x)) \int_{t-\varepsilon}^t \int_{\mathbb{R}^d} \Lambda(t - s, x - y)M(ds, dy)$$

Conditioned on $\mathscr{F}_{t-\varepsilon}$, $u^\varepsilon(t, x)$ has a Gaussian distribution with variance dominated by $C|h|\varepsilon^{-\gamma}$. The $L^2(\Omega)$-difference $\mathbb{E}[(u(t, 0) - u^\varepsilon(t, 0))^2]$ is bounded by $C\varepsilon^\delta(\varepsilon^{\gamma_1} + \varepsilon^{\gamma_2})$. So

$$\left| \int_{\mathbb{R}} \Delta_h^n \phi(y) P_{u(t,x)}(dy) \right| \leq C_n \|\phi\|_{\mathcal{C}_b^\alpha} \left(|h|^n \varepsilon^{-\frac{\gamma n}{2}} + \varepsilon^{\frac{\alpha(\gamma_1+\delta)}{2}} + \varepsilon^{\frac{\alpha(\gamma_2+\delta)}{2}} \right)$$
$$\leq C_n \|\phi\|_{\mathcal{C}_b^\alpha} |h|^{\alpha\rho\bar{\gamma}/2}.$$

A clever choice for ε is: $\varepsilon = t|h|^{\rho/\gamma}$ with $\rho \in (0, 2)$.
Optimizing over α, ρ: best order of Sobolev space is (almost) $1 - \bar{\gamma}^{-1}$.

Absolute continuity for SPDEs IV - An example

Example: Stochastic Wave Equation

Consider the SWE in any spatial dimension with $I_0 \equiv 0$ and Gaussian noises with Riesz kernel covariance $\mu(d\xi) = |\xi|^{-d+\beta} d\xi$.
Then

$$\gamma = 3 - \beta, \quad \gamma_1 = 3 - \beta, \quad \gamma_2 = 3 \quad \text{and} \quad \delta = 2 - \beta.$$

So $\bar{\gamma} = (5 - 2\beta)/(3 - \beta) > 1$. Therefore $u(t, x)$ is absolutely continuous with density in $B_{1,\infty}^s(\mathbb{R})$ for all $0 < s < (2 - \beta)/(5 - 2\beta)$.
For finite measures μ all this holds with $\beta = 0$.

Other situations where this approach might work:

1. Stochastic heat equation. The existence and smoothness of a density are done by Marquez-Mellouk-Sarra. But slightly weaker conditions on σ and b with this approach.

2. Stochastic wave equation with $\sigma(u) = u$. This is current work with D. Conus, first very preliminary results are available.

Logarithmic asymptotics of densities I - Setting

Now we consider a family of SPDEs indexed by $\varepsilon \in (0, 1]$

$$Lu^\varepsilon(t, x) = b(u^\varepsilon(t, x)) + \varepsilon\sigma(u^\varepsilon(t, x))\dot{F}(t, x).$$

This translates to the mild formulation

$$u^\varepsilon(t, x) = I_0(t, x) + \varepsilon \int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - y)\sigma(u^\varepsilon(s, y))M(ds, dy)$$
$$+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - y)b(u^\varepsilon(s, y))dyds.$$

As $\varepsilon \downarrow 0$, u^ε converges uniformly in $L^2(\Omega)$ to the solution of the PDE $Lu^0 = b(u)$. For several examples of SPDEs a **large deviations principle** holds, which quanitifes the speed of this convergence. So $P_{u^\varepsilon(t,x)} \to \delta_{u^0(t,x)}$.

Question

Assume that $P_{u^\varepsilon(t,x)} = p_{t,x}^\varepsilon \lambda^1$, then formally $p_{t,x}^\varepsilon(y) \to \delta_{u^0(t,x)}(y)$. But at what speed?

Logarithmic asymptotics of densities II - Assumptions

We consider

$$u^\varepsilon(t,x) = I_0(t,x) + \varepsilon \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y)\sigma(u^\varepsilon(s,y))M(ds,dy)$$
$$+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y)b(u^\varepsilon(s,y))dyds,$$

and assume

- Λ is a nonnegative distribution and the integrability conditions hold,
- $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |I_0(t,x)| < \infty$,
- there exist $C, \gamma, t_0 > 0$ such that for all $t \in [0, t_0]$

$$Ct^\gamma \leq \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(s)(\xi)|^2 \mu(d\xi)ds,$$

- $b, \sigma \in \mathcal{C}_b^\infty(\mathbb{R})$,
- $\inf_{x\in\mathbb{R}^d} |\sigma(x)| \geq \sigma_0 > 0$,
- for all $(t,x) \in (0, T] \times \mathbb{R}^d$, $(u^\varepsilon(t,x))_{\varepsilon\in(0,1]}$ satisfies a LDP in $\mathbb{R}$ with rate function J.

Logarithmic asymptotics of densities III - Main theorem

Theorem (Sanz-Solé & S. (2014))

Fix $(t, x) \in (0, T] \times \mathbb{R}^d$.

1. Assume all of the above. Then for any $y \in \mathbb{R}$,

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \log p_{t,x}^\varepsilon(y) \leq -J(y).$$

2. Assume all but the last assumption from above. Fix $y \in \mathbb{R}$ in the interior of the topological support $P_{u^\varepsilon(t,x)}$. Then

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \log p_{t,x}^\varepsilon(y) \geq -I(y),$$

with $I(y) := \inf \left\{ \frac{1}{2} \|h\|_{\mathcal{H}_T}^2; h \in \mathcal{H}_T, \Phi_{t,x}^h = y \right\}$, and

$$\Phi_{t,x}^h = I_0(t, x) + \left\langle \Lambda(t - \cdot, x - *)\sigma(\Phi_{\cdot,*}^h), h \right\rangle_{\mathcal{H}_T}$$
$$+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - y) b(\Phi_{s,y}^h) dy ds.$$

Logarithmic asymptotics of densities IV - Comments

The lower bound is only meaningful if $I(y) < \infty$. For this we show:

Theorem (Sanz-Solé & S. (2014))

Under slightly stronger assumptions on the growth of the $\|\Lambda\|_{\mathcal{H}_t}$-norm, the topological support of $P_{u^\varepsilon(t,x)}$ is the closure of the set $\{\Phi^h_{t,x}; h \in \mathcal{H}_T\}$.

With this, we can show.

Proposition (Sanz-Solé & S. (2014))

Under these assumptions and $\sigma, b \in \mathcal{C}^1_b(\mathbb{R})$, for all y in the interior of the topological support of $P_{u^\varepsilon(t,x)}$, $I(y) < \infty$.

Problem: we cannot verify this for the SHE!

Proposition (Sanz-Solé & S. (2014))

Assume the initial assumptions and that b is bounded. Then
$\{y \in \mathbb{R}; I(y) < \infty\} = \mathbb{R}$.

Result by E. Nualart: same holds with σ, instead of b, bounded.

Part II: SPDEs - Existence and Uniqueness of Solution

4. Random-field solutions to linear hyperbolic stochastic partial differential equations with variable coefficients

Hyperbolic SPDEs I - Definition

In this part we consider **linear hyperbolic SPDEs**

$$L(t, x, \partial_t, \nabla_x)u(t, x) = \gamma(t, x) + \sigma(t, x)\dot{F}(t, x),$$

where

- $\gamma, \sigma : \mathbb{R}^{1+d} \to \mathbb{R}$ are in $L^2([0, T]; L^\infty(\mathbb{R}^d))$, with spatial Fourier transforms in $L^2([0, T]; L^1(\mathbb{R}^d))$
- F is the same Gaussian noise with spatial correlation as above,
- $L = L(t, x, \partial_t, \nabla_x)$ is a hyperbolic PDO with variable coefficients.

We use again the mild formulation and **define** the solution to that SPDE as

$$u(t, x) = I_0(t, x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y)\sigma(s, y)M(ds, dy)$$
$$+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y)\gamma(s, y)dyds.$$

Here

- I_0 is a term accounting for the initial conditions, pointwise finite
- Λ is the fundamental solution to the associated PDE $Lu = \gamma$.

Hyperbolic SPDEs II - Problems and central ideas

Main problem: what is a suitable concept for the fundamental solution?
In the case of constant coefficients: solution (via Fourier transform) of

$$Lu = \delta_{0,0}.$$

This is no longer possible, instead, we use **microlocal analysis**. This is a truly hyperbolic theory, for parabolic SPDEs with variable coefficients, use heat kernel estimates.

We will compute families of Fourier integral operators, whose Schwartz kernels will be the replacement for the fundamental solution. For 2nd order hyperbolic PDEs we would have for instance

$$u(t) = T_0(t)u_0 + T_1(t)u_1 + \int_0^t T_2(t,s)\gamma(s)ds.$$

Another problem: We cannot tell whether these Schwartz kernels will be functions, distributions, nonnegative etc.

Consequence: we can only treat linear hyperbolic SPDEs!

Consequence: need to give meaning to the stochastic and pathwise integral!

Hyperbolic SPDEs III - Definition of the integrals

The integrability conditions we have seen earlier are replaced by

Standing hypotheses

For $(t, s, x) \in \Delta_T \times \mathbb{R}^d$, $\Lambda(t, s, x)$ is a function with values in $\mathcal{S}'_r(\mathbb{R}^d)$, such that $\xi \mapsto \mathcal{F}\Lambda(t, s, x)(\xi)$ is a function, $(t, s, x, \xi) \mapsto \mathcal{F}\Lambda(t, s, x)(\xi)$ is measurable, and

$$\int_0^T \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\Lambda(t, s, x)(\xi + \eta)|^2 \mu(d\xi) \left(\int_{\mathbb{R}^d} |\mathcal{F}\sigma(s)(\eta)| d\eta \right)^2 ds < \infty,$$

$$\int_0^T \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(t, s, x)(\eta)|^2 \left(\int_{\mathbb{R}^d} |\mathcal{F}\gamma(s)(\eta)| d\eta \right)^2 ds < \infty,$$

and some (technical) regularity conditions.

Theorem (Ascanelli & S. (2014))

Under the standing hypotheses, the stochastic and pathwise integrals are well-defined as $L^2(\Omega)$-limits of stochastic and pathwise integrals of approximating step processes.

Hyperbolic SPDEs IV - Strictly hyperbolic SPDEs

Consider SPDEs with PDOs of the form

$$L = \partial_t^2 - \sum_{j,k=1}^d a_{j,k}(t,x)\partial_{x_j}\partial_{x_k} - \sum_{j=1}^d b_j(t,x)\partial_{x_j} - c(t,x),$$

with $a_{j,k} \in C^1([0,T]; \mathcal{C}_b^\infty(\mathbb{R}^d))$ and $b_j, c \in \mathcal{C}([0,T]; \mathcal{C}_b^\infty(\mathbb{R}^d))$. Assume that

$$\sum_{j,k=1}^d a_{j,k}(t,x)\xi_j\xi_k \geq C|\xi|^2,$$

for some $C > 0$. Assume for the initial conditions that $u_0 \in H^r(\mathbb{R}^d)$ and $u_1 \in H^{r-1}(\mathbb{R}^d)$, where $2r > d$. Suppose that

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{1 + |\xi + \eta|^2}\mu(d\xi) < \infty.$$

Then, for some time horizon $0 < \tilde{T} \leq T$, there exists a unique solution to the SPDE with PDO L. Similarly for higher-order SPDEs.

Hyperbolic SPDEs V - Weakly hyperbolic SPDEs

Consider the SPDE

$$
\begin{cases}
\left(\partial_t^2 - t^k \partial_x^2 + c t^{k\rho} \partial_x \right) u(t,x) = \dot{F}(t,x), \\
u(0,x) = 0, \ \partial_t u(0,x) = 0,
\end{cases}
$$

for some constant $c > 0$, $k \in \mathbb{N}$, $k \geq 2$ and $\rho = 2^{-1} - k^{-1}$. Here the phenomenon of **loss of derivatives** in Sobolev spaces occurs! This means that

$$
E(t,s) : H^r \to H^{r-\delta},
$$

where E is the fundamental solution operator. This translates into stricter conditions on the covariance measures, and their spectral measures μ, i.e.

$$
\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi + \eta|^2)^{1-\delta}} \mu(d\xi) < \infty.
$$

Problem: the $\delta > 0$ increases with c!
Conclusion: Nonexistence of solution for quite simple equations!

Part III: Stochastic Integration

5. Integration Theory for infinite dimensional volatility modulated Volterra processes

Stochastic Integration I - Definitions

Let in the following

$$X(t) = \int_0^t g(t, s)\sigma(s)\delta B(s),$$

where

- B is a cylindrical Wiener process on $\mathcal{H}_1$,
- σ is an $L(\mathcal{H}_1, \mathcal{H}_2)$-valued process, not necessarily adapted, and
- g is a deterministic function such that $g(t, s) \in L(\mathcal{H}_2, \mathcal{H}_2)$.

Particular examples for this type of processes are

- Gaussian processes in Hilbert spaces,
- Ambit processes,
- Mild solutions to SDE and SPDE

$$X(t) = \int_0^t g(t - s)\sigma(X(s))dB(s).$$

Stochastic Integration II - Formal definition

Set

$$\mathcal{K}_g(Y)(t,s) := Y(s)g(t,s) + \int_s^t \big(Y(u) - Y(s)\big)g(du,s).$$

Motivated by a computation for differentiable integrands Y, we set:

Definition

Fix $t \in [0,T]$. We say that a stochastic process $(Y(s))_{s \in [0,t]}$ belongs to the domain of the stochastic integral with respect to X if

1. $(Y(u) - Y(s))_{u \in (s,t]}$ is integrable w.r.t. $g(du,s)$ almost surely,
2. $\mathcal{K}_g(Y)(t,\cdot)\sigma(\cdot)1_{[0,t]}(\cdot) \in \mathrm{Dom}(\delta)$ for all $t \in [0,T]$, and
3. $\mathcal{K}_g(Y)(t,s)$ is Malliavin differentiable for all $s \in [0,t]$ and the Hilbert-valued stochastic process $s \mapsto \mathrm{tr}_{\mathcal{H}} D_s\big(\mathcal{K}_g(Y)(t,s)\big)\sigma(s)$ is Bochner integrable on $[0,t]$ almost surely.

We write $Y \in \mathcal{I}^X(0,t)$ and set

$$\int_0^t Y(s)dX(s) = \int_0^t \mathcal{K}_g(Y)(t,s)\sigma(s)\delta B(s) + \mathrm{tr}_{\mathcal{H}} \int_0^t D_s\big(\mathcal{K}_g(Y)(t,s)\big)\sigma(s)ds.$$

Stochastic Integration III - Calculus rules

Basic rules: Without much effort, one can show familiar calculus rules for this stochastic integral, such as **linearity**, **local operator**, **compatibility** with projections, and

$$\int_0^t \sum_{j=0}^{n-1} Z_j 1_{(t_j, t_{j+1}]}(s) dX(s) = \sum_{j=0}^{n-1} Z_j \big(X(t_{j+1}) - X(t_j) \big).$$

Connection to semimartingale integral: In some cases, one can also show an equality with the classic semimartingale integral.

SDEs driven by X: Consider the SDE driven by X with additive noise

$$dY(t) = -AY(t)dt + FdX(t).$$

One can compute its solution to be equal to

$$Y(t) = \int_0^t e^{-(t-s)A} F dX(s),$$

if we assume that $u \mapsto e^{-(u-s)A} F$ is $g(du, s)$-integrable.

Stochastic Integration IV - Itô formula

Theorem (Benth & S. (2014))

Let $F : \mathcal{H}_2 \to \mathcal{H}_3$ be twice Fréchet differentiable. Furthermore assume that g satisfies the semimartingale condition. Assume that Y and σ are twice Malliavin differentiable, $Y(s)g(s,s)\sigma(s) \in \mathbb{L}^{2,p}(\mathcal{H}, \mathcal{H}_2)$ for some $p > 4$ and

$$\int_0^s Y(s)\frac{\partial g}{\partial s}(s,u)\sigma(u)\delta B(u) + \operatorname{tr}_{\mathcal{H}}\left(D_s(Y(s))g(s,s)\sigma(s)\right)$$
$$+ \operatorname{tr}_{\mathcal{H}}\int_0^s D_u(Y(s))\frac{\partial g}{\partial s}(s,u)\sigma(u)du \in \mathbb{L}^{1,4}(\mathcal{H}_2).$$

Then $F'(Z)Y \in \mathcal{I}^X(0,t)$ for all $t \in [0,T]$ and

$$F(Z_t) = F(0) + \int_0^t F'(Z(s))Y(s)dX(s)$$
$$- \frac{1}{2}\operatorname{tr}_{\mathcal{H}}\int_0^t F''(Z_s)\left(Y(s)g(s,s)\sigma(s)\right)\left(Y(s)g(s,s)\sigma(s)\right)ds.$$

Stochastic Integration V - Random-field integral

An alternative is to consider

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} g(t,s;x,y)\sigma(s,y)M(\delta s, dy),$$

where

- g is a deterministic function,
- σ is a a random field,
- M is a worthy martingale measure, white in time, with covariation measure Γ.

We define the kernel

$$\mathcal{K}_g(h)(t,s,y) := \int_{\mathbb{R}^d} h(s,z)g(t,s;dz,y) + \int_s^t \int_{\mathbb{R}^d} \big(h(u,z) - h(s,z)\big)g(du,s;dz,y)$$

and set for $Y \in \mathcal{I}^X([0,t] \times \mathbb{R}^d)$

$$\int_0^t \int_{\mathbb{R}^d} Y(s,y)X(ds,dy) = \int_0^t \int_{\mathbb{R}^d} \mathcal{K}_g(Y)(t,s,y)\sigma(s,y)M(\delta s,dy)$$

$$+ \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D_{s,y-z}\mathcal{K}_g(Y)(t,s,y)\sigma(s,y)dy\Gamma(dz)ds.$$

Stochastic Integration VI - Connection of the integrals

Definition

Assume for all $(s, y) \in [0, T] \times \mathbb{R}^d$ that the function $(t, x) \mapsto g(t, s, x, y)$ has bounded variation on $[u, v] \times \mathbb{R}^d$ for all $0 \leq s < u < v \leq t$. Assume

1. Y is integrable w.r.t. $g(t, s; dz, y)$ a.s. and (t, s, y)-a.e.,

2. $(Y(u, z) - Y(s, z))_{u \in (s, t] \times \mathbb{R}^d}$ is $g(du, s; dz, y)$-integrable a.s. and (s, y)-a.e.,

3. $(s, y) \mapsto \mathcal{K}_g(Y)(t, s, y)\sigma(s, y)1_{[0, t]}(s)$ is integrable w.r.t. M,

4. $\mathcal{K}_g(Y)(t, s, y)$ is Malliavin differentiable w.r.t. $D_{s, y-z}$ and $(s, y, z) \mapsto D_{s, y-z}(\mathcal{K}_g(Y)(t, s, y))\sigma(s, y)$ is $\lambda|_{[0, T]} \otimes \lambda|_{\mathbb{R}^d} \otimes \Gamma$-integrable on $[0, t] \times \mathbb{R}^d \times \mathbb{R}^d$ a.s.

Proposition (Benth & S. (2014))

Let $Y \in \mathcal{I}^X([0, t] \times \mathbb{R}^d)$. Then $Y \in \mathcal{I}^X([0, t])$ with $\mathcal{H}_1 = \mathcal{H}$, $\mathcal{H}_3 = \mathbb{R}$ and

$$\int_0^t \int_{\mathbb{R}^d} Y(s, y)X(ds, dy) = \int_0^t Y(s)dX(s).$$

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The standing (technical) hypotheses

- Let ϕ denote a nonnegative function in $\mathcal{C}_0^\infty(\mathbb{R}^d)$, with support included in the unit ball of $\mathbb{R}^d$, satisfying $\int_{\mathbb{R}^d} \phi(x)dx = 1$. For all such ϕ and all $0 \leq a \leq b \leq T$, we have $\int_a^b (\Lambda(s) * \phi)(x)ds \in \mathcal{S}(\mathbb{R}^d)$, and

$$\int_{\mathbb{R}^d} \int_a^b |(\Lambda(s) * \phi)(x)|ds dx < \infty.$$

- $t \mapsto \mathcal{F}\Lambda(t)$ satisfies the first integrability condition, and

$$\lim_{h\downarrow 0} \int_0^T \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{s<r<s+h} |\mathcal{F}\Lambda(r)(\xi+\eta) - \mathcal{F}\Lambda(s)(\xi+\eta)|^2 \mu(d\xi) \, ds = 0.$$

- $t \mapsto \mathcal{F}\Lambda(t)$ satisfies the second integrability condition, and

$$\lim_{h\downarrow 0} \int_0^T \sup_{\eta \in \mathbb{R}^d} \sup_{s<r<s+h} |\mathcal{F}\Lambda(r)(\eta) - \mathcal{F}\Lambda(s)(\eta)|^2 ds = 0.$$