Densities for the Navier–Stokes equations with noise

Marco Romito

Università di Pisa

Universitat de Barcelona
March 25, 2015

Summary

The equations

Consider the Navier–Stokes equations,

$$\begin{cases} \dot{u} + (u \cdot \nabla)u + \nabla p = \nu \Delta u + \dot{\eta}, \\ \operatorname{div} u = 0, \\ u(0) = x, \end{cases}$$

either on the torus $\mathbb{T}_3$ with periodic boundary conditions (and zero mean) or on a bounded domain with Dirichlet boundary conditions. Here $\dot{\eta}$ is Gaussian noise which is white in time and coloured in space.

For instance assume periodic boundary conditions,

$$\dot{\eta}(t, y) = S\, dW = \sum_{\mathbf{k} \in \mathbf{Z}^3_\star} \sigma_{\mathbf{k}} \dot{\beta}_{\mathbf{k}}(t) e_{\mathbf{k}}(y),$$

and $e_{\mathbf{k}}(y) = e^{i\mathbf{k} \cdot y}$ are the Fourier exponential.

What we look for and why

Theorem

*Under **suitable** assumptions on the covariance of the driving noise, **suitable** finite–dimensional projections of any martingale weak solution of the 3D Navier–Stokes equations have a density with respect to the Lebesgue measure.*

- Why densities?
  - probabilistic form of regularity (hot issue for NSE),
  - better understanding of the evolution of the flow,
  - ideas related to uniqueness in law,
  - quantifying uncertainty.
- Why do we consider finite dimensional projections:
  - there is a reference measure,
  - easier: curse of regularity solved,
  - ideas related to long time behaviour [MatPar2006] [HaiMat2006]

Existence of densities

We focus on the following problem

$$dx = b(x)\,dt + dB$$

How to prove the existence of a density:

- Girsanov transformation,
- the Fokker-Planck equation,
- Malliavin calculus,
- ?????

A standard method

The idea is to use integration by parts. Having a smooth density would give,

$$\mathbb{E}\Big[\frac{\partial\phi}{\partial h}(x_t)\Big] = \int \frac{\partial\phi}{\partial h}(y) f_t(y)\,dy = -\int \phi(y)\frac{\partial f_t}{\partial h}(y)\,dy.$$

So, if we know that

$$\Big|\mathbb{E}\Big[\frac{\partial\phi}{\partial h}(x_t)\Big]\Big| \leqslant c|h|\,\|\phi\|_\infty$$

by duality there is a density.

How to prove this?

Integration by parts

Assume that we can find a variation of the noise H that compensates the variation h, that is $h = \mathcal{D}_H x_t$. Then

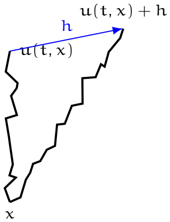
$$\mathbb{E}\Big[\frac{\partial\phi}{\partial h}(x_t)\Big] = \mathbb{E}[D\phi(x_t)h] = \mathbb{E}[D\phi(x_t)\mathcal{D}_H x_t] = $$
$$= \mathbb{E}[\mathcal{D}_H\phi(x_t)] = \mathbb{E}\Big[\phi(x_t)\int H\,dW\Big]$$

by the **chain rule** and **integration by parts**.

How to find H?: essentially one needs to (pseudo)invert the map

$$\Psi H = \int_0^t H(s)\mathcal{D}_s x_t\,ds$$

that is $H = \Psi^\star \mathcal{M}_t^{-1} h$.



Malliavin calculus and Navier-Stokes

We are dealing with a process u such that

$$du - \nu\Delta u\,dt + (u\cdot\nabla)u\,dt + \nabla p = \mathcal{S}\,dW,$$

whose Malliavin derivative $\mathcal{D}_H u$ satisfies

$$\frac{d}{dt}\mathcal{D}_H u - \nu\Delta\mathcal{D}_H u + (u\cdot\nabla)\mathcal{D}_H u + (\mathcal{D}_H u\cdot\nabla)u + \nabla q = \mathcal{S}H,$$

NO!

- Trying to prove invertibility of the Malliavin matrix forces to use the dynamics of the linearisation.
- A decent estimate on the Malliavin derivative is the same as an estimate on the difference of two solutions!

MAYBE

- the dynamics is well defined and good for short times,
- an invertible covariance allows to prove invertibility without using the dynamics.

It falls in the class of equations with non-regular coefficients,

- Fournier-Printemps (starting point for what we'll explain here),
- Bally and Caramellino (based on interpolation),
- Kohatsu-Higa and co-workers.

Primer on Besov spaces

Set

$$(\Delta_h^1 f)(x) = f(x+h) - f(x), \qquad (\Delta_h^n f)(x) = \Delta_h^1(\Delta_h^{n-1} f)(x)$$

and given any integer $n > s$,

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \Big(\int_{|h|\leqslant 1} \frac{\|\Delta_h^n f\|_{L^p}^q}{|h|^{sq}} \frac{dh}{|h|^n}\Big)^{\frac{1}{q}} \qquad \Big[+ \sup_{|h|\leqslant 1} \frac{\|\Delta_h^n f\|_{L^p}}{|h|^s}\Big]$$

Some nice properties of Besov spaces

- different (more general) definition in terms of Littlewood-Paley decomposition,
- $B^s_{p,p} = W^{s,p}$, $B^s_{\infty,\infty} = C^s$, $s \geqslant 0$ non–integer,
- $(B^s_{p,q})' = B^{-s}_{p',q'}$, $q < \infty$,
- $(I - \Delta) : B^s_{p,q} \longrightarrow B^{s-2}_{p,q}$ isomorphism,

Smoothing lemma

Theorem (deterministic fractional integration by parts)

If μ is a finite measure on $\mathbf{R}^d$ and there are $s > 0$, an integer $m > s$, and $\alpha \in (0,1)$ such that for every $\phi \in C_c^\infty(\mathbf{R}^d)$ and every $h \in \mathbf{R}^d$,

$$\Big|\int_{\mathbf{R}^d} \Delta_h^m \phi(x)\, \mu(dx)\Big| \leqslant c|h|^s \|\phi\|_{C_b^\alpha},$$

then μ has a density $f \in B^r_{1,\infty}$ for every $r < s - \alpha$.

We use the above lemma to prove existence of a density for the solution x of our simple model problem at time $t = 1$

$$dx = b(x)\, dt + dB.$$

We look for an estimate

$$\mathbb{E}[\phi(x_1 + h) - \phi(x_1)] \approx |h|^s \|\phi\|_{C_b^\alpha}$$

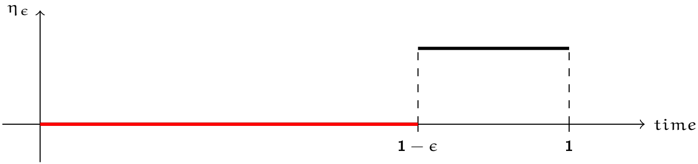
Fact 1: exploiting the short times

Recall: wish to estimate

$$\mathbb{E}[\phi(x_1 + h) - \phi(x_1)] \approx |h|^s \|\phi\|_{C_b^\alpha}$$

Consider

$$dx^\epsilon = \eta_\epsilon(t) b(x^\epsilon) + dB.$$



$$\begin{cases} dx \quad = b(x \quad) + dB, \\ dx^\epsilon = b(x^\epsilon) + dB, \end{cases}$$

[fournier–printemps (2010)]

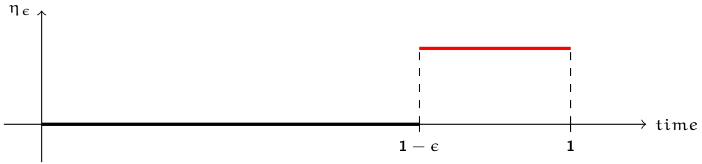
Fact 1: exploiting the short times

Recall: wish to estimate

$$\mathbb{E}[\phi(x_1 + h) - \phi(x_1)] \approx |h|^s \|\phi\|_{C_b^\alpha}$$

Consider

$$dx^\epsilon = \eta_\epsilon(t) b(x^\epsilon) + dB.$$



$$\begin{cases} dx \;\; = b(x \;\;) + dB, \\ dx^\epsilon = b(x^\epsilon) + dB, \end{cases}$$

[fournier–printemps (2010)]

Decomposition of the error

Hence

- $s \leqslant 1-\epsilon$: $\quad x^\epsilon(s) = x(s)$,
- $s \geqslant 1-\epsilon$: $\quad x^\epsilon(s)$ is the one–step explicit Euler approximation of x, starting at time $1-\epsilon$ with time step ϵ.

and

$$\begin{aligned}\mathbb{E}[\phi(x_1+h)] - \mathbb{E}[\phi(x_1)] &= \mathbb{E}[\phi(x_1+h)] - \mathbb{E}[\phi(x_1^\epsilon+h)]\\ &\quad + \mathbb{E}[\phi(x_1^\epsilon+h)] - \mathbb{E}[\phi(x_1^\epsilon)]\\ &\quad + \mathbb{E}[\phi(x_1^\epsilon)] - \mathbb{E}[\phi(x_1)]\end{aligned}$$

where

- **numerical error**
- **probabilistic error**

The numerical error

There is not much that can be done here,

$$\mathbb{E}[\phi(x_1^\epsilon)] - \mathbb{E}[\phi(x_1)] \leqslant [\phi]_\alpha \mathbb{E}\Big[\|x_1^\epsilon - x_1\|^\alpha\Big]$$

and

$$|x_1 - x_1^\epsilon| = \Big|\int_{1-\epsilon}^1 b(x_s)\,ds\Big| \leqslant \|b\|_\infty \epsilon,$$

so that

$$\textbf{numerical error} \approx \epsilon^\alpha$$

How to improve it?

The numerical error can be made smaller using a higher order numerical method.

More regularity of the drift b needed.

Fact 2: inverting the covariance (a.k.a. the probabilistic error)

Absorb the variation in the noise

$$\begin{aligned} x_1^\epsilon(t) + h &= \text{i. c.} + \int_{1-\epsilon}^1 dB_s + h \\ &= \text{i. c.} + \int_{1-\epsilon}^1 (dB_s + H\,ds) \end{aligned}$$

with an obvious value of $H \approx \frac{h}{\epsilon}$ and by the Girsanov formula (actually Cameron–Martin here!)

$$\textbf{prob. error} \approx \mathbb{E}\big[(G_1 - 1)\phi(x_1^\epsilon)\big] \approx \frac{|h|}{\sqrt{\epsilon}}\|\phi\|_\infty$$

How to improve it? Look for higher order increments $\Delta_h^n \phi(x_1)$ that yield the same numerical error but

$$\textbf{prob. error} \approx \Big(\frac{|h|}{\sqrt{\epsilon}}\Big)^n.$$

A balance of regularity

Putting all together,

$$\text{total error} \approx \textbf{probabilistic} + \textbf{numeric} \approx \frac{|h|}{\sqrt{\epsilon}}\|\phi\|_\infty + [\phi]_\alpha \epsilon^\alpha$$

and by optimizing in ϵ,

$$\mathbb{E}[\phi(x_1 + h)] - \mathbb{E}[\phi(x_1)] \approx |h|^{\frac{2\alpha}{2\alpha+1}} \|\phi\|_{C^\alpha}$$

- "α derivatives" given for the estimate,
- "$\frac{2\alpha}{2\alpha+1}$ derivatives " obtained
- use the smoothing lemma with $n = 1$, $\alpha \to \alpha$ and $s = \frac{2\alpha}{2\alpha+1}$.

Other functionals

The same idea can be used on quantities that **derive** from an equation (although might not solve one), such as in Navier–Stokes the balance of energy:

$$\mathcal{E}_t = \frac{1}{2}\int |u(t,x)|^2\,dx + \nu\int_0^t\int |\nabla u(s,x)|^2\,dx\,ds$$

To keep it simple, let us think again at $x \in \mathbf{R}^d$,

$$dx = b(x)\,dt + dB, \qquad \text{and} \qquad \mathcal{E}_t = \sum_i (x_t^i)^2$$

It is immediate to check that the corresponding energy computed for x^ϵ has a generalized χ^2 distribution.
Apparently odd: it turns out that the regularity depends on the number of degrees of freedom of the χ^2 (here d), and only in infinite dimension the unconditioned result is recovered.

Time regularity

Let f be the density of the solution of

$$dx = b(x)\,dt + dB$$

Theorem

$\|f(t) - f(s)\|_{B^{\alpha}_{1,\infty}} \lesssim |t-s|^{\beta/2}$, *for all* $\alpha + \beta < 1$

- $\|f(t) - f(s)\|_{L^1} \lesssim |t-s|^{\frac{1}{2}-}$, more challenging, through Girsanov transformation,
- $[f(t) - f(s)]_{B^{\alpha}_{1,\infty}} \lesssim |t-s|^{\beta/2}$ easier:

$$\|\Delta^n_h(f(t) - f(s))\|_{L^1} \lesssim \begin{cases} \|\Delta^n_h f(t)\|_{L^1} + \|\Delta^n_h f(s)\|_{L^1} & |h| \ll |t-s|, \\ \|f(t) - f(s)\|_{L^1} & |t-s| \ll |h| \end{cases}$$

The associated Fokker–Planck equation

Consider again

$$dx = b(x)\,dt + dB, \qquad x(0) = x_0$$

with $b \in L^\infty$. We know that the density $f \in B^{1-}_{1,\infty}$. In fact we can show that $f \in B^{1-}_{\infty,\infty}$. The density solves

$$\begin{cases} \partial_t f = \frac{1}{2}\Delta f - \nabla \cdot (bf), \\ f(0) = \delta_{x_0}, \end{cases}$$

that is

$$f(t,x) = p_t(x - x_0) + \int_0^t \nabla p_{t-s} \star (bf)(x)\,dt,$$

where p_t is the heat kernel.

Besov bounds with the Fokker–Planck equation

$$f(t,x) = p_t(x - x_0) + \int_0^t \nabla p_{t-s} \star (bf)(x)\,dt,$$

hence

$$\begin{aligned}\|\Delta_h^2 f(t,x)\|_{L^1} &= \Big\|\Delta_h^2 p_t(x - x_0) + \int_0^t (\Delta_h^2 \nabla p_{t-s}) \star (bf)(x)\,dt\Big\|_{L^1} \\ &\leqslant \|\Delta_h^2 p_t\|_{L^1} + \|b\|_\infty \|f\|_{L^1} \int_0^t \|\Delta_h^2 \nabla p_{t-s}\|_{L^1}\,ds\end{aligned}$$

and conclude with estimates for the heat kernel

Hölder bounds with the Fokker–Planck equation

To see the Hölder bound, consider for simplicity the equation for the stationary solution (density of the invariant measure):

$$\frac{1}{2}\Delta f - \nabla \cdot (bf) = 0,$$

that is, if g is the Poisson kernel ($g(x) = |x|^{2-d}$, $d \geqslant 3$),

$$\begin{aligned} f(x) &= \nabla g \star (bf) = (\nabla g \mathbb{1}_{B_\epsilon(0)}) \star (bf) + (\nabla g \mathbb{1}_{B_\epsilon(0)^c}) \star (bf) \\ &\lesssim \|b\|_\infty \|f\|_\infty \epsilon^d + \epsilon^{1-d} \|b\|_\infty \|f\|_{L^1} \end{aligned}$$

and choose $\epsilon = (2\|b\|_\infty)^{-1/d}$. The Hölder norm follows likewise.

Watch out! Needs to know that already $\|f\|_\infty < \infty$.

Some open problems

- Local estimate (see the problem with the energy).
- Probabilistic proof of the Fokker–Planck approach.
- Degenerate noise completely open (at least in this way).