Stochastic travelling waves and computation

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Joint with : V.Thümmler, E. Tzitzili.

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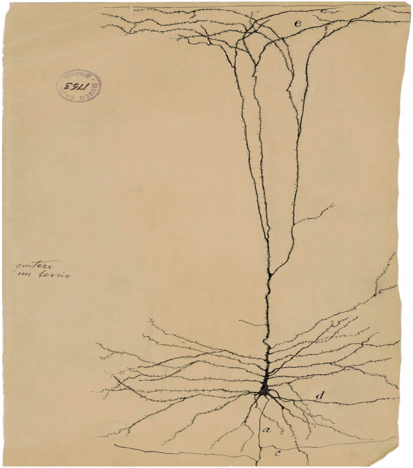

Ramon y Cajal (1888)

Nagumo equation

Reduced model for voltage wave propagation in neural axon

$$u_t = u_{xx} + u(1-u)(u-\alpha) \quad u(x,t) \in \mathbb{R},\ x \in \mathbb{R},\ t > 0$$

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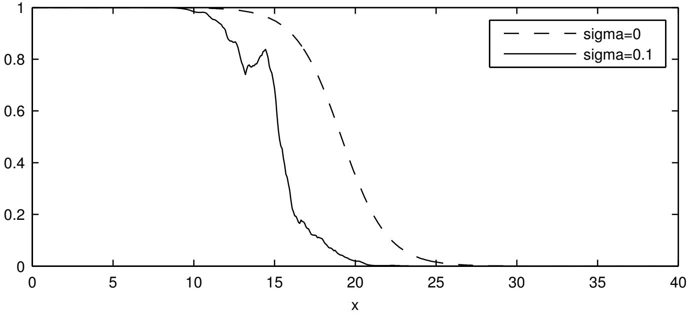

(Stochastic) FitzHugh–Nagumo equation

Deterministic voltage wave propagation in neural axon

$$u_t = u_{xx} + u(1-u)(u-\alpha) - z$$

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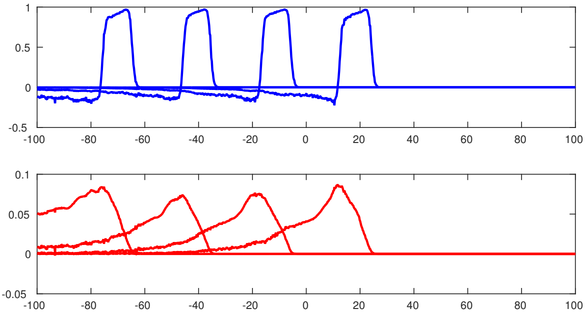

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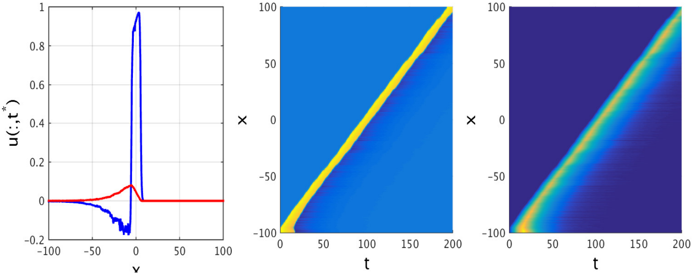

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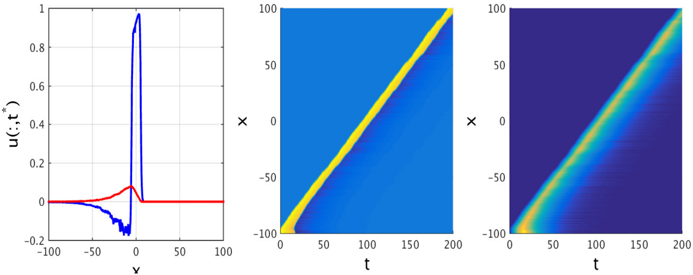

- Numerical Scheme for Simulation of Stratonovich System...

Stratonovich exponential integrator

Stratonovich SDE (1D)

$$du = [Au + f(u)]dt + g(u) \circ d\beta$$

has mild solution

$$u(t) = e^{tA}u(0) + \int_0^t e^{(t-s)A} f(u(s))ds + \int_0^t e^{(t-s)A} g(u(s)) \circ d\beta(s).$$

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$$u(\Delta t) = e^{\Delta t A}u(0) + \int_0^{\Delta t} e^{(\Delta t - s)A} f(u(s))ds + \int_0^{\Delta t} e^{(\Delta t - s)A} g(u(s)) \circ d\beta.$$

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From deterministic integral

$$u(\Delta t) = e^{\Delta t A}u(0) + A^{-1}(e^{\Delta t A} - 1) f(u(0)) + \int_0^{\Delta t} e^{(\Delta t - s)A} g(u(s)) \circ d\beta(s).$$

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Need to estimate u^{n+1} for $g(u^{n+1})$.

Stratonovich exponential integrator

In practice predict $g(u^{n+1})$ by an Euler–Maruyama type step

- Semi-implicit EM predictor

$$\tilde{u}^{n+1} = u^n + \left[Au^{n+1} + f(u^n)\right]\Delta t + g(u^n)\Delta b^{n+1}.$$

$$u^{n+1} = e^{\Delta t A}u^n + \phi_1(A)f(u^n) + \frac{1}{2}e^{(\frac{\Delta t}{2})A}\left(g(u^n) + g(\tilde{u}^{n+1})\right)\Delta b^{n+1}.$$

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Developed in one-dimension - similar for $u \in \mathbb{R}^d$ - or $u(x,t) \in H$.

- Ito SPDE case:

Parabolic : [L.& Rougemont, Jentzen et al, L. & Tambue, Larsson et al.]

Wave equation : [Cohen & Quer-Sardanyons]

Strong Convergence [L. & Efthalia Tzitzili]

$$du = [Au + f(u)]\, dt + G(u) \circ dW$$

Assumptions

- $W(t) = [\beta_1(t), \ldots, \beta_m(t)]$.

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- Milstein version :

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Finite Element approximation in space

$$du_h = [A_h u_h + f(u_h)]\, dt + G(u_h) \circ dW_h$$

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- $W(x,t) = \sum_{n \in \mathbb{Z}} \sqrt{q_n} \phi_n(x) \beta_n(t)$, β_n iid Brownian motions.

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Let $u_h^n \approx u(t_n)$ by Stochastic Exponential Integrator
Then, for $\epsilon > 0$

$$\left(E\left[\|u(t_n) - u_h^n\|^2\right]\right)^{1/2} \leq C\Delta t^{1/2-\epsilon}.$$

(Hiding space dependence)

Elements of the Proofs

$$\left(E\left[\|u(t_n)-u_n\|^2\right]\right)^{1/2} \le C\Delta t^{1/2-\epsilon}.$$

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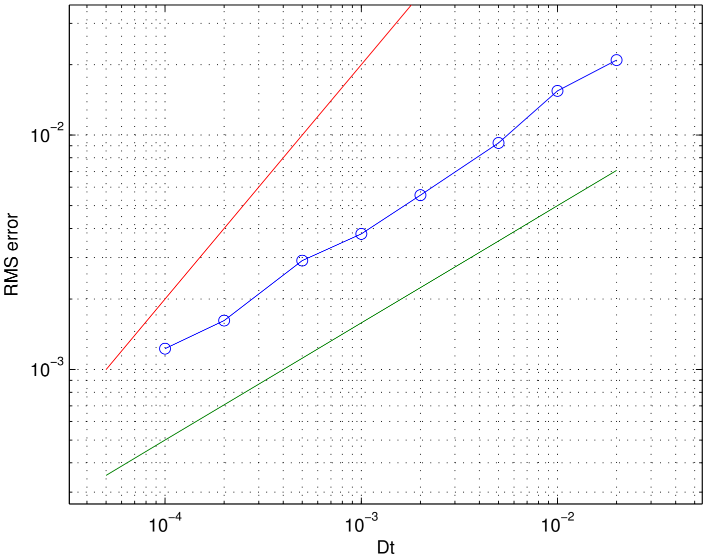
- For diagonal noise

$$\left(E\left[\|u(t_n)-u_n\|^2\right]\right)^{1/2} \leq C\Delta t^{1-\epsilon}.$$

Non-diagonal noise : $\left(E\left[\|u(t_n) - u_n\|^2\right]\right)^{1/2} \leq C\Delta t^{1/2-\epsilon}$.



Reference lines have slope 1/2 (green) and 1 (red)

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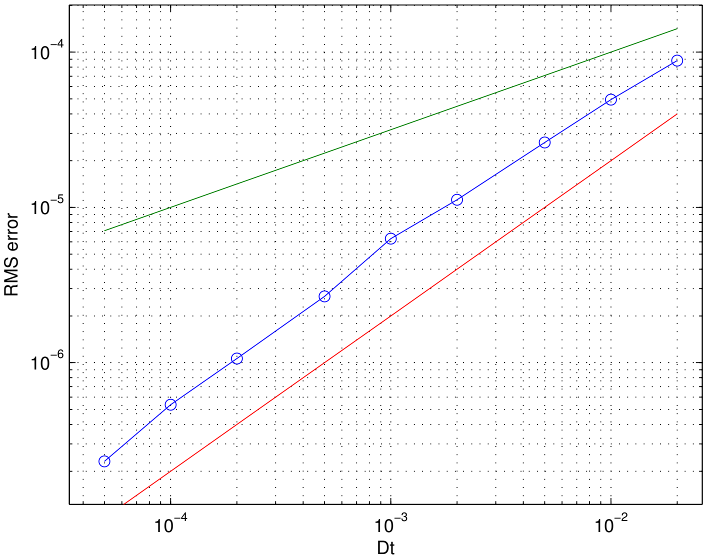

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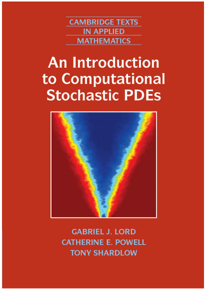
Advert Break



- Stochastic Travelling Wave

Deterministic travelling waves $u_t = u_{xx} + f(u)$

- TW with wave speed λ

Into co-moving frame $u(x,t) = u(x - \lambda t, t)$

$$u_t = u_{xx} + \lambda u_x + f(u), \quad x \in \mathbb{R}, \quad t \geq 0$$

of which the travelling wave u is a stationary solution $(u_t = 0)$.

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- Get a PDE with algebraic constraint : PDAE

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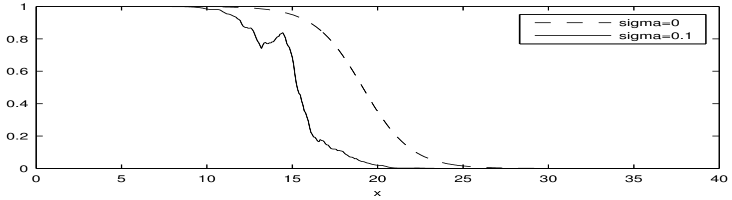
Stochastic Travelling wave



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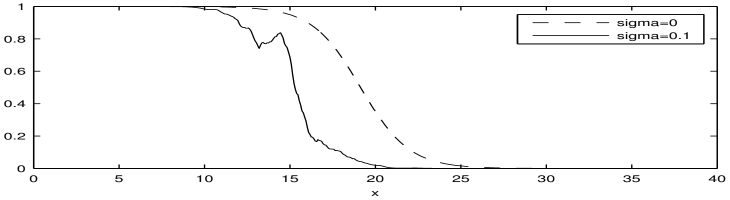

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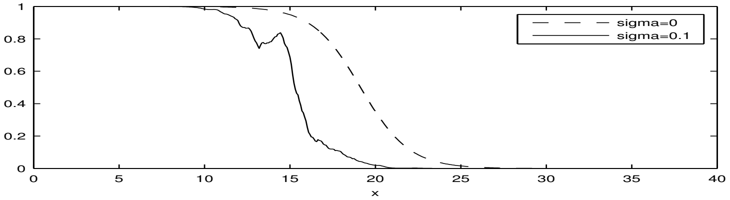

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Freezing a Stochastic Travelling wave

$$du = \Big[u_{xx} + f(u)\Big]\, dt + g(u) \circ dW(t)$$

1) Add advection term to freeze wave

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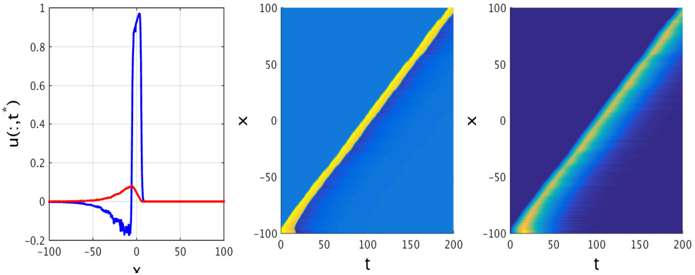
- $\lambda(t)$ "instantaneous" wave speed
- Wavespeed : $\Lambda(t) = \frac{1}{t}\int_0^t \lambda(s)ds$

Single realisation: SPDE /SPDAE



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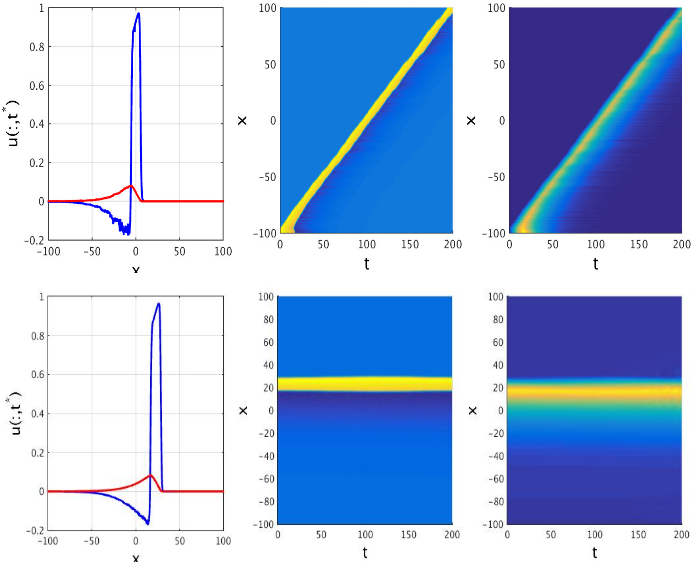

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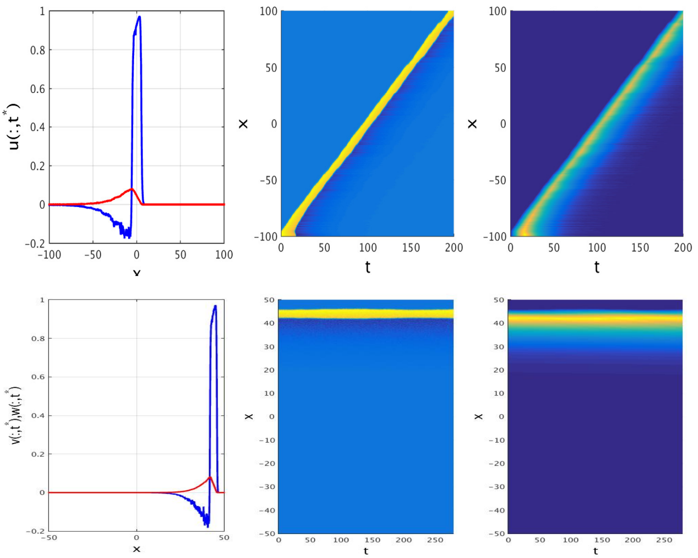

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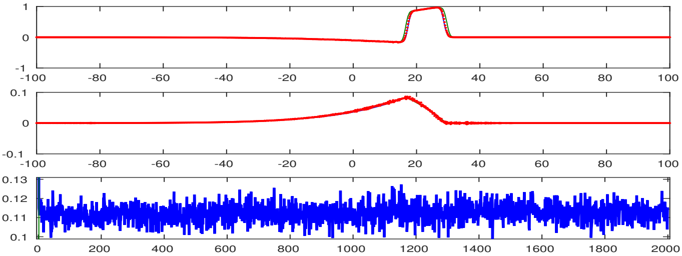
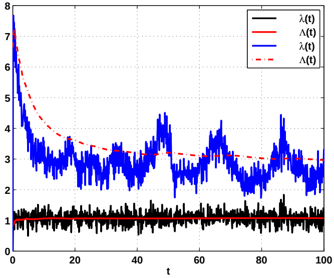


Wave speeds vs time



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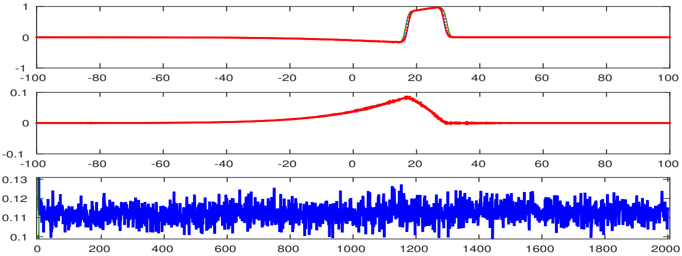

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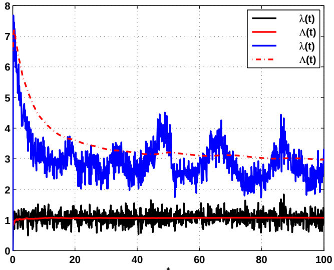
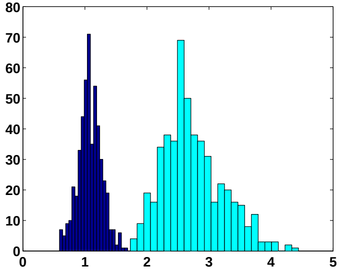




Distributions instantaneous wave speeds λ

Expected values

Compare :

- SPDAE : Λ^{fix}
- SPDE : Reference Λ_{fit}
- SPDE : Level set Λ_c

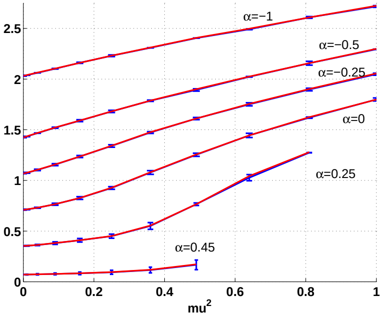
	Λ or Λ^{fix}	Λ_c
SPDE	1.08588 ± 0.19680	$1.08381 \pm 2.81\text{e-}03$
SPDAE	1.08951 ± 0.19512	$-4.0\text{e-}05 \pm 2.0\text{e-}5$

Based on 100 realizations

Correlation length $\xi = 0.1$

Different nonlinearities α.

Wave speeds as noise intensity increases



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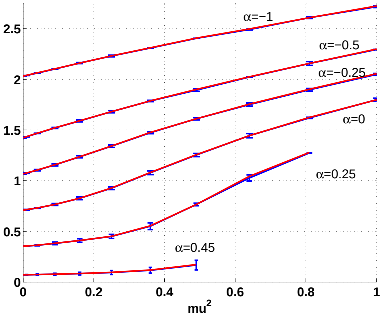

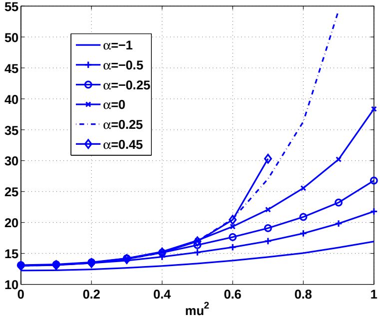

Width of wave

Strong Errors for SPDAEs?

No analysis ...

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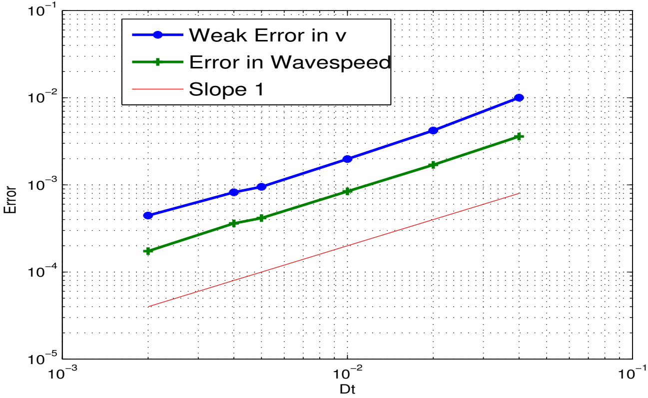
Estimate numerically :



time step Δt

Summary:

New exponential integrator for Stratonovich SDEs

▶ Stochastic Differential Algebraic Equations ?

Stochastic travelling wrt reference function $\hat{u}$.

Some adavantages of freezing method

- Efficient : smaller domain
- Does not rely on deterministic wave/ small noise
- Faster convergence than via level set c.

▶ But - large advection terms: numerically challenging.

Thank You

Refs

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[2] Emma Coutts and GJL. J. Compt. Neuroscience. 2013.

[3] Iyabo Adamu and GJL. J. Comp. Math. 2012

[4] GJL and Efthalia Tzitzili. In prep.