A Fourier analysis based approach of integration

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Integration

Background: understand $\int f dg$ for trajectories of stochastic processes; these are rough, e.g. only α-Hölder continuous as for Brownian motion: $\alpha < \frac{1}{2}$: rough path analysis

$f, g : [0,1] \to \mathbb{R}$; Riemann-Stieltjes' theory: g of bounded variation with (signed) interval measure m_g on the Borel sets of $[0,1]$:

$$\int_0^t f(s)dg(s) = \int_0^t f(s)dm_g(s).$$

f of bounded variation with interval measure m_f, integration by parts:

$$\int_0^t f(s)dg(s) = f(t)g(t) - f(0)g(0) - \int_0^t g(s)dm_f(s).$$

Remark: obvious tradeoff between regularity of f and of g.

Young's integral: f is α-, g β-Hölder, and $\alpha + \beta > 1$, $\int f dg$ defined.
Aim: Present Fourier based approach to *Young's integral,* embedded in new approach of *rough paths.*

Application: rough SPDE

Application goal: approach SPDE with rough path techniques in the spirit of Hairer's paper on regularity structures

E. g.: on torus

$$\frac{\partial}{\partial t}u(t,x) = -Au(t,x) + g(u(t,x))Du(t,x) + \xi(t,x),$$

with $u : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}^n$, $-A = -(-\Delta)^\sigma$ fractional Laplacian with $\sigma > \frac{1}{2}$, D spatial gradient, ξ space-time white noise

Fourier decomposition and Young's integral

Fourier decomposition of Hölder continuous functions f studied by Ciesielski (1961):

$$f(t) = \sum_{p \geq 0, 1 \leq m \leq 2^p}^{\infty} \langle H_{pm}, df \rangle G_{pm}(t)$$

with piecewise linear $G_{pm}, p \geq 0, 0 \leq m \leq 2^p$ (Schauder functions).

Then define

$$\int_0^t f(s) dg(s) = \sum_{p,m,\ q,n}^{\infty} \langle H_{pm}, df \rangle \langle H_{qn}, dg \rangle \int_0^t G_{pm}(s) dG_{qn}(s).$$

Lit: Baldi, Roynette '92: LDP; Ciesielski, Kerkyacharian, Roynette '93: calculus on Besov spaces; Roynette '93: BM on Besov spaces

Haar and Schauder functions

Define the **Haar functions** for $p \geq 0$, $1 \leq m \leq 2^p$

$$H_{pm}(t) = \sqrt{2^p}\mathbf{1}_{\left[\frac{m-1}{2^p}, \frac{2m-1}{2^{p+1}}\right)}(t) - \sqrt{2^p}\mathbf{1}_{\left[\frac{2m-1}{2^{p+1}}, \frac{m}{2^p}\right)}(t)$$

and $H_{00} = 1, H_{p0} = 0, p \geq 0$.

Haar functions form an orthonormal basis of $L^2([0,1])$.

The primitives of the Haar functions

$$G_{pm}(t) = \int_0^t H_{pm}(s)ds, \quad t \in [0,1], p \geq 0, 1 \leq m \leq 2^p,$$

are the **Schauder functions.**

$(H_{pm})_{p\geq 0, 0\leq m\leq 2^p}$ is orthonormal basis. So if $f = \int_0^\cdot \dot{f}(s)ds$ with $\dot{f} \in L^2([0,1])$ (write $f \in \mathcal{H}$)

$$f(t) = \int_0^t \sum_{p\geq 0, 0\leq m\leq 2^p} \langle H_{pm}, \dot{f}\rangle H_{pm}(s)ds = \sum_{p\geq 0, 0\leq m\leq 2^p} \langle H_{pm}, \dot{f}\rangle G_{pm}(t)$$

Ciesielski's isomorphism

Two observations: $t^0_{pm} = \frac{m-1}{2^p}, t^1_{pm} = \frac{2m-1}{2^{p+1}}, t^2_{pm} = \frac{m}{2^p}$; then

$$\langle H_{pm}, \dot{f} \rangle = \int H_{pm} df = \sqrt{2^p} \left[2f(t^1_{pm}) - f(t^0_{pm}) - f(t^2_{pm}) \right].$$

Hence

$$|\int H_{pm} df| \le c 2^{p(\frac{1}{2}-\alpha)} |f|_\alpha.$$

Since $||G_{pm}||_\infty = 2^{-p/2-1}$, Schauder functions of one family with disjoint support

$$\left\| \sum_{p \ge K} \sum_{m=0}^{2^p} \left(\int H_{pm} df \right) G_{pm} \right\|_\infty \le C 2^{-\alpha K} |f|_\alpha.$$

Thus series representation extends to closure of $\mathcal{H}$ w.r.t. $| \cdot |_\alpha$.

This is C^α, the space of α-Hölder continuous functions.

Ciesielski's isomorphism

Define

$$\chi_{pm} = 2^{\frac{p}{2}} H_{pm}, \;\; \varphi_{pm} = 2^{\frac{p}{2}} G_{pm}, \; p \geq 0, 0 \leq m \leq 2^p.$$

Then for $p \geq 0, 1 \leq m \leq 2^p$

$$f = \sum_{pm} \langle H_{pm}, df \rangle \, G_{pm} = \sum_{pm} \langle 2^{-p} \chi_{pm}, df \rangle \varphi_{pm} = \sum_{pm} f_{pm} \varphi_{pm}, \quad ||\varphi_{pm}||_\infty = \frac{1}{2},$$

with $f_{pm} = \langle 2^{-p} \chi_{pm}, df \rangle = 2f(t^1_{pm}) - f(t^0_{pm}) - f(t^2_{pm})$.

Since φ_{pm} vanishes at t^j_{pm} for $j = 0, 2$, this implies that

$$f_p = \sum_{q \leq p} \sum_{m=0}^{2^q} f_{qm} \varphi_{qn}$$

is the linear interpolation of f on the dyadic points $t^i_{pm}, i = 0, 1, 2, m = 0, ..., 2^m$.

Ciesielski's isomorphism

According to observation on previous slide

$$||f||_\alpha = \sup_{pm} 2^{p\alpha} |f_{pm}| = \sup_{pm} 2^{p(\alpha - \frac{1}{2})} |\langle H_{pm}, df \rangle| \sim |f|_\alpha.$$

Ciesielski: Map

$$T^\alpha : C^\alpha \to \ell^\infty, \qquad f \mapsto (2^{p\alpha} \; f_{pm})_{p \geq 0, 1 \leq m \leq 2^p}$$

isomorphism between a

function space and a *sequence space*.

Can be extended to **Besov spaces** $\mathrm{B}^\alpha_{p,q}$ normed by $|| \cdot ||_{\alpha,p,q}$: for a function $f : [0,1] \to \mathbf{R}, 0 < \alpha < 1, 1 \leq p, q \leq \infty, t \in [0,1]$

$$\omega_p(t,f) = \sup_{|y| \leq t} [\int_0^1 |f(x+y) - f(x)|^p dx]^{\frac{1}{p}}, \; ||f||_{\alpha,p,q} = ||f||_p + [\int_0^1 (\frac{\omega_p(t,f)}{t^\alpha})^q \frac{1}{t} dt]^{\frac{1}{q}}.$$

Lit: Ciesielski, Kerkyacharian, Roynette '93: study of Brownian motion on Besov spaces, **stochastic integral.**

Back to integration

Let now $f \in C^\alpha$, $g \in C^\beta$.

Then we may write

$$f = \sum_{p,m} f_{pm} \varphi_{pm}, \qquad g = \sum_{p,m} g_{pm} \varphi_{pm}.$$

The Schauder functions are piecewise linear, thus of bounded variation. Therefore it is possible to define

$$\begin{aligned} \int_0^t f(s) dg(s) &= \sum_{p,m,\ q,n} f_{pm} g_{qn} \int_0^t \varphi_{pm}(s) d\varphi_{qn}(s) \\ &= \sum_{p,m,\ q,n} f_{pm} g_{qn} \int_0^t \varphi_{pm}(s) \chi_{qn}(s) ds. \end{aligned}$$

To study the behaviour of the integrals on the rhs as functions of t we have to control for $i, j\ p, m\ q, n$

$$\langle 2^{-i} \chi_{ij}, \varphi_{pm} \chi_{qn} \rangle.$$

Universal estimate, Paley-Littlewood packages

Lemma 1
For $i, p, q \geq 0, 0 \leq j \leq 2^i, 0 \leq m \leq 2^p, 0 \leq n \leq 2^q$

$$|\langle 2^{-i}\chi_{ij}, \varphi_{pm}\chi_{qn}\rangle| \leq 2^{-2(i \vee p \vee q)+p+q},$$

except in case $p < q = i$, in which we have

$$|\langle 2^{-i}\chi_{ij}, \varphi_{pm}\chi_{qn}\rangle| \leq 1.$$

For $f = \sum_{pm} f_{pm}\varphi_{pm}$ as above let

$$\Delta_p f = \sum_{m=0}^{2^p} f_{pm}\varphi_{pm}, \quad S_p f = \sum_{q \leq p} \Delta_q f.$$

According to *Ciesielski's isomorphism*

$$f \in C^\alpha \text{ iff } ||f||_\alpha = \sup_p ||(2^{p\alpha}||\Delta_p f||_\infty)||_{l^\infty} < \infty.$$

Key lemma in Paley-Littlewood language

Corollary 1
f, g continuous functions. For $i, p, q \geq 0, 0 \leq j \leq 2^i, 0 \leq m \leq 2^p, 0 \leq n \leq 2^q$

$$||\Delta_i(\Delta_p f \Delta_q g)||_\infty \leq 2^{-(i \vee p \vee q) - i + p + q} ||\Delta_p f||_\infty ||\Delta_q g||_\infty,$$

except in case $p < q = i$, in which we have

$$||\Delta_i(\Delta_p f \Delta_q g)||_\infty \leq ||\Delta_p f||_\infty ||\Delta_q g||_\infty.$$

For $p > i$ or $q > i$ we have

$$\Delta_i(\Delta_p f \Delta_q g) = 0.$$

Decomposition of the integral

The corollary indicates that different components of the integral have different smoothness properties. We may write

$$\begin{aligned}\int f dg &= \sum_{p,q} \int \Delta_p f d\Delta_q g \\ &= \sum_{p<q} \int \Delta_p f d\Delta_q g + \sum_{p\geq q} \int \Delta_p f d\Delta_q g \\ &= \sum_{q} \int S_{q-1} f d\Delta_q g + \sum_{p} \int \Delta_p f d\Delta_p g + \sum_{p} \int \Delta_p f dS_{p-1} g.\end{aligned}$$

In view of the second part of Corollary 1, we expect the first part to be rougher. Integration by parts gives

$$\begin{aligned}\sum_q \int S_{q-1} f d\Delta_q g &= \sum_q S_{q-1} f \Delta_q g - \sum_q \int \Delta_q g dS_{q-1} f \\ &= \pi_<(f,g) - \sum_q \int \Delta_q g dS_{q-1} f.\end{aligned}$$

Decomposition of the integral

$\pi_<(f,g)$: *Bony paraproduct*

Defining further

$$L(f,g) = \sum_p (\Delta_p f dS_{p-1} g - \Delta_p g dS_{p-1} f),$$

(*antisymmetric Lévy area*)

$$S(f,g) = \sum_p \Delta_p f d\Delta_p g = c + \frac{1}{2} \sum_p \Delta_p f \Delta_p g$$

(*symmetric part*)

we have

$$\int f dg = \pi_<(f,g) + S(f,g) + L(f,g).$$

The Young integral

In case the Hölder regularity coefficients of f and g are large enough, the three components of the integral behave well.

According to the Corollary, we estimate for $i \geq 0$

$$||\Delta_i f \Delta_i g||_\infty \leq ||\Delta_i f||_\infty ||\Delta_i g||_\infty \leq 2^{-(\alpha+\beta)i} ||f||_\alpha ||g||_\beta.$$

This implies for any $\alpha, \beta \in]0,1[$

$$||S(f,g)||_{\alpha+\beta} \leq C||f||_\alpha ||g||_\beta.$$

Similarly

$$||\pi_<(f,g)||_\beta \leq C||f||_\infty ||g||_\beta.$$

and, **but only if** $\alpha + \beta > 1$

$$||L(f,g)||_{\alpha+\beta} \leq C||f||_\alpha ||g||_\beta.$$

The Young integral

We can summarize the findings above.

Thm (Young's integral)
Let $\alpha, \beta \in (0,1)$ be such that $\alpha + \beta > 1$, and let $f \in C^\alpha$ and $g \in C^\beta$. Then

$$I(f, dg) := \sum_{p,q} \int_0^\cdot \Delta_p f d\Delta_q g \in C^\beta \quad \text{and} \quad \|I(f,dg)\|_\beta \lesssim \|f\|_\alpha \|g\|_\beta.$$

Furthermore

$$\|I(f,dg) - \pi_<(f,g)\|_{\alpha+\beta} \lesssim \|f\|_\alpha \|g\|_\beta.$$

It is important to note that we get a version of the Lévy area only in case $\alpha + \beta > 1$. If f and g arise in the context of Brownian motion, we usually have only $\alpha, \beta < \frac{1}{2}$, and Lévy area has to be given externally.

Beyond Young's integral: an example

The following example illustrates for $\alpha + \beta < 1$ the Lévy area may fail to exist, and indicates what may be missing. Let $f, g\colon [-1,1] \to \mathbb{R}$ be given by

$$f(t) := \sum_{k=1}^{\infty} a_k \sin(2^k \pi t) \quad \text{and} \qquad g(t) := \sum_{k=1}^{\infty} a_k \cos(2^k \pi t),$$

where $a_k := 2^{-\alpha k}$ and $\alpha \in [0,1]$. For $m \in \mathbb{N}$ let $f^m, g^m\colon [-1,1] \to \mathbb{R}$ be the mth partial sum of the series. f^m, g^m are α-Hölder continuous uniformly in m. For $s, t \in [-1,1]$ and $k \in \mathbb{N}$ such that $2^{-k-1} \leq |s-t| \leq 2^{-k}$ with C independent of m by simple calculation:

$$|f^m(t) - f^m(s)| \leq C|t-s|^{\alpha}, \quad |g^m(t) - g^m(s)| \leq C|t-s|^{\alpha}.$$

Hence also f, g α-Hölder continuous.

Beyond Young's integral: an example

Lévy's area for (f^m, g^m) given by

$$\int_{-1}^{1} g^m(s)df^m(s) - \int_{-1}^{1} f^m(s)dg^m(s)$$

$$= \sum_{k,l=1}^{m} a_k a_l \int_{-1}^{1} \big(\sin(2^k\pi s)\sin(2^l\pi s)2^l\pi + \cos(2^l\pi s)\cos(2^k\pi s)2^k\pi\big)ds$$

$$= \sum_{k,l=1}^{m} a_k a_l \Big(2^l\pi \int_{-1}^{1} \frac{1}{2}(\cos((2^k-2^l)\pi s) - \cos((2^k+2^l)\pi s))ds$$

$$+2^k\pi \int_{-1}^{1} (\cos((2^k-2^l)\pi s) + \cos((2^k+2^l)\pi s))ds\Big)$$

$$= 2\sum_{k=1}^{m} a_k^2 2^k\pi = 2\sum_{k=1}^{m} 2^{(1-2\alpha)k}\pi.$$

This diverges as m tends to infinity for $\alpha \leq \frac{1}{2}$. Hence (f, g) possesses no Lévy area.

Beyond Young's integral: an example

Note that for $-1 \leq s \leq t \leq 1$, and $0 \neq f^g(s) \in \mathbb{R}$ by trigonometry

$$|f(t) - f(s) - f^g(s)(g(t) - g(s))|$$
$$= \left|2 \sum_{k=1}^{\infty} a_k \sin(2^{k-1}\pi(s-t)) \sqrt{1 + f^g(s)^2} \sin[2^{k-1}\pi(s+t) + \arctan((f^g(s))^{-1})]\right|.$$

Hölder regularity for $s = 0, t = 2^{-n}$, $f^g(0) > 0$ ($f^g(0) < 0$ analogous): quantity $\geq$

$$\left|2 \sum_{k=1}^{n} a_k \sin(2^{k-1-n}\pi) \sqrt{1 + (f^g(0))^2} \sin[2^{k-1-n}\pi + \arctan((f^g(0))^{-1})]\right|$$
$$\geq 2^{-\alpha n} \sin\left(\frac{\pi}{2} + \arctan((f^g(0))^{-1})\right)$$
$$\neq \mathcal{O}(|t-s|^{2\alpha}).$$

Beyond Young's integral

Hölder regularity at 0 not better than α; hence f not controlled by g for $\alpha < \frac{1}{2}$ in the sense of following notion.

(para)controlled path formalizes heuristics of *fractional Taylor expansion.*

For $\alpha > 0$ let $x \in C^\alpha$. Then

$$\mathbf{D}_x^\alpha = \left\{ f \in C^\alpha : \exists f^x \in C^\alpha \text{ s.t. } f^\sharp = f - \pi_<(f^x, x) \in C^{2\alpha} \right\}.$$

$f \in \mathbf{D}_x^\alpha$ is called *controlled* by x, f^x *derivative of* f w.r.t. x. On $\mathbf{D}_x^\alpha$ define norm

$$||f||_{x,\alpha} = ||f||_\alpha + ||f^x||_\alpha + ||f^\sharp||_{2\alpha}.$$

If $\alpha > 1/3$, then since $3\alpha > 1$ the term $L(f - \pi_<(f^x, x), x)$ is well defined. It suffices to make sense of $L(\pi_<(f^x, x), x)$. This is done by *commutator estimate*:

$$||L(\pi_<(f^x, x), x) - \int_0^\cdot f^x(s) dL(x, x)(s)||_{3\alpha} \leq ||f^x||_\alpha ||x||_\alpha^2,$$

and the integral is well defined **provided** $L(x, x)$ **exists.**

Beyond Young's integral

Thm (Rough path integral)
Let $\alpha \in (1/3, 1)$, $\alpha \neq 1/2$, $\alpha \neq 2/3$. Let $x \in C^\alpha$, $f, g \in \mathbf{D}_x^\alpha$. Assume that the Lévy area

$$L(x, x) := \lim_{N \to \infty} \big(L(S_N x^k, S_N x^\ell)\big)_{1 \leq k \leq d, 1 \leq \ell \leq d}$$

converges uniformly, such that $\sup_N \|L(S_N x, S_N x)\|_{2\alpha} < \infty$. Then

$$I(S_N f, dS_N g) = \sum_{p \leq N} \sum_{q \leq N} \int_0^{\cdot} \Delta_p f(s) d\Delta_q g(s)$$

converges in $C^{\alpha-\varepsilon}$ for all $\varepsilon > 0$. Denote the limit by $I(f, dg)$. Then $I(f, dg) \in \mathbf{D}_x^\alpha$ with derivative $f g^x$, and

$$||I(f, dg)||_{x,\alpha} \lesssim ||f||_{x,\alpha} \big(1 + ||g||_{x,\alpha}\big) \big(1 + ||x||_\alpha + ||x||_\alpha^2 + ||L(x, x)||_{2\alpha}\big).$$

An approximation of the Lévy area

Aim: approximate Lévy's area by dyadic martingales.

Filtration:

$$\mathcal{F}_q = \sigma(\chi_{2^k+l} : k \le q, l \le 2^k - 1), \qquad q \ge 0.$$

Martingales:

$$M_q^f = \sum_{p \le q} \sum_{m < 2^p} \langle \chi_{pm}, df \rangle \chi_{pm},$$

$$N_q^g = \sum_{p \le q} \sum_{m < 2^p} \langle \chi_{pm}, dg \rangle \chi_{pm}.$$

Rademacher functions: $r_q = 2^{-q/2} \sum_{n < 2^q} \chi_{qn}$ and the associated martingale $R_q = \sum_{p \le q} r_q$.

An approximation of the Lévy area

Discrete time stochastic integral of X, Y:

$$(X \cdot Y)_n = \sum_{k \leq n} X_{k-1} \Delta Y_k = \sum_{k \leq n} X_{k-1}(Y_k - Y_{k-1}).$$

Then

$$\begin{aligned} I(S_k f, dS_k g) =& \sum_{0<q<k} 2^{-q-2} \mathbf{E} \left[\Delta R_q (N^g_{q-1} \Delta M^f_q - M^f_{q-1} \Delta N^g_q) \right] \\ & + \frac{1}{2} f_{00} g_{00} + O(2^{-\alpha k}) \\ & = \sum_{0<q<k} 2^{-q-2} \mathbf{E} \left[\Delta [R, N^g \cdot M^f - M^f \cdot N^g]_q \right] \\ & + \frac{1}{2} f_{00} g_{00} + O(2^{-\alpha k}). \end{aligned}$$

Expectation is w.r.t. Lebesgue measure on $[0, 1]$.

An approximation of the Lévy area

A discrete analogue of Lévy's area appears naturally.

Geometric interpretation:

Proposition 1. *M, N discrete time processes. Denote linear interpolation by X, Y respectively:*

$$X_s = M_{k-1} + (s - (k-1))\Delta M_k, \quad s \in [k-1, k],$$

similarly for Y. Then discrete time Lévy area

$$\frac{1}{2}\{((M - M_0) \cdot N)_n - ((N - N_0) \cdot M)_n\}$$

equals the area between the curve $\{(X_s, Y_s) : s \leq n\}$ and line chord from (M_0, N_0) to (M_n, N_n).

Construction of the Lévy area

For a d-dimensional process $X = (X^1, \dots, X^d)$ we construct the area $L(X,X) = L(X^i, X^j)_{1 \le i,j \le d}$. Assume the components are independent. Let $R(s,t) = (\mathbf{E}(X_s^i X_t^j))_{1 \le i,j \le d}$. Increment of R over rectangle $[s,t] \times [u,v]$

$$R_{[s,t]\times[u,v]} = R(t,v) + R(s,u) - R(s,v) - R(t,u) = (\mathbf{E}(X_{s,t}^i X_{u,v}^j))_{1 \le i,j \le d}.$$

Let us make the following assumptions.

(ρ-var) There exist $\rho \in [1,2)$ and $C > 0$ such that for all $0 \le s < t \le 1$ and for every partition $s = t_0 < t_1 < \cdots < t_n = t$ of $[s,t]$

$$\sum_{i,j=1}^{n} |R_{[t_{i-1},t_i]\times[t_{j-1},t_j]}|^\rho \le C|t-s|.$$

(HC) The process X is hypercontractive, i.e. for every $m, n \in \mathbf{N}$ and every $p > 2$ there exists $C_{p,m,n} > 0$ such that for every polynomial $P : \mathbb{R}^n \to \mathbb{R}$ of degree m, for all $i_1, \cdots, i_n$, and for all $t_1, \dots, t_n \in [0,1]$

$$\mathbf{E}(|P(X_{t_1}^{i_1}, \dots, X_{t_n}^{i_n})|^{2p}) \le C_{p,m,n} \mathbf{E}(|P(X_{t_1}^{i_1}, \dots, X_{t_n}^{i_n})|^2)^p.$$

Construction of the Lévy area

Lemma 3
Assume that the stochastic process $Y : [0,1] \to \mathbb{R}$ satisfies (ρ-var). Then for all p and for all $0 \leq M \leq N \leq 2^p$

$$\sum_{m_1, m_2 = M}^{N} |\mathbf{E}(X_{pm_1} X_{pm_2})|^{\rho} \lesssim (N - M + 1) 2^{-p}.$$

Lemma 4
Let $Y, Z : [0,1] \to \mathbb{R}$ be independent continuous processes, both satisfying (ρ-var) for some $\rho \in [1, \infty]$. Then for all $i, p \geq 0$ and all $q < p$, and for all $0 \leq j \leq 2^i$

$$\mathbf{E}\left[\left|\sum_{m=0}^{2^p} \sum_{n=0}^{2^q} X_{pm} Y_{qn} \langle 2^{-i} \chi_{ij}, \varphi_{pm} \chi_{qn} \rangle\right|^2\right] \lesssim 2^{(p \vee i)(1/\rho - 4)} 2^{(q \vee i)(1 - 1/\rho)} 2^{-i} 2^{p(4 - 3/\rho)} 2^{q/\rho}$$

Construction of the Lévy area

Thm 2
Let $X : [0,1] \to \mathbb{R}^d$ be a continuous stochastic process with independent components, and assume that X satisfies (ρ-var) for some $\rho \in [1,2)$ and (HC). Then for every $\alpha \in (0, 1/\rho)$ almost surely

$$\sum_{N \geq 0} \|L(S_N X, S_N X) - L(S_{N-1} X, S_{N-1} X)\|_\alpha < \infty,$$

and therefore the limit $L(X,X) = \lim_{N\to\infty} L(S_N X, S_N X)$ is almost surely an α-Hölder continuous process.

Condition (HC) is fulfilled by all Gaussian processes, also by all processes in fixed Gaussian chaos (Hermite processes), (ρ-var) by fractional Brownian motion or bridge of Hurst index $H > \frac{1}{4}$.