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On stochastic partial differential equations of parabolic type

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I. Filtering problem

II. Equations of nonlinear filtering

III. Some problems in filtering theory

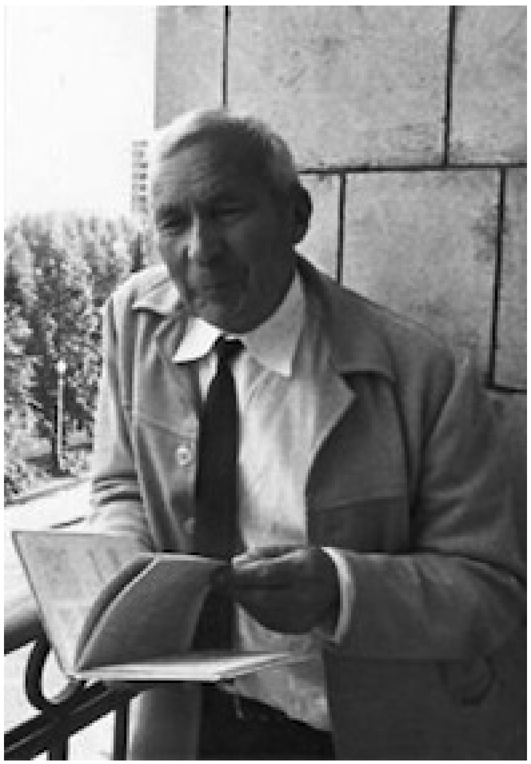
- Innovation problem

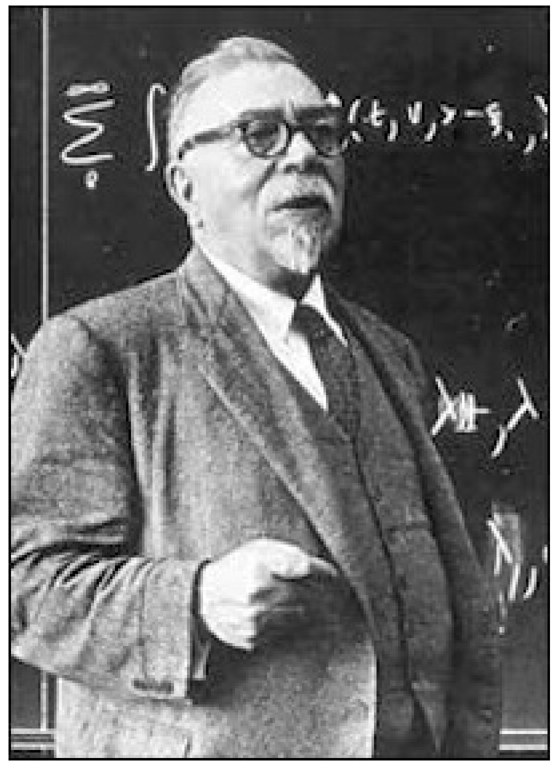
- Robustness of the filter

- Accelerated numerical schemes

I. Filtering theory

1. **Estimating future values of signals**
in the framework of stationary processes





2. **State-space filtering model**

$$dX_t = bX_t\,dt + \theta\,dW_t, \qquad X_0 = \xi \sim N(m, R)$$

$$dY_t = BX_t\,dt + dV_t, \qquad Y_0 = \eta \sim N(\bar{m}, \bar{R}),$$

$(W_t, V_t)_{t\geq 0}$ is multi-dimensional Wiener process, b, B and θ are given matrices.

Task:

At t estimate $\varphi(X_t)$ from observations $(Y_s)_{s\in[0,t]} =: \mathcal{Y}_t$. with smallest mean square error; i.e., let $\widehat{\varphi}$ denote the optimal estimate

$$E|\widehat{\varphi} - \varphi(X_t)|^2 = \min_f E|f(\mathcal{Y}_t) - \varphi(X_t)|^2 .$$

Clearly,

$$\widehat{\varphi} = P_t(\varphi) := E(\varphi(X_t)|Y_s, s \le t) = \int_{\mathbb{R}^d} \varphi(x) P_t(dx)$$

$$= \int_{\mathbb{R}^d} \varphi(x) \pi_t(x)\, dx,$$

where

$$P_t(dx) := P(X_t \in dx | Y_s, s \le t) = p_t(x)\, dx.$$

Note that (X_t, Y_t) is Gaussian process. Hence given $\mathcal{Y}_t$, $P_t(dx)$ is also Gaussian. It is sufficient to determine its mean, m_t and its covariance C_t, or equivalently,

$$m_t = E(X_t|Y_s, s \le t), \quad \gamma_t = E\{(X_t - m_t)(X_t - m_t)^*\}.$$

$$dm_t = bm_t\,dt + \gamma_t B^*\,d\bar{V}_t, \quad m_0 = EX_0$$

$$\dot{\gamma}_t = a\gamma_t + \gamma_t a^* - \gamma_t B^* B \gamma_t + \theta\theta^*, \quad \gamma_0 = E\{(X_0 - m_0)(X_0 - m_0)^*\},$$

$$\bar{V}_t := Y_t - \int_0^t B m_s\,ds.$$

Remark. $(\bar{V}_t)_{t\ge 0}$ is called innovation process. It is a Wiener process.

Interpretation:

$$\bar{V}_{t+\Delta t} - \bar{V}_t = Y_{t+\Delta t} - Y_t - Hm_t\,\Delta t$$

carries new information, relative to $(Y_s)_{s\in[0,t]}$. Clearly, for each $t \geq 0$

$$\sigma(\bar{V}_s : s \leq t) \subset \sigma(Y_s : s \leq t).$$

Innovation problem: $\sigma(\bar{V}_s : s \leq t) = \sigma(Y_s : s \leq t)$? Does the innovation process carry the same information as the observation process?

3. Nonlinear filtering

State: $dX_t = b(Z_t)\,dt + \theta(Z_t)\,dW_t + \rho(Z_t)\,dV_t, \qquad X_0 = \xi$

Observation: $dY_t = B(Z_t)\,dt + dV_t, \quad Y_0 = 0,$

where $Z_t := (X_t, Y_t) \in \mathbb{R}^{d+d_1}$, (W, V) is a multidimensional Wiener; b and B are Lipschitz continuous vector fields, and θ and ρ are Lipschitz continuous matrix fields on $\mathbb{R}^{d+d_1}$; ξ is a random vector, independent of (W, V).

As we know, for functions of X_t, $\varphi(X_t)$, the optimal estimator, given the observations $(Y_t)_{s\in[0,t]}$, is

$$P_t(\varphi) := E(\varphi(Z_t)|Y_s, s \le t).$$

Innovation process: $\bar{V}_t = Y_t - \int_0^t P_s(B)\,ds$.

Questions:

1. Innovation problem: $\sigma(\bar{V}_s, s \leq t) = \sigma(Y_s, s \leq t)$?

2. How to compute $P_t(\varphi)$?

3. Robustness of the filter: does the computation of $P_t(\varphi)$ depend continuously on the observations?

4. How to calculate $P_t(\varphi)$ numerically?

1. Innovation problem

Assumption 1. There is $\delta > 0$ such that $\rho\rho^* \geq \delta I$ for all $\lambda \in \mathbb{R}^d$, $z \in \mathbb{R}^{d+d_1}$, where

$$(\rho\rho^*)^{ij} := \rho^{ir}\rho^{jr}.$$

Assumption 2. ξ has a probability density $\pi_0 \in H^1_p$ for some $p \geq 2$,

Assumption 3. $(1 + |x|^2)^\alpha p_0 \in L_2$, for some $\alpha > d/2$.

Theorem 1.(N.V. Krylov, I.G.) Let Assumptions 1-3 hold. Then $\sigma(\bar{V}_s, s \leq t) = \sigma(Y_s, s \leq t)$.

Generalizes a result of N.V. Krylov 1979.

2. Equations of nonlinear filtering

Some notation:

$$b_t(x) := b(x, Y_t), \quad \rho_t(x) = \rho(x, Y_t), \quad \theta_t(x) := \theta(x, Y_t),$$

$$B_t(x) = B(x, Y_t), \quad a_t(x) := (\rho_t \rho_t^*(x) + \theta_t \theta_t^*(x))/2, \quad x \in \mathbb{R}^d$$

$$L_t = a_t^{ij}(x) D_i D_j + b_t(x)^i D_i, \quad M^r = \theta_t^{ir}(x) D_i + B_t^i(x)$$

Filtering equation:

$$dP_t(\varphi) = P_t(L\varphi)\, dt + \{P_t(M^r\varphi) - P_t(B_t^r)P_t(\varphi)\}\, d\bar{V}_t^r \quad (1)$$
$$P_0(\varphi) = E(\varphi(X_0)), \quad (2)$$

where $\bar{V}$ is the innovation process. If $\pi_t(x) := P_t(dx)/dx$ exists, then it satisfies

$$d\pi_t(x) = L_t^* \pi_t(x)\, dt + \{M_t^{r*}\pi_t(x) - (\pi_t, B_t^r)\pi_t(x)\}\, d\bar{V}_t^r,$$
$$\pi_0(x) = P(X_0 \in dx)/dx =: p_0.$$

(*Kushner-Shiryayev equation*)

Idea of the proof of Thm1: Consider the system

$$\begin{aligned} d\pi_t(x) =& L_t^* \pi_t(x)\, dt \\ &+ \{M_t^{r*} p_t(x) - (\pi_t, B_t^r)\pi_t(x)\}\, d\bar{V}_t^r, \quad \pi_0 = p_0 \\ dY_t =& (\pi_t, B_t)\, dt + d\bar{V}_t^r, \quad Y_0 = 0. \end{aligned}$$

Step 1. Approximation $\pi^{(0)} = p_0$, $Y^{(0)} = 0$, $n = 1, 2, ...$

$$d\pi_t^{(n)} = L^{(n)} \pi_t^{(n)}\, dt + \{(M^{(n)} \pi_t^n - (\pi^{(n-1)}, B^n)\pi^n\}\, d\bar{V}_t^r$$

$$dY_t^{(n)} = (\pi_t^{(n-1)}, B_t^{(n)})\, dt + d\bar{W}_t.$$

Step 2. $\exists$ $(\pi_t^{(n)}, Y_t^{(n)})$, which is $\sigma(\bar{V}_s : s \leq t)$-measurable.

Step 3. Show that in probability

$$\sup_{t\in[0,T]} |\pi_t^{(n)} - \pi_t|_{L_2} \to 0, \qquad \sup_{t\in[0,T]} |Y_t^{(n)} - Y_t| \to 0.$$

Existence and analytic property of the density π_t.

The *Kushner-Shiryayev equation* can be transform into a linear SPDE (*Zakai equation*):

$$\begin{aligned} du_t(x) =& L^* u_t(x)\, dt + M^{r*} u_t(x)\, dY_t^r \\ u_0(x) =& p_0(x) \end{aligned}$$

Under Assumptions 1 and 2 $\exists$ a (generalised) solution $(u_t)_{t \in [0,T]} \in L_p([0,T], H_p^1)$, $(u_t, 1) > 0$ and

$$\pi_t(x) = \frac{u_t(x)}{(u_t, 1)}.$$

3. Robustness of the filter

Let $Y^{(n)}$ be continuous processes of bounded variation, $Y_0^{(n)} = 0$.
Question: $\sup_{t \le T} |Y_t^{(n)} - Y_t| \to 0 \Rightarrow \pi_t^{(n)} \to \pi_t$?

Consider

$$
\begin{aligned}
du_t^{(n)}(x) =& L^{*(n)} u_t^{(n)}(x)\, dt + M^{r*(n)} u_t^{(n)}(x)\, dY_t^{r(n)}, \\
u_0(x) =& p_0(x)
\end{aligned}
$$

Define $A_t^{(n)ij} := \int_0^t Y_s^{(n)i}\, dY_s^{(n)j} - \int_0^t Y_s^{(n)j}\, dY_s^{(n)i}$,

$$A_t^{ij} := \int_0^t Y_s^i\, dY_s^j - \int_0^t Y_s^j\, dY_s^i, \quad i, j = 1, ..., d_1.$$

$$S_t^{(n)ij} := \int_0^t (Y_s^i - Y_s^{(n)i}) dY_s^{(n)j}.$$

Theorem 2. (I.G. 1988) Let $m \geq 0$. Let

(i) $\sup_{t\in[0,T]} |Y_t^{(n)} - Y_t| \to 0$, $\sup_{t\in[0,T]} |S_t^{(n)} - S_t| \to 0$,

$$||S^{(n)}||_T = o(\ln n)$$

(ii) $b(x,y)$, $\rho(x,y)$ sufficiently smooth in x;
$B(x,y)$, $\theta(x,y)$ are sufficiently smooth in (x,y),

(iii) $p_0 \in H_2^k$ with sufficiently high k.

Then $\sup_{t\in[0,T]} |u_t^{(n)} - \tilde{u}_t|_{H_2^m} = 0$ (in probability),

where $\tilde{u}$ is the solution of

$$d\tilde{u}_t = \tilde{L}_t \tilde{u}\, dt + M_t^{r*} \tilde{u}_t\, dY_t^r, \quad \tilde{u}_0 = p_0,$$

$$\tilde{L}_t := L_t^* + \tfrac{1}{2} M_t^{r*} M_t^{r*} + \sum_r N_t^{r*}, \quad N_t^r = \theta_{y^r}^{ir}(x, Y_t) D_i + B_{y^r}^r(x, Y_t).$$

Consider with $\mathcal{L}_t^{(n)} := L_t^{(n)} - \frac{1}{2} M_t^{r(n)*} M_t^{r(n)*} + \sum_r N_t^{r(n)*}$,

$$du_t^{(n)}(x) = \mathcal{L}_t^{(n)} u_t^{(n)}(x)\,dt + M^{r*(n)} u_t^{(n)}(x)\,dY_t^{r(n)}$$
$$u_0(x) = p_0(x).$$

Then $\exists$ H^m-valued unique solutions u and $u^{(n)}$.

$$\sup_{t\in[0,T]} |u_t^{(n)} - u_t|_{H_2^m} \to 0, \qquad \sup_{t\in[0,T]} |\pi_t^{(n)} - \pi_t|_{H_2^m} \to 0,$$

where $\pi^{(n)} := u^{(n)}/(u^{(n)}, 1)$.

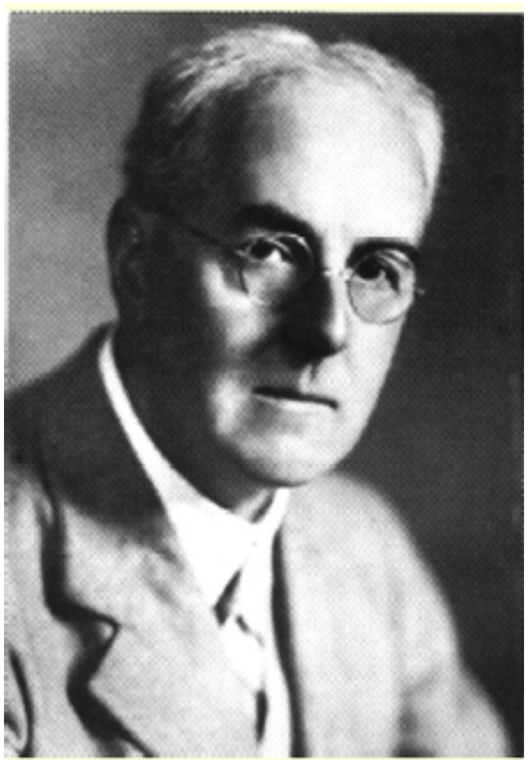
Theorem 3. (P. Stinga, I.G. 2013) For some $\kappa > 0$ assume

$$\sup_{t\le T} |Y_t^{(n)} - Y_t| = O(n^{-\kappa}), \quad \sup_{t\le T} |S_t^{(n)} - S_t| = O(n^{-\kappa}) \quad (a.s.).$$

Then for any $\gamma < \kappa$

$$\sup_{t\le T} |u_t^{(n)} - u_t|_{H_2^m} = O(n^{-\gamma}), \quad |\pi_t^{(n)} - \pi_t|_{H_2^m} = O(n^{-\gamma}).$$

4. Accelerated numerical schemes



Richardson's idea:
Assume for $q(h) \approx q$ and for $h \to 0$ we have

$$q(h) = q + q_1 h + O(h).$$

Then $\bar{q}(h) := 2q(h/2) - q(h) = q + O(h^2)$.

More generally, assume

$$q(h) = q + q_1 h + q_2 \frac{h^2}{2} + ... + q_k \frac{h^k}{k!} + O(h^{k+1}),$$

then

$$\bar{q}(h) := \lambda_0 q(h) + \lambda_1 q(h/2) + ... + \lambda_k q(h/2^k) = q + O(h^{k+1}),$$

with constants $\lambda_0, \lambda_1, ..., \lambda_k$, defined by

$$(\lambda_0, \lambda_1, ..., \lambda_k) = (1, 0, ..., 0) V^{-1},$$

where V^{-1} is the inverse of $V = (V^{ij})$,

$$V^{ij} = 2^{-(i-1)(j-1)}, \quad j = 1, 2, ..., k+1.$$

Accelerated finite difference schemes

Notation: For $h > 0$, vectors e_i, $x \in \mathbb{R}^d$

$$\delta_{h,e_i}\varphi(x) := \frac{1}{h}(\varphi(x+he_i)-\varphi(x)), \quad \delta_i^h\varphi(x) := \frac{1}{2}(\delta_{h,e_i}+\delta_{-h,e_i})$$

$$D_i \rightsquigarrow \delta_i^h, \quad L_t^* \rightsquigarrow L^h, \quad M^{r*} \rightsquigarrow M^{h,r}$$

Consider

$$\begin{aligned} du_t^h(x) =& L_t^h u_t^h(x)\,dt + M_t^{h,r} u_t^h(x)\,dY_t^r \\ u_0^h(x) =& p_0(x) \end{aligned}$$

$t \in [0, T]$, $x \in \mathbb{G}_h := h\mathbb{Z}^d$.

Infinite system of SDEs

Truncated finite difference schemes

For $R > 0$ let $\zeta_R \in C_0^\infty(\mathbb{R}^d)$, s.t., $\zeta_R(x) = 1$ if $|x| \leq R$, $\zeta_R(x) = 0$ if $|x| \geq \rho > R$.

Set $L_R^h := \varphi_R L^h$, $M_R^{h,r} := \varphi_R M^{h,r}$, $p_0^R = p_0 \zeta_R$.

Consider

$$
\begin{aligned}
du_t^{h,R}(x) =& L_{R,t}^h u_t^{h,R}(x)\, dt + M_{R,t}^{h,r} u_t^{h,R}(x))\, dY_t^r \\
u_0^{h,R}(x) =& p_0^R(x)
\end{aligned}
$$

for $t \in [0, T]$ and $x \in \mathbb{G}_{h,\rho} := \mathbb{G}_h \cap \{|x| \leq \rho\}$.

Finite system of linear SDEs. $\exists$ a unique solution $u^{h,R}$. Set $v^{h,R} := \sum_{j=0}^k \lambda_j u^{h/2^j,R}$.

Theorem 4.(M. Gerencsér, N.V. Krylov, I.G.) Let $k \geq 0$. Assume

(i) b, ρ, θ, B are sufficiently smooth

(ii) $p_0 \in H_2^k$

Then the Zakai equation has a *classical solution* u, and for $R > 0$, $\kappa \in (0,1)$, $q > 0$

$$E \sup_{t\in[0,T]} \sup_{x\in\mathbb{G}_h\cap\{|x|\leq\kappa R\}} |v_{R,t}^h(x) - u_t(x)|^q \leq N_1 h^{q(k+1)} + N_2 e^{-\nu R^2}$$

with positive constants N_1, N_2, ν, independent of h.

Theorem 5. Assume (i), (ii) and $(1+|x|^2)^\alpha p_0 \in L_2$ for some $\alpha > d$. Then for

$$\bar{\pi}_t^{R,h}(x) := \frac{v^{h,R}(x)}{(v^{h,R},1)}$$

we have

$$E \sup_{t\in[0,T]} \sup_{x\in\mathbb{G}_h\cap\{|x|\leq\kappa R\}} |\pi^h_{R,t}(x)-\pi_t(x)|^q \leq N_1 h^{q(k+1)}+N_2 e^{-\nu R^2}$$

with positive constants N_1, N_2, ν, independent of h.

Summary: Via problems from filtering theory we showed a tiny bit of the theory and its applications. of the theory of parabolic SPDEs.

Moltes Gràcies !