Barcelona Probability Seminar

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On numerical solutions of degenerate stochastic PDEs

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Based on joint work with N.V. Krylov and M. Gerencsér

I. Solvability of degenerate SPDEs in L_p spaces

II. Stochastic finite diff. schemes

- Solvability and estimates in L_p-spaces
- Rate of convergence, accelerated schemes
- Truncated schemes

I. Solvability of degenerate SPDEs in L_p spaces

$$du_t = (Lu_t + f_t)\,dt + (M^r u_t + g_t^r)\,dw_t^r, \qquad (1)$$

on $H_T := [0, T] \times \mathbb{R}^d$, with

$$u_0(x) = \psi(x), \quad x \in \mathbb{R}^d, \qquad (2)$$

with independent $(\mathcal{F}_t)_{t \geq 0}$-Wiener processes $(w^r)_{r=1}^{\infty}$,

$$L = a^{ij} D_{ij} + b^i D_i + c, \quad M^r = \sigma^{ri} D_i + \nu^r,$$

$$a = a_t^{ij}(\omega, x), \quad b = b_t^i(\omega, x), \quad c = c_t(\omega, x),$$

$$(\sigma_t^{ir}(\omega, x))_{r=1}^{\infty} \in l_2, \quad (\nu_t^r(\omega, x))_{r=1}^{\infty} \in l_2,$$

$$\psi \in L_p, \quad f_t \in L_p, \quad (g_t^r)_{r=1}^{\infty} \in L_p,$$

$t \in [0, T]$, $i, j = 1, 2, ..., d$, $x \in \mathbb{R}^d$, $p \geq 2$.

Assumption 1 (stochastic parabolicity).

$$\hat{a} \geq 0, \quad \omega \in \Omega,\, t \in [0,T],\, x \in \mathbb{R}^d.$$

$$\hat{a}^{ij} := a^{ij} - \frac{1}{2}\sum_{r=1}^{\infty} \sigma^{ir}\sigma^{jr}, \quad i,j = 1,2,...,d$$

Degenerate parabolic SPDE

Aim: Existence, uniqueness, regularity
and finite difference approximations of the solutions
in L_p spaces under stochastic parabolicity

Historical remarks:

(a) Deterministic PDEs: $du_t = (Lu_t + f_t)\, dt, \quad (a^{ij}) \geq 0$

• Solvability in L_2-spaces

O. A. Oleĭnik (1965), O.A. Oleĭnik–V.A. Radkevič (1966)

• Solvability in L_p spaces (for SPDEs):

N. V. Krylov–B.L. Rozovskii (1982),

N.V. Krylov–I.G. (2003)

M. Gerencsér–N.V. Krylov–I.G. (2014)

Our first aim is to improve the results of Krylov-Rozovskii (1982) and N.V. Krylov–I.G. (2003) and also to extend them to systems of SPDEs.

Definition. A W^1_p-valued $u = (u_t)_{t\in[0,T]}$ is a solution to (1)-(2) if almost surely

$$\int_0^T |u_t|^p_{W^1_p}\, dt < \infty,$$

$$\begin{aligned}(u_t,\varphi) =&(\psi,\varphi) + \int_0^t \{-(a^{ij}D_i u_s, D_j\varphi) \\ &+ (\widehat{b}^i D_i u_s + c u_s + f_s, \varphi)\}\, ds \\ &+ \int_0^t (M^r u_s + g^r_s, \varphi)\, dw^r_s\end{aligned}$$

for $t \in [0,T]$, $\varphi \in C^\infty_0(\mathbb{R}^d)$, where $\widehat{b}^i = b^i - D_j a^{ij}$, $(,)$ is the inner product in $L_2(\mathbb{R}^d)$.

Let $K \geq 0$ be a constant, $p \geq 2$.

Assumption 2.

$$|a|_{C^2} \leq K, \quad |b|_{C^1} \leq K, \quad |c|_{C^1} \leq K,$$

$$|\sigma|_{C^2} \leq K, \quad |\nu|_{C^2} \leq K.$$

Assumption 3. $\psi \in W_p^1$, $\mathcal{F}_0$-measurable;

$$\mathcal{K}^p(T) := \int_0^T |f_t|_{W_p^1}^p + |g_t|_{W_p^2}^p \, dt < \infty \, (a.s.).$$

Theorem 1. (M. Gerencsér, N.V. Krylov, I.G. 2014) Let Assumptions 1-3 hold. Then $\exists!$ solution u. Moreover, u is weakly continuous W_p^1-valued, strongly continuous L_p-valued process, and for every $q > 0$

$$E \sup_{t\in[0,T]} |u_t|^q_{W_p^1} \leq N(E|\psi|^q_{W_p^1} + E\mathcal{K}^q(T)) \tag{3}$$

with $N = N(p, q, d, K, T)$.

More regular data–more regular solution:

Let $m \geq 1$, $\bar{m} = \max(m, 2)$

Assumption 2_m.

$$|a|_{C^{\bar{m}}} \leq K, \quad |b|_{C^m} \leq K, \quad |c|_{C^m} \leq K,$$

$$|\sigma|_{C^{m+1}} \leq K, \quad |\nu|_{C^{m+1}} \leq K.$$

Assumption 3_m**.** $\psi \in W^m_p$, $\mathcal{F}_0$-measurable;

$$\mathcal{K}^p_m(T) := \int_0^T |f_t|^p_{W^m_p} + |g_t|^p_{W^{m+1}_p} \, dt < \infty.$$

Theorem 2. Let Assumptions 1, 2_m and 3_m hold. Then u is weakly continuous W^m_p-valued, strongly continuous W^{m-1}_p-valued process, and for every $q > 0$

$$E \sup_{t \in [0,T]} |u_t|^q_{W^m_p} \leq N(E|\psi|^q_{W^{m+1}_p} + E\mathcal{K}^q_m(T)) \qquad (4)$$

with $N = N(m, p, q, d, K, T)$.

To prove Theorems 1 and 2 we need to estimate the L_p norm of D_1u,...D_du. Since $Du := (D_1u, ..., D_du)$ satisfies a system of SPDEs, it is natural to consider systems of parabolic SPDEs, and Thm1 is proved for a class of parabolic system od SPDEs.

II. Stochastic Finite Difference Schemes

Consider (1)-(2). The finite difference approximations (in spatial variables) are defined as follows:
For a finite set $\Lambda \subset \mathbb{R}^d \setminus \{0\}$, and $h > 0$ define

$$\mathbb{G}_h = \{h(\lambda_1 + ... + \lambda_n) : \lambda_i \in \Lambda \cup \{-\Lambda\}, n = 1, 2, ...\},$$

$$\delta_{h,\lambda}\varphi(x) = \frac{1}{h}(\varphi(x + h\lambda) - \varphi(x)), \quad \delta^h_\lambda = \frac{1}{2}(\delta_{h,\lambda} + \delta_{-h,\lambda});$$

$$\begin{aligned} du^h_t(x) =& (L^h_t u^h_t(x) + f_t(x))\,dt \\ & + (M^{h,r}_t u_t(x) + g^r_t(x))\,dw^r_t \qquad (5) \\ u^h_0(x) =& \psi(x) \qquad (6) \end{aligned}$$

for $t \in [0, T]$ and $x \in \mathbb{G}_h$, where

$$L^h = \sum_{\lambda,\gamma\in\Lambda} \alpha^{\lambda\gamma}\delta^h_\lambda\delta^h_\gamma + \sum_{\lambda\in\Lambda} \beta^\lambda\delta^h_\lambda + \sum_{\lambda\in\Lambda} (\mathfrak{p}^\lambda\delta_{h,\lambda} + \mathfrak{q}^\lambda\delta_{h,-\lambda}) + c, \tag{7}$$

$$M^{h,r} = \sum_{\lambda\in\Lambda} \mu^{\lambda r}\delta^h_\lambda + \nu^r, \quad r = 1, 2, ..., \tag{8}$$

with bounded real-valued $\alpha^{\lambda\gamma} = \alpha^{\gamma\lambda}$, β^λ, $\mathfrak{p}^\lambda$, $\mathfrak{q}^\lambda$; bounded l_2-valued $\sigma^\lambda = (\sigma^{\lambda r})_{r=1}^\infty$, $\nu^\lambda = (\nu^{\lambda r})_{r=1}^\infty$, on $\Omega \times H_T$, for all $\lambda, \gamma \in \Lambda$.

Aim: Show the existence of an expansion of u^h in h:

$$u_t^h(x) = \sum_{j=0}^{k} \frac{h^j}{j!} u_t^{(j)}(x) + h^{k+1} r_t^h(x), \tag{9}$$

where $u^{(0)} = u$, $u^{(1)}$,...,$u^{(k)}$ are random fields on H_T, independent of h, and r^h is a random field on H_T s.t.

$$E \sup_{t \le T} \sup_{x \in \mathbb{G}_h} |r_t^h(x)|^q \le N \tag{10}$$

with a constant N independent of h.

• expansion with $k = 0$ gives

$$|u_t^h(x) - u_t(x)| \le h r_t^h(x),$$

and the estimate on r yields

$$E \sup_{t \in [0,T], x \in \mathbb{G}_h} |u_t^h(x) - u_t(x)|^q \le N h^q$$

• expansion with $k = 1$: $u^h = u + h u^{(1)} + h^2 r^h$ gives

$$v^h := 2u^{h/2} - u^h = u + h^2(2r^{h/2} - r^h),$$

which yields

$$E \sup_{t \in [0,T], x \in \mathbb{R}^d} |v_t^h(x) - u_t(x)|^q \le N h^{2q}.$$

- For $k \geq 1$ define

$$(\lambda_0, \lambda_1, ..., \lambda_k) = (1, 0, ..., 0)V^{-1},$$

where V^{-1} is the inverse of the $(k+1)\times(k+1)$ Vandermonde matrix $V = (V^{ij})$ given by $V^{ij} = 2^{-(i-1)(j-1)}$. Set

$$v^h = \sum_{i=0}^{k} \lambda_i u^{h/2^i}.$$

Then expansion (9) gives

$$E \sup_{t\in[0,T]} \sup_{x\in\mathbb{G}_h} |v_t^h(x) - u_t(x)|^q \leq N_1 h^{q(k+1)}.$$

To obtain a power series expansion we make the following assumptions:

Assumption 0. (consistency)

$$a^{ij} = \sum_{\lambda,\gamma\in\Lambda} \alpha^{\lambda\gamma}\lambda^i\gamma^j, \quad b^i = \sum_{\lambda\in\Lambda} (\beta^\lambda + \mathfrak{p}^\lambda - \mathfrak{q}^\lambda)\lambda^i,$$

$$\sigma^{ir} = \sum_{\lambda\in\Lambda} \mu^{\lambda r}\lambda^i.$$

Assumption I.

(i) $\sum_{\lambda,\gamma\in\Lambda}(\alpha^{\lambda\gamma} - \frac{1}{2}\mu^{\lambda r}\mu^{\gamma r})z_\lambda z_\gamma \geq 0, \quad \forall z_\lambda \in \mathbb{R},\ \lambda \in \Lambda;$

(ii) $\mathfrak{p}^\lambda \geq 0,\ \mathfrak{q}^\lambda \geq 0 \quad \forall\lambda \in \Lambda.$

Assumption II. $m \geq 1$, $\bar{m} = \max(m, 2)$

$$|\alpha^{\lambda\gamma}|_{C^{\bar{m}}} \leq K, \quad |\beta^{\lambda}|_{C^{m}} \leq K, \quad |c|_{C^{m}} \leq K,$$

$$|(\mu^{\lambda r})_{r=1}^{\infty}|_{C^{m+1}} \leq K, \quad |(\nu^{r})_{r=1}^{\infty}|_{C^{m+1}} \leq K.$$

Assumption III. $\psi \in W_p^m$, $\mathcal{F}_0$-measurable;

$$\mathcal{K}_m^p(T) = \int_0^T |f_t|_{W_p^m}^p + |g_t|_{W_p^{m+1}(l_2)}^p \, dt < \infty.$$

Theorem 4. Let $p = 2$. Let $k \geq 0$ be an integer, let Assumptions 0-III hold with

$$m > 2k + 3 + \frac{d}{2}.$$

Then expansion (9) holds for $h > 0$, $t \in [0, T]$, $x \in \mathbb{G}_h$ with a continuous random field $(r_t^h(x))_{(t,x) \in H_T}$ satisfying (10) with $N = N(K, T, k, d, \Lambda)$.

Truncated Finite Difference Schemes

Let $\varepsilon > 0$ and nonnegative $\zeta \in C_0^\infty(\mathbb{R}^d)$, s.t., $\zeta(x) = 1$ if $|x| \le 1$, $\zeta(x) = 0$ if $|x| \ge \rho > 1$.

Set $\varphi_R(x) := \varphi(x/R)$, $L_R^h := \varphi_R L^h$, $M_R^{h,r} := \varphi_R M^{h,r}$, $f_R := \varphi_R f$, $\quad g_R := \varphi_R g$

Consider

$$\begin{aligned} du_t^h(x) = & (L_{R,t}^h u_t^h(x) + f_{R,t}(x))\, dt \\ & + (M_{R,t}^{h,r} u_t(x) + g_{R,t}^r(x))\, dw_t^r \qquad (11) \\ u_0^h(x) = & \psi_R(x) \qquad (12) \end{aligned}$$

for $t \in [0, T]$ and $x \in \mathbb{G}_{h,R\rho} := \mathbb{G}_h \cap \{|x| \le R\rho\}$.

Set $v^h := \sum_{j=0}^k \lambda_j u^{h/2^j}$.

Theorem 6. Let $p = 2$, $k \geq 0$, and Assumptions 0-III hold with

$$m > 2k + 3 + \frac{d}{2}.$$

Then for $R > 0$, $\kappa \in (0, 1)$, $q > 0$

$$E \sup_{t \in [0,T]} \sup_{x \in \mathbb{G}_h \cap \{|x| \leq \kappa R\}} |v^h_{R,t}(x) - u_t(x)|^q \leq N_1 h^{q(k+1)} + N_2 e^{-\theta R^2}$$

with positive constants N_1, N_2, θ, independent of h.
If k is odd and $\mathfrak{p}^\lambda = \mathfrak{q}^\lambda = 0$ for $\lambda \in \Lambda$, then

$$m > 2k + 2 + \frac{d}{2}$$

is sufficient.

Thank You!

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