Multiple points of the Brownian sheet in critical dimensions

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Based on joint work with:

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Overview

- Introduction to the Brownian sheet
- Hitting points, multiple points
- Intersection equivalence
- Main result: absence of multiple points in critical dimensions
- Method of proof

Fix two positive integers d and N.

An N-parameter Brownian sheet with values in $\mathbb{R}^d$ is a Gaussian random field $B = (B^1, \ldots, B^d)$, defined on a probability space $(\Omega, \mathcal{F}, P)$, with parameter set $\mathbb{R}_+^N$, continuous sample paths and covariances

$$\mathrm{Cov}(B^i(\mathbf{s}),\ B^j(\mathbf{t})) = \delta_{i,j}\ \prod_{\ell=1}^{N} (s_\ell \wedge t_\ell),$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise, $\mathbf{s}, \mathbf{t} \in \mathbb{R}_+^N$, $\mathbf{s} = (s_1, \ldots, s_N)$ and $\mathbf{t} = (t_1, \ldots, t_N)$.

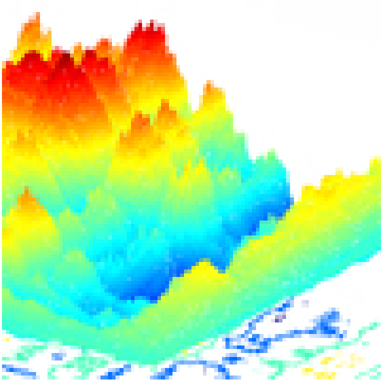
The case $N = 1$: Brownian motion $B = (B(t),\ t \in \mathbb{R}_+)$.

The case $N > 1$: multi-parameter extension of Brownian motion.

A few references: Orey & Pruitt (1973), R. Adler (1978), W. Kendall (1980), J.B. Walsh (1986), D. & Walsh (1992), Khoshnevisan & Shi (1999)

D. Khoshnevisan, *Multiparameter processes,* Springer (2002).

A sample path of the Brownian sheet $N = 2$, $d = 1$



Basic questions, $N = 1$.

(1) Hitting points. For which d is $P\{\exists t \in \mathbb{R}_+^* : B(t) = x\} > 0$ ($x \in \mathbb{R}^d$)?

(2) Polar sets. For $I \subset \mathbb{R}_+$, let $B(I) = \{B(t) : t \in I\}$ be the range of B over I. Which sets $A \subset \mathbb{R}^d$ are polar ($P\{B(I) \cap A \neq \emptyset\} = 0$) ?

(3) Hitting probabilities. For $A \subset \mathbb{R}^d$, what are bounds on

$$P\{B(I) \cap A \neq \emptyset\}?$$

(4) What is the Hausdorff dimension of the range of B? $\dim B(\mathbb{R}_+) = 2 \wedge d$ a.s.

(5) What is the Hausdorff dimension of level sets of B?

$$L(x) = \{t \in \mathbb{R}_+ : B(t) = x\}$$

If $d = 1$, then $\dim L(x) = \frac{1}{2}$ a.s.

Given $k \geqslant 2$ independent Brownian motions $B^1, \dots, B^k$ with values in $\mathbb{R}^d$, what is the probability that their sample paths have a non-empty intersection? Is

$$P_{(y_1,\dots,y_k)}\{\exists (t_1,\dots,t_k) \in (\mathbb{R}_+)^k : B^1(t_1) = \cdots = B^k(t_k)\} > 0?$$

(y_ℓ distinct) Equivalently, is

$$P_{(y_1,\dots,y_k)}\{B^1(\mathbb{R}_+) \cap \cdots \cap B^k(\mathbb{R}_+) \neq \emptyset\} > 0?$$

$d \geqslant 5$: 2 independent BM's do not intersect (Kakutani 1944).

$d \geqslant 4$: 2 independent BM's do not intersect.

$d = 3$: $P_{(x_1,x_2)}\{B^1(\mathbb{R}_+) \cap B^2(\mathbb{R}_+) \neq \emptyset\} > 0$ (Dvoretzky, Erdős & Kakutani 1950), but 3 independent BM's do not intersect (& Taylor 1957).

$d = 2$: For all $k \geqslant 2$, sample paths of k independent BM's intersect (D-E-K 1954).

$x \in \mathbb{R}^d$ is a k-multiple point of B if there exist distinct $t_1, \ldots, t_k \in \mathbb{R}_+$ such that

$$B(t_1) = \cdots = B(t_k) = x.$$

We let M_k denote the (random, possibly empty) set of all k-multiple points of B.

Existence of multiple points. Given a positive integer k, is

$$P\{M_k \neq \emptyset\} > 0?$$

Existence of multiple points within a given set. Given a positive integer k and a subset $A \subset \mathbb{R}^d$, is

$$P\{M_k \cap A \neq \emptyset\} > 0?$$

Relationship between self-intersections and intersections of independent motions

Case of Brownian motion (or of Markov processes).

Given a Brownian motion B, a double point (or 2-multiple point) occurs if there are two disjoint intervals I_1 and I_2 in $\mathbb{R}_+$ such that

$$B(I_1) \cap B(I_2) \neq \emptyset. \tag{1}$$

Suppose I_1 precedes I_2. By the Markov property of B, given the position of B at the right endpoint of I_1, $B|_{I_2}$ is conditionally independent of $B|_{I_1}$, so the study of (1) is essentially the study of intersections of two independent Brownian motions.

For random fields, this type of Markov property is absent, so other methods are needed.

Case $N > 1$: the Brownian sheet

(1) (Orey & Pruitt, 1973). Points are polar for the (N, d)-Brownian sheet if and only if $d \geqslant 2N$.

(2) (J. Rosen, 1984) If $d(k-1) < 2kN$, then the Hausdorff dimension of the set of k-multiple points is $kN - d(k-1)/2$.

(3) (Khoshnevisan, 1997). The (N, d)-Brownian sheet does not have k-multiple points if $d(k-1) > 2kN$.

Remark. The critical case $d(k-1) = 2kN$ remained open.

(4) Double points (D. et al, Annals Probab. 2012). In the case $k = 2$ and for critical dimensions $d = 4N$, the (N, d)-Brownian sheet does not have double points.

Theorem 1 (D. & Mueller, today's talk)

Absence of k-multiple points, $k \geqslant 2$: If N, d and k are such that $(k-1)d = 2kN$, then an (N, d)-Brownian sheet has no k-multiple points.

Non-critical dimensions

Case $N = 1$, $d \geqslant 3$: standard Brownian motion with values in $\mathbb{R}^d$ does not hit points.

Explanation. Let $t_k = 1 + k2^{-2n}$. Fix $x \in \mathbb{R}^d$. Then

$$
\begin{aligned}
P\{\exists t \in [1,2] : B(t) = x\} &= P\left(\bigcup_{k=1}^{2^{2n}} \{\exists t \in [t_{k-1}, t_k] : B(t) = x\}\right) \\
&\leqslant \sum_{k=1}^{2^{2n}} P\{\exists t \in [t_{k-1}, t_k] : B(t) = x\} \\
&\sim \sum_{k=1}^{2^{2n}} P\{\|B_{t_k} - x\| \leqslant n2^{-n}\} \\
&\leqslant \sum_{k=1}^{2^{2n}} c(n2^{-n})^d \\
&= c2^{2n} c(n2^{-n})^d \\
&= cn^d 2^{(2-d)n} \\
&\to 0 \qquad \text{as } n \to +\infty \text{ (because } d \geqslant 3).
\end{aligned}
$$

Let $(V(x),\ x \in \mathbb{R}^k)$ be a centered continuous Gaussian random field with values in $\mathbb{R}^d$ with i.i.d. components: $V(x) = (V_1(x), \dots, V_d(x))$. Set

$$\sigma^2(x, y) = E[(V_1(x) - V_1(y))^2].$$

Let I be a "rectangle". Assume the two conditions:
(C1) There exists $0 < c < \infty$ and $H_1, \dots, H_k \in]0, 1[$ such that for all $x \in I$,

$$c^{-1} \leqslant \sigma^2(0, x) \leqslant c,$$

and for all $x, y \in I$,

$$c^{-1} \sum_{j=1}^{k} |x_j - y_j|^{2H_j} \leqslant \sigma^2(x, y) \leqslant c \sum_{j=1}^{k} |x_j - y_j|^{2H_j}$$

(H_j is the Hölder exponent for coordinate j).

(C2) There is $c > 0$ such that for all $x, y \in I$,

$$\mathrm{Var}(V_1(y) \mid V_1(x)) \geqslant c \sum_{j=1}^{k} |x_j - y_j|^{2H_j}.$$

Theorem 1 (Biermé, Lacaux & Xiao, 2007)

Fix $M > 0$. Set

$$Q = \sum_{j=1}^{k} \frac{1}{H_j}.$$

Assume $d > Q$. Then there is $0 < C < \infty$ such that for every compact set $A \subset B(0, M)$,

$$C^{-1}\ Cap_{d-Q}(A) \leqslant P\{V(I) \cap A \neq \emptyset\} \leqslant C\mathcal{H}_{d-Q}(A).$$

Special case obtained by D., Khoshnevisan and E. Nualart (2007); see also D. & Sanz-Solé (2010).

This results tells us what sort of inequality to aim for when we have information about Hölder exponents.

Notice the Hausdorff measure appearing on the right-hand side.

Measuring the size of sets: capacity

Capacity. $\mathrm{Cap}_\beta(A)$ denotes the Bessel-Riesz capacity of A:

$$\mathrm{Cap}_\beta(A) = \frac{1}{\inf_{\mu \in \mathcal{P}(A)} \mathcal{E}_\beta(\mu)},$$

$$\mathcal{E}_\beta(\mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k_\beta(x-y)\mu(dx)\mu(dy)$$

and

$$k_\beta(x) = \begin{cases} \|x\|^{-\beta} & \text{if } 0 < \beta < d, \\ \ln(\frac{1}{\|x\|}) & \text{if } \beta = 0, \\ 1 & \text{if } \beta < 0. \end{cases}$$

Examples. If $A = \{z\}$, then:

$$\mathrm{Cap}_\beta(\{z\}) = \begin{cases} 1 & \text{if } \beta < 0, \\ 0 & \text{if } \beta \geqslant 0. \end{cases}$$

If A is a subspace of $\mathbb{R}^d$ with dimension $\ell \in \{1, \ldots, d-1\}$, then:

$$\mathrm{Cap}_\beta(A) \begin{cases} > 0 & \text{if } \beta < \ell, \\ = 0 & \text{if } \beta \geqslant \ell. \end{cases}$$

Another measure of the size of sets: Hausdorff measure

For $\beta \geqslant 0$, the β-dimensional Hausdorff measure of A is defined by

$$\mathcal{H}_\beta(A) = \lim_{\epsilon \to 0^+} \inf \left\{ \sum_{i=1}^\infty (2r_i)^\beta : A \subseteq \bigcup_{i=1}^\infty B(x_i\,, r_i),\ \sup_{i \geq 1} r_i \leqslant \epsilon \right\}.$$

When $\beta < 0$, we define $\mathcal{H}_\beta(A)$ to be infinite.

Note. For $\beta_1 > \beta_2 > 0$,

$$\mathrm{Cap}_{\beta_1}(A) > 0 \Rightarrow \mathcal{H}_{\beta_1}(A) > 0 \Rightarrow \mathrm{Cap}_{\beta_2}(A) > 0.$$

Example. $A = \{z\}$

$$\mathcal{H}_\beta(\{z\}) = \begin{cases} \infty & \text{if } \beta < 0 \\ 1 & \text{if } \beta = 0 \\ 0 & \text{if } \beta > 0. \end{cases}$$

Remark. For $\beta = 0$,

$$\mathrm{Cap}_0(\{z\}) = 0 < \mathcal{H}_0(\{z\}) = 1.$$

When $d = Q$ and $A = \{x\}$, the inequality

$$C^{-1}\ \mathrm{Cap}_{d-Q}(A) \leqslant P\{V(I) \cap A \neq \emptyset\} \leqslant C\mathcal{H}_{d-Q}(A)$$

is equivalent to

$$0 \leqslant P\{x \text{ is polar}\} \leqslant 1,$$

which is uninformative!

Let

$$X(\mathbf{t}^1, \dots, \mathbf{t}^k) = (B(\mathbf{t}^1) - B(\mathbf{t}^2), B(\mathbf{t}^2) - B(\mathbf{t}^3), \dots, B(\mathbf{t}^{k-1}) - B(\mathbf{t}^k)).$$

Then

$$B(\mathbf{t}^1) = \cdots = B(\mathbf{t}^k) \iff X(\mathbf{t}^1, \dots, \mathbf{t}^k) = 0 \in (\mathbb{R}^d)^{k-1},$$

Existence of k-multiple points for B is equivalent to hitting 0 for X.

Remark. The probability density function of $X(\mathbf{t}^1, \dots, \mathbf{t}^k)$ is related to the joint probability density function of $(B(\mathbf{t}^1), \dots, B(\mathbf{t}^k))$, so the difficulty lies in getting properties of the joint density.

Intersection-equivalence

Let $\mathcal{T}_N^k$ denote the set of parameters $(\mathbf{t}^1, \dots, \mathbf{t}^k)$ with $\mathbf{t}^i \in]0, \infty[^N$ such that no two $\mathbf{t}^i = (t_1^i, \dots, t_N^i)$ and $\mathbf{t}^j = (t_1^j, \dots, t_N^j)$ $(i \neq j)$ share a common coordinate:

$$\mathcal{T}_N^k = \{(\mathbf{t}^1, \dots, \mathbf{t}^k) \in \ (]0, \infty[^N)^k : t_\ell^i \neq t_\ell^j, \text{ for all } 1 \leq i < j \leq k \\ \text{and } \ell = 1, \dots, N\}.$$

Theorem 2

Let $A \subset \mathbb{R}^d$ be a Borel set. For all $k \in \{2, 3, \dots\}$, we have

$$P\{\exists(\mathbf{t}^1, \dots, \mathbf{t}^k) \in \mathcal{T}_N^k : B(\mathbf{t}^1) = \dots = B(\mathbf{t}^k) \in A\} > 0$$

if and only if

$$P\{\exists(\mathbf{t}^1, \dots, \mathbf{t}^k) \in \mathcal{T}_N^k : W_1(\mathbf{t}^1) = \dots = W_k(\mathbf{t}^k) \in A\} > 0,$$

where $W_1, \dots, W_k$ are independent N-parameter Brownian sheets with values in $\mathbb{R}^d$.

Using Theorem 1

Theorem 3 (Khoshnevisan and Shi, 1999)

Fix $M > 0$ *and* $0 < a_\ell < b_\ell < \infty$ $(\ell = 1, \dots, N)$. *Let*

$$I = [a_1, b_1] \times \cdots \times [a_N, b_N].$$

There exists $0 < C < \infty$ *such that for all compact sets* $A \subset B(0, M)$,

$$\frac{1}{C}\ Cap_{d-2N}(A) \leqslant P\{B(I) \cap A \neq \emptyset\} \leqslant C\ Cap_{d-2N}(A). \tag{2}$$

(Cap denotes Bessel-Riesz capacity)

Theorem 4 (Peres, 1999)

Property (2) *implies that if* $W_1, \dots, W_k$ *are independent* N*-parameter Brownian sheets with values in* $\mathbb{R}^d$ *and* $d > 2N$, *then*

$$\frac{1}{C}\ Cap_{k(d-2N)}(A) \leqslant P\{W_1(I_1) \cap \cdots \cap W_k(I_k) \cap A \neq \emptyset\} \leqslant C\ Cap_{k(d-2N)}(A)$$

In particular, the r.h.s. is 0 if $A = \mathbb{R}^d$ and $k(d-2N) = d$, i.e. $(k-1)d = 2kN$.

Theorem 5

Let B be a Brownian sheet and let $W_1, \ldots, W_k$ be independent Brownian sheets. Fix $M > 0$ and k boxes $R_1, \ldots, R_k$, where, for each coordinate axis, the projections of the R_i onto this coordinate axis are pairwise disjoint. Then, for all $(R_1, \ldots, R_k) \in \mathcal{R}_M$, the random vectors

$$(B|_{R_1}, \ldots, B|_{R_k}) \qquad \text{and} \qquad (W_1|_{R_1}, \ldots, W_k|_{R_k})$$

(with values in $(C(R_1, \mathbb{R}^d) \times \cdots \times C(R_k, \mathbb{R}^d))$) have mutually absolutely continuous probability distributions.

Comments.

(1) There is no convenient Markov property to use.

(2) If the projections of the boxes are not disjoint, then this property would fail.

(3) D. et al 2012: used quantitative estimates on the conditional distribution of $B|_{R_k}$ given $(B|_{R_1}, \ldots, B|_{R_{k-1}})$. This was only achieved for certain configurations of boxes, which limited the final result to double points (2 boxes).

First main ingredient in the proof of Theorem 5

Fix $M > 0$. Define the one-parameter filtration $\mathcal{G} = (\mathcal{G}_u,\ u \in [0, M])$ by

$$\mathcal{G}_u = \sigma\left\{B(t_1, \dots, t_{N-1}, v) : (t_1, \dots, t_{N-1}) \in \mathbb{R}_+^{N-1},\ v \in [0, u]\right\}.$$

Let $(Z(\mathbf{s}),\ \mathbf{s} \in \mathbb{R}_+^{N-1} \times [0, M])$ be an $\mathbb{R}^d$-valued adapted random field: for all $\mathbf{s} \in \mathbb{R}_+^{N-1} \times [0, M]$, $Z(\mathbf{s})$ is $\mathcal{G}_{s_N}$-measurable. For $u \in [0, M]$, define

$$L_u = \exp\left(\int_{\mathbb{R}_+^{N-1} \times [0,u]} Z(\mathbf{s}) \cdot dB(\mathbf{s}) - \frac{1}{2}\int_{\mathbb{R}_+^{N-1} \times [0,u]} \|Z(\mathbf{s})\|^2\, d\mathbf{s}\right).$$

Theorem 6 (Cameron-Martin-Girsanov)

If $(L_u, u \in [0, M])$ is a martingale with respect to $\mathcal{G}$, then the process $(\tilde{B}(\mathbf{t}),\ \mathbf{t} \in \mathbb{R}_+^{N-1} \times [0, M])$ defined by

$$\tilde{B}(t_1, \dots, t_N) = B(t_1, \dots, t_N) - \int_{[0,t_1] \times \cdots \times [0,t_N]} Z(s_1, \dots, s_N)\, ds_1 \cdots ds_N$$

is an $\mathbb{R}^d$-valued Brownian sheet under the probability measure Q, where Q is defined by

$$\frac{dQ}{dP} = L_M.$$

Using Girsanov to create independence

Case $N = 1$. Let $(B_t,\ t \in \mathbb{R}_+)$ be a Brownian motion. Show that

$$\text{Law}((B_t,\ t \in [1,2]),\ (B_t,\ t \in [3,4]),\ (B_t,\ t \in [5,6]))$$

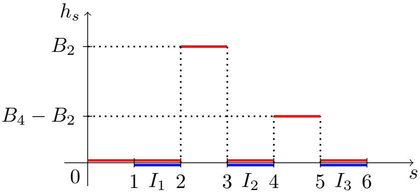
is mutually absolutely continuous w.r.t.

$$\text{Law}((W_t^{(1)},\ t \in [1,2]),\ (W_t^{(2)},\ t \in [3,4]),\ (W_t^{(3)},\ t \in [5,6]))$$

where $(W_t^{(j)})$ are independent Brownian motions.

By Girsanov's theorem: $(B_t,\ t \in [0,6]) \sim \left(\tilde{B}_t = B_t - \int_0^t h_s\, ds,\ \ t \in [0,6]\right)$.

Define $(h_s,\ s \in [0,6])$ by



$$\tilde{B}_t = \begin{cases} B_t, & t \in I_1 \\ B_t - B_2, & t \in I_2 \\ B_t - B_4, & t \in I_3 \end{cases}$$

Set $W_t^{(j)} = \tilde{B}_t,\ t \in I_i$. Then the $(W_t^{(j)},\ t \in I_j)$ are independent.

Observation. For $t \in I_3$, $\tilde{B}_t = B_t - E(B_t \mid \mathcal{F}(I_1) \vee \mathcal{F}(I_2))$.
For $t \in I_2$, $\tilde{B}_t = B_t - E(B_t \mid \mathcal{F}(I_1))$.

Case $N = 2$.

Fix $k \geqslant 2$ and consider k boxes $R_1, \dots, R_k$ with disjoint projections on each coordinate axis:

$$R_j = I_j^1 \times I_j^2, \qquad j = 1, \dots, k,$$

where, for $\ell = 1, 2$, the intervals $I_1^\ell, \dots, I_k^\ell$ are pairwise disjoint. We assume that

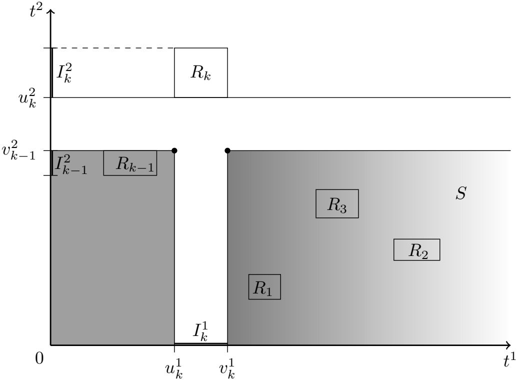
$$I_1^2 < I_2^2 < \cdots < I_k^2.$$

Let

$$S = (I_k^1)^c \times [0, \sup I_{k-1}^2].$$

Notice that for $j = 1, \dots, k-1$, $R_j \subset S$, and there is some space between S and R_k.

Illustration



Then for $\mathbf{t} \in R_k$, $E(B(\mathbf{t}) \mid \mathcal{F}(S)) = \tilde{B}(\mathbf{t})$, where

$$\tilde{B}(t^1, t^2) = \frac{t^1 - u_k^1}{v_k^1 - u_k^1} B(v_k^1, v_{k-1}^2) + \frac{v_k^1 - t^1}{v_k^1 - u_k^1} B(u_k^1, v_{k-1}^2).$$

Creating independent while preserving absolute continuity

For $s^1 \in]0, v_k^1[$ and $s^2 \in]v_{k-1}^2, u_k^2[$, let

$$Z(s^1, s^2) = \frac{\partial^2}{\partial s^1 \partial s^2} \left(\frac{s^2 - v_{k-1}^2}{u_k^2 - v_{k-1}^2} \, \frac{s^1 \wedge u_k^1}{u_k^1} \, \tilde{B}(s^1 \vee u_k^1, v_{k-1}^2) \right),$$

and $Z(s^1, s^2) = 0$ otherwise.
Define

$$\hat{B}(\mathbf{t}) = B(\mathbf{t}) - \int_{[\mathbf{0},\mathbf{t}]} Z(s^1, s^2)\, ds^1 ds^2.$$

Facts.

(a) For $\mathbf{t} \in R_k$,

$$\hat{B}(\mathbf{t}) = B(\mathbf{t}) - E(B(\mathbf{t}) \mid \mathcal{F}(S)),$$

and the r.h.s. is independent of $\mathcal{F}(S)$ (Gaussian process).

(b) $\hat{B} = B$ on $R_1 \cup \cdots \cup R_{k-1}$.

(c) $\mathrm{Law}(\hat{B}) \sim \mathrm{Law}(B)$.

Concluding the proof

We proceed by induction:

$$\begin{aligned}
\text{Law}(B|_{R_1}, \dots, B|_{R_k}) &\sim \text{Law}(\hat{B}|_{R_1}, \dots, \hat{B}|_{R_{k-1}}, \hat{B}|_{R_k}) \\
&\sim \text{Law}(\hat{B}|_{R_1}, \dots, \hat{B}|_{R_{k-1}}, W_k|_{R_k}) \\
&\sim \text{Law}(B|_{R_1}, \dots, B|_{R_{k-1}}, W_k|_{R_k})
\end{aligned}$$

where W_k and B are independent. Then we use induction to replace the other terms:

$$\sim \text{Law}(W_1|_{R_1}, \dots, W_{k-1}|_{R_{k-1}}, W_k|_{R_k}),$$

where $W_1, \dots, W_k$ are independent Brownian sheets.

Use the results of Khoshnevisan & Shi and Peres to conclude.

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