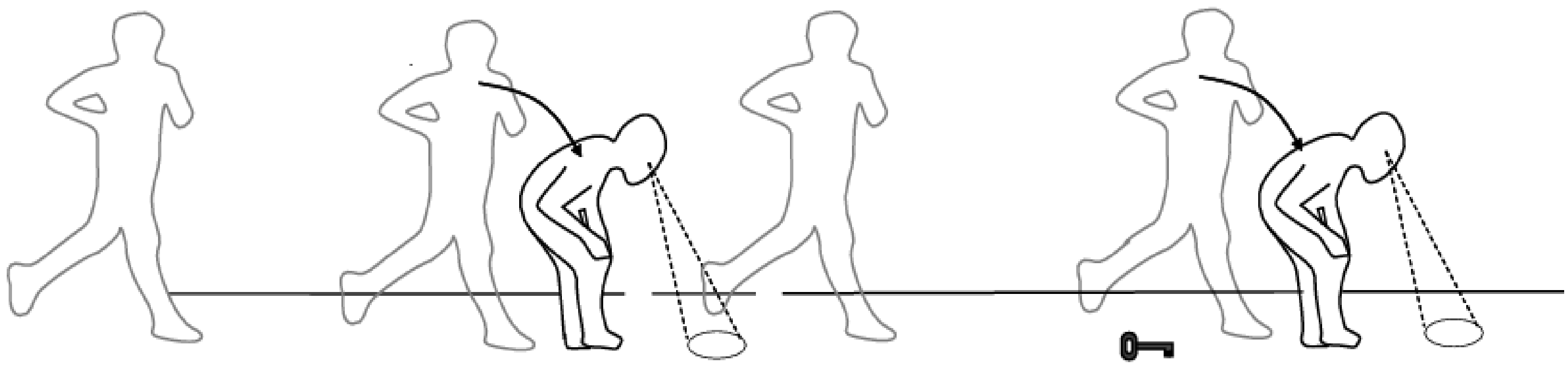
THE OPTIMAL WALK



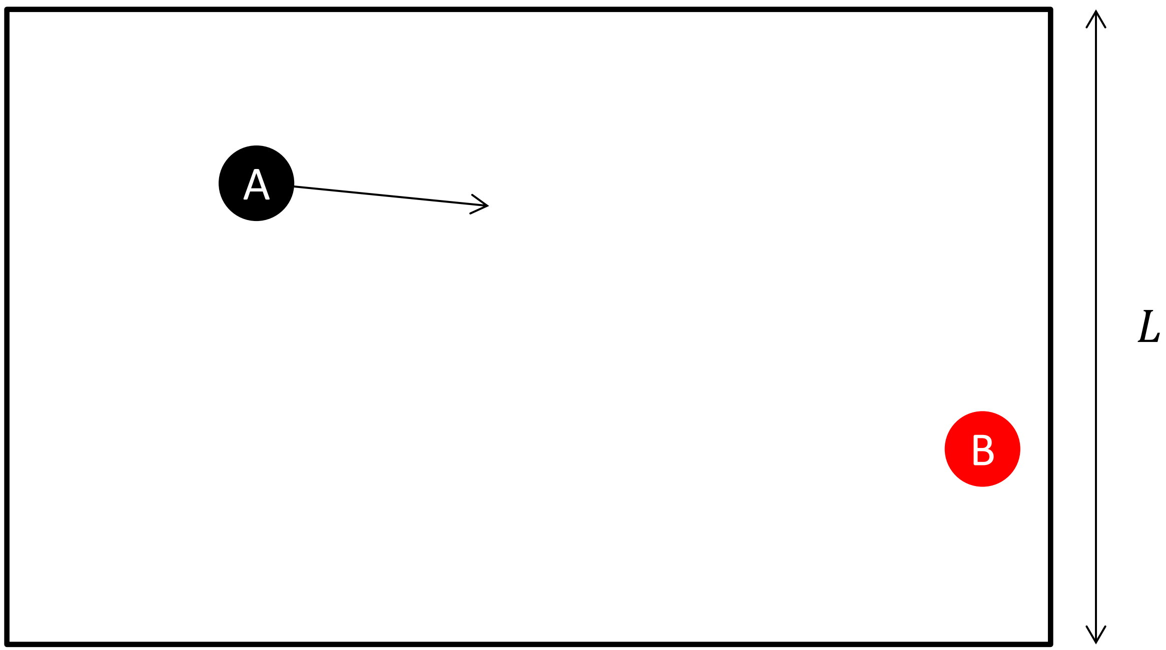
TO THE RANDOM WALK

Daniel Campos
(Universitat Autònoma de Barcelona)

Contents

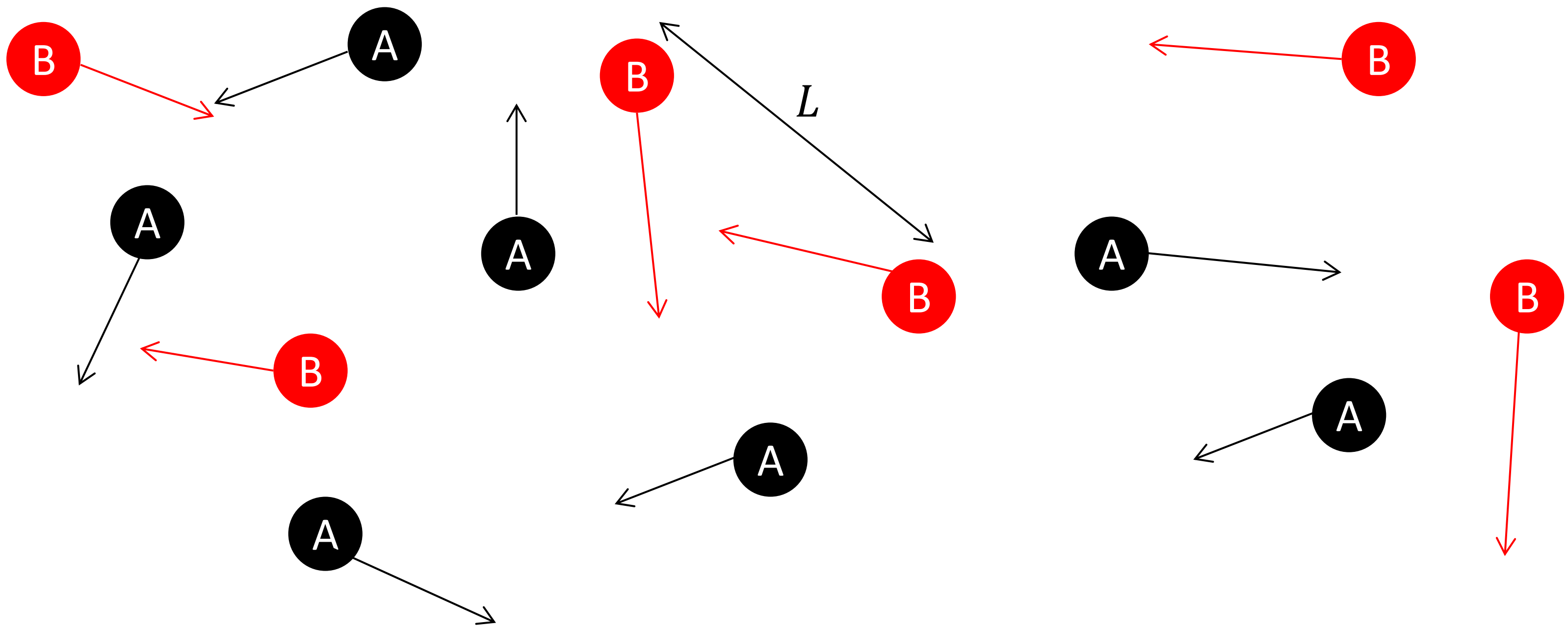
1. *Introduction* *(Random search theory: applications and tools)*
2. *The optimal walk to the (Lévy) walk*
3. *The optimal walk to the (intermittent) walk*
4. *The optimal walk to the (myopic) walk*
5. *The optimal walk to the (mortal) walk*
6. *The optimal walk to the (systematic?) walk*

The first-passage problem:



How do we measure search efficiency?

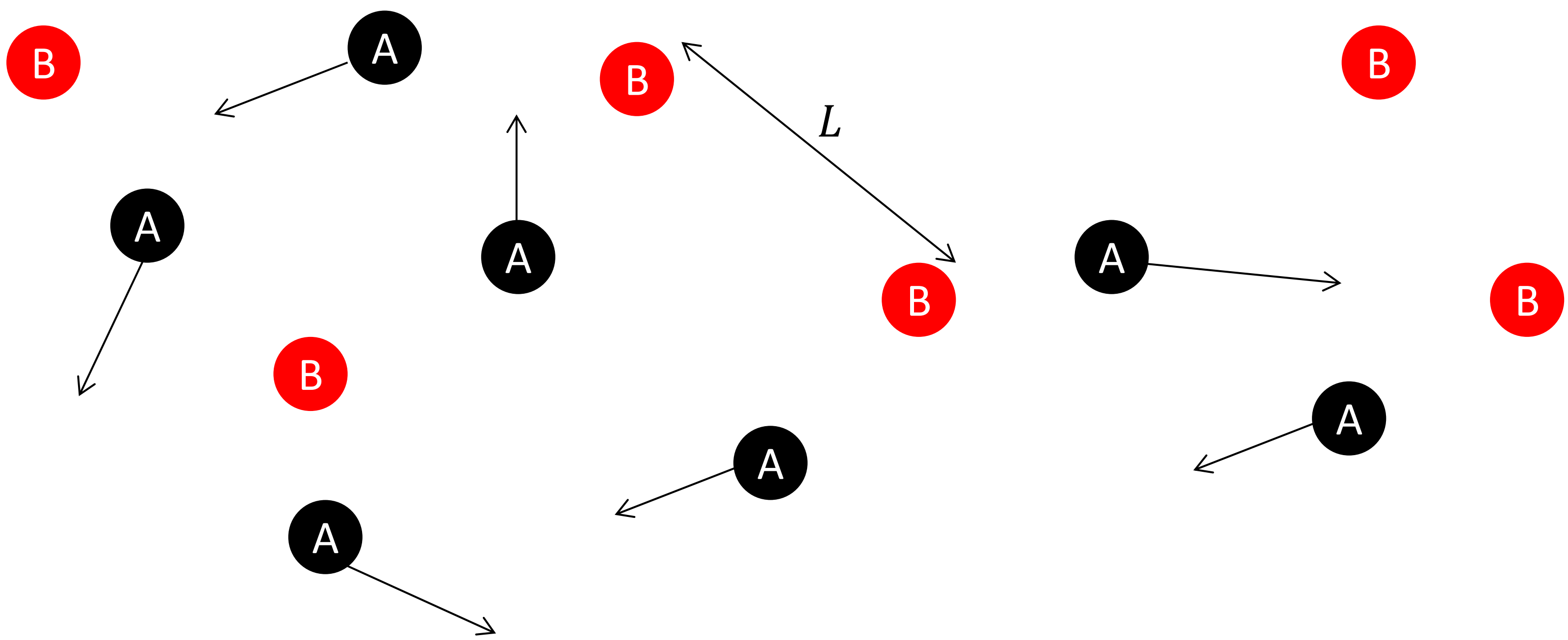
i) Time distribution to the first passage: $f(t; x_0)$
ii) Mean time to the first passage: $\langle T \rangle = \int_0^\infty t f(t; x_0) dt$
iii) First-passage probability up to time t_m: $S(t_m) = \int_0^{t_m} f(t; x_0) dt$



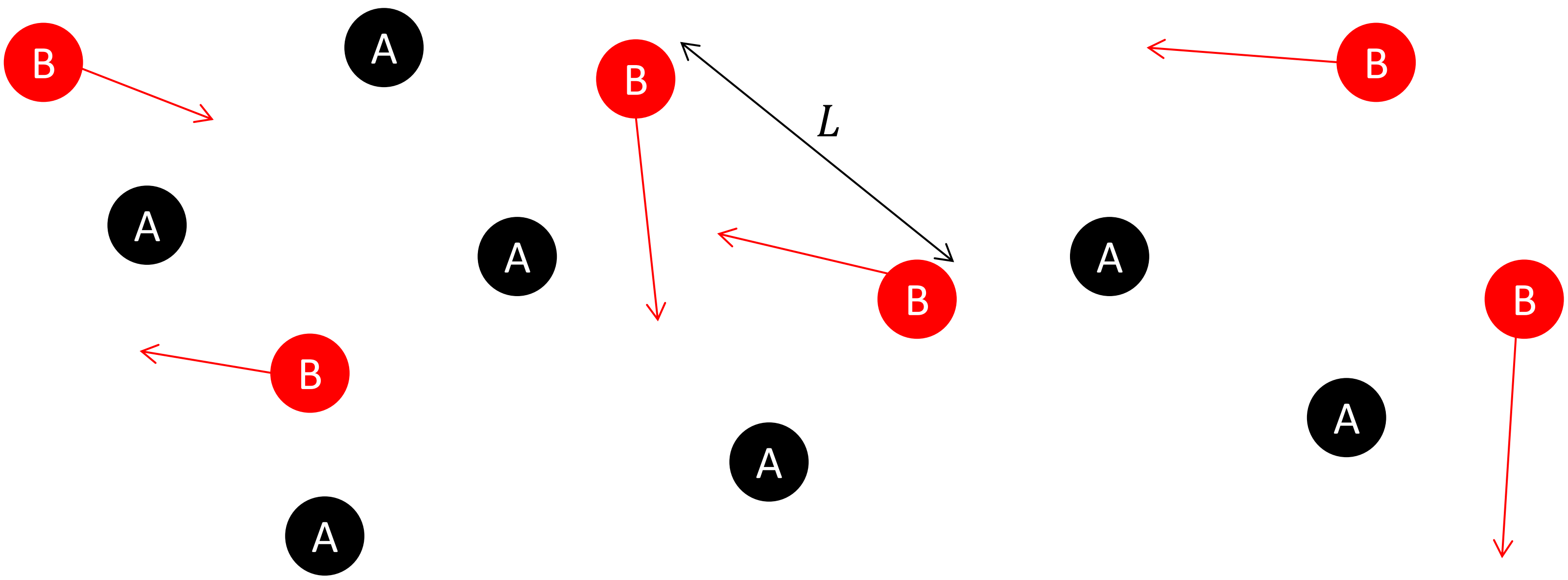
<u>Random search theory:</u> $A + B \rightarrow A$

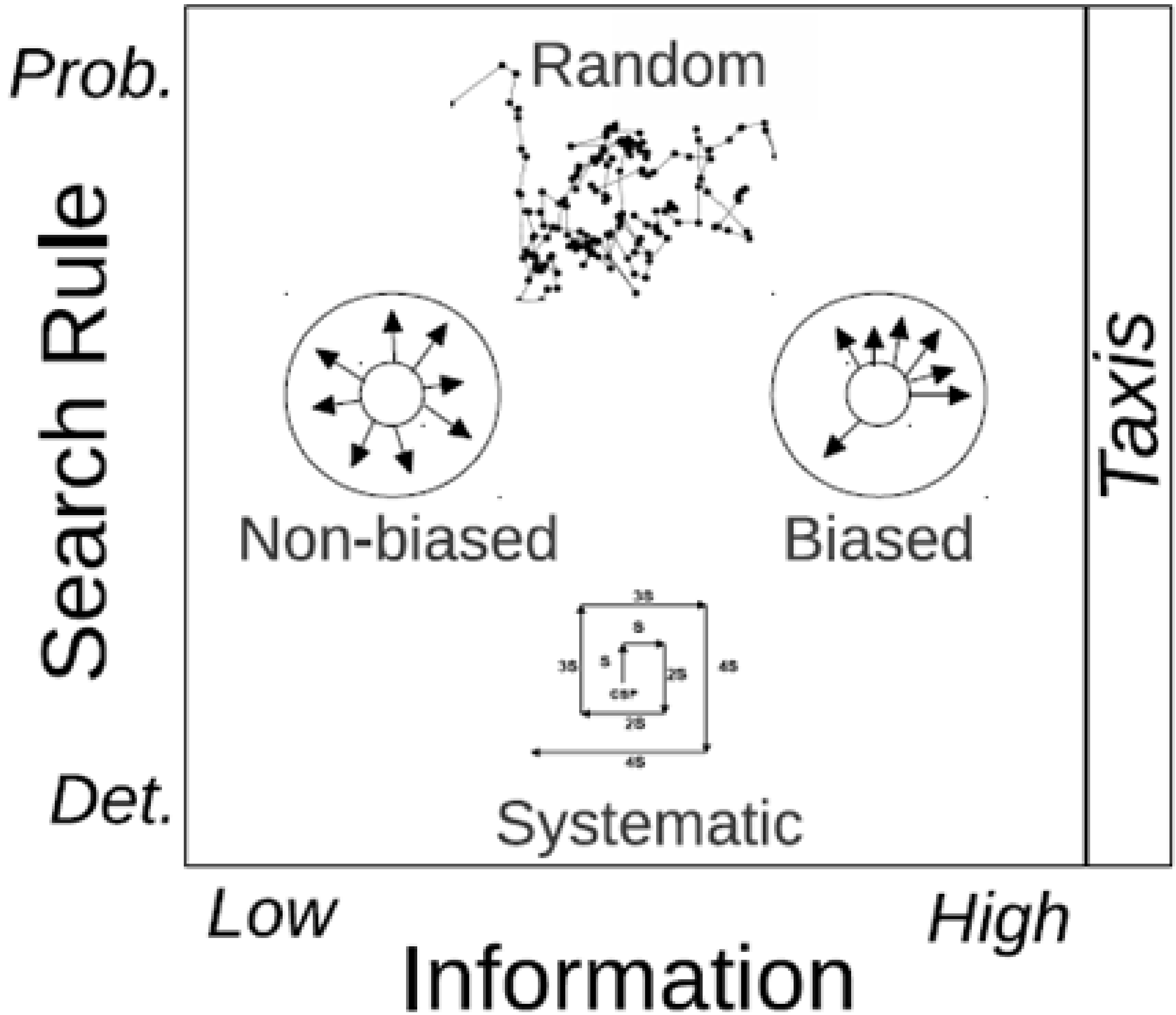
We will describe the position of the i-th particle through a stochastic process $X_i(t)$.

The target problem:



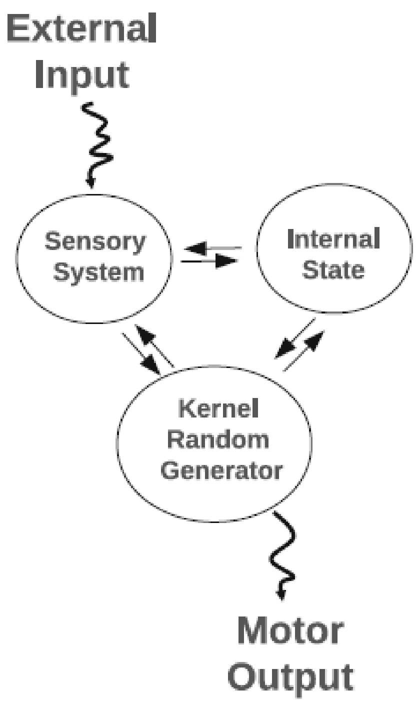
The trapping problem:

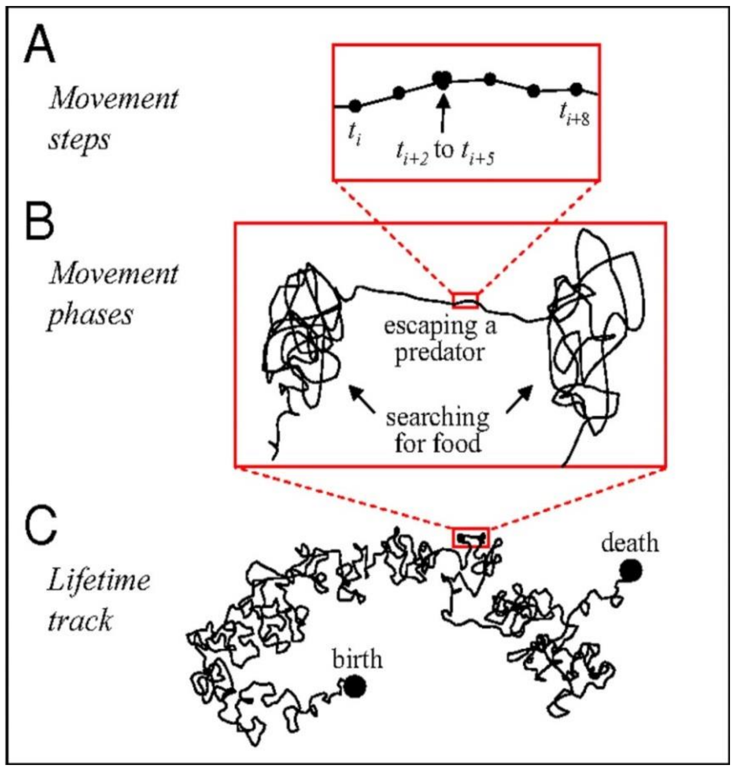


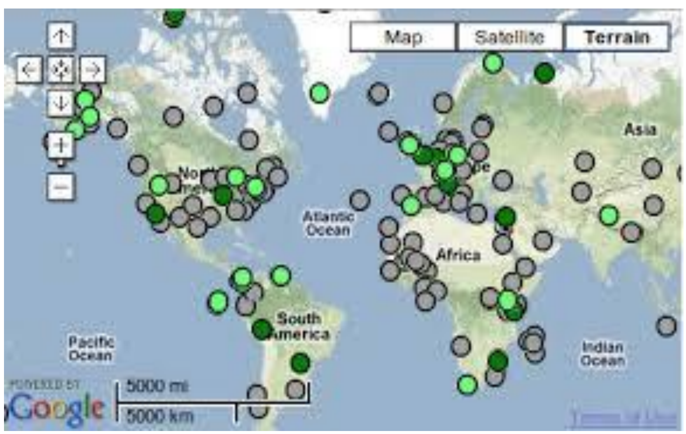

Prob.
Det.
Search Rule
Random
Non-biased
Biased
Systematic
Taxis
Low
High
Information

1. Introduction (Random search theory: applications and tools)

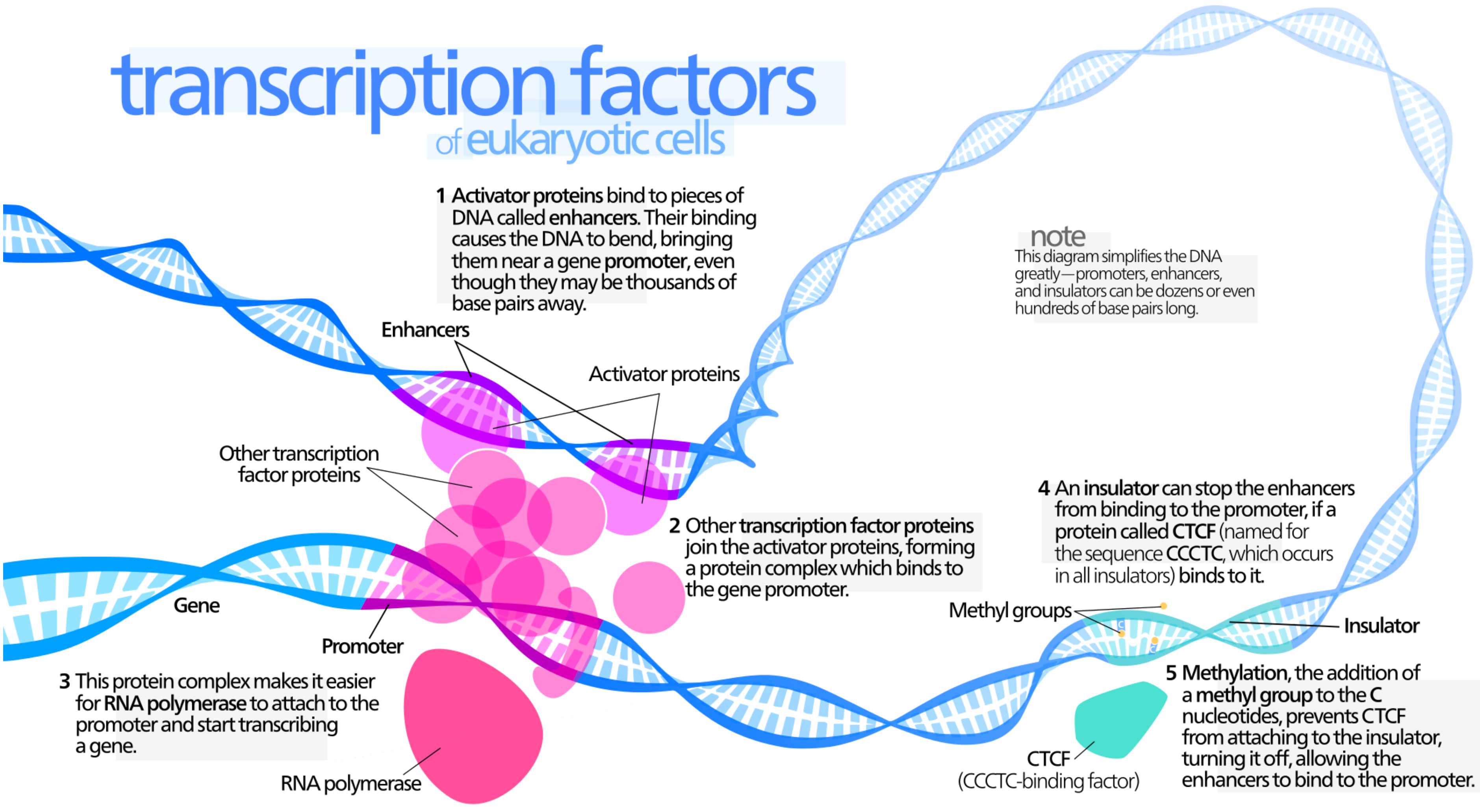
Movement ecology represents a new area of ecology which requires a detailed data processing of individual animal trajectories (obtained through telemmetry, GPS,…).







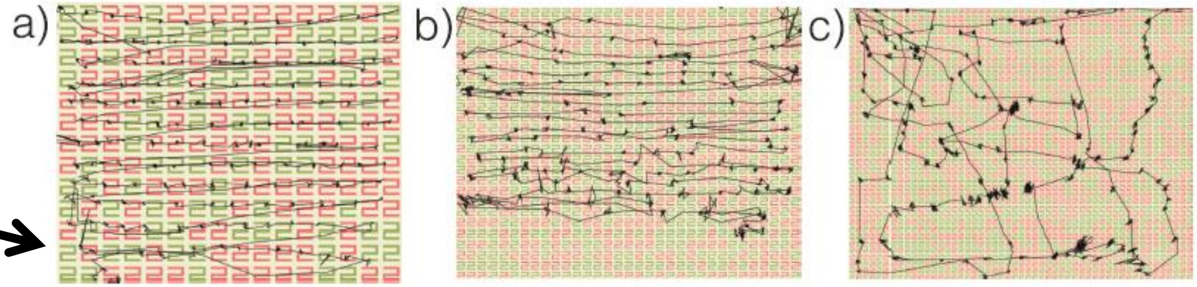
Search at the microscopic level

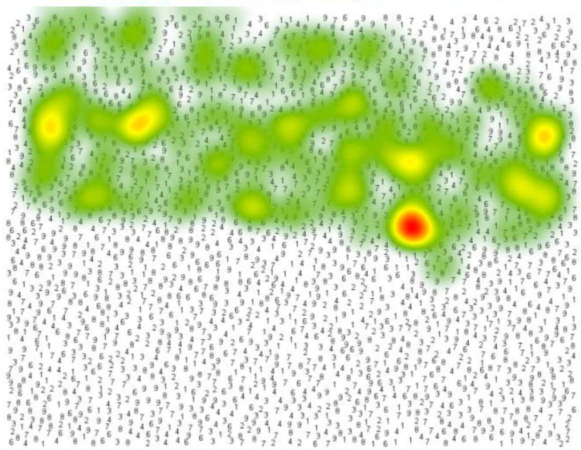


1. Introduction (Random search theory: applications and tools)

Human searches:

Everyday experiences



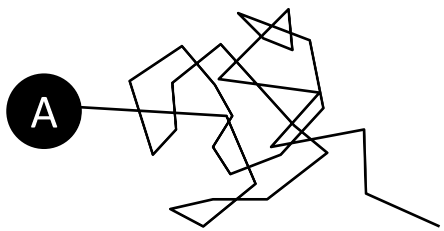


Experiments

SAR applications

Types of motion (I): 'Pure' diffusion model

A **Wiener process** $W(t)$ is defined as a stationary process whose increments $W(t_2) - W(t_1)$ follow a Gaussian distribution with zero mean and variance $|t_2 - t_1|$.



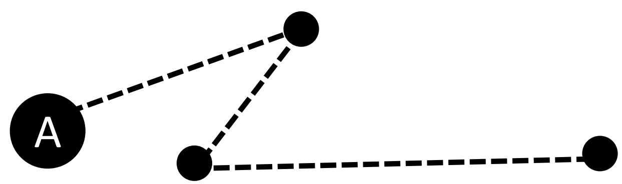
If we assume that $X(t) = x_0 + \sqrt{2D}W(t)$ then:

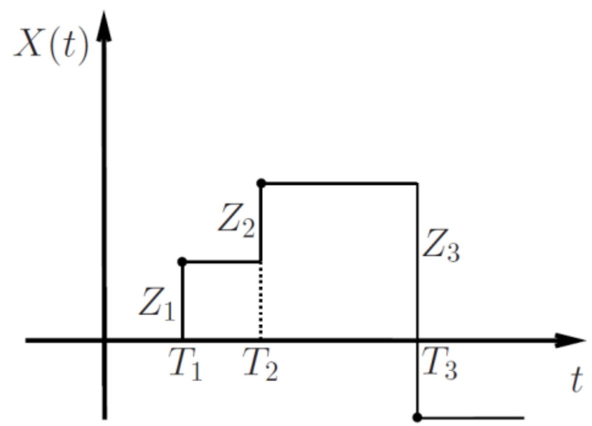
i) The probability density $p(x,t)$ follows a Gaussian distribution with $\langle X \rangle = x_0$ and $\langle X^2 \rangle = 2Dt + {x_0}^2$
ii) It becomes impossible to define a characteristic speed for A
iii) The problem of infinite propagation signals emerge

…but the advantage is that we can describe $X(t)$ as a Gaussian (stable) process.

Types of motion (II): 'Jump' model

We define the position of the particle after n jumps as: $X_n = \sum_{i=1}^{n} Z_i$
...and the time it takes to perform these n jumps as: $T_n = \sum_{i=1}^{n} \Theta_i$





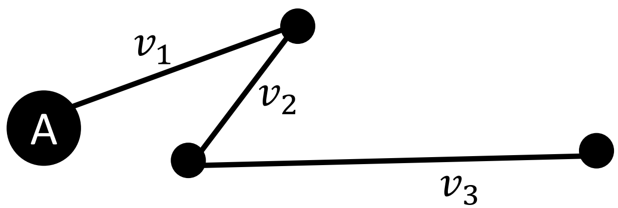
...where Z_i and Θ_i each are i.i.d. random variables distributed, respectively, according to

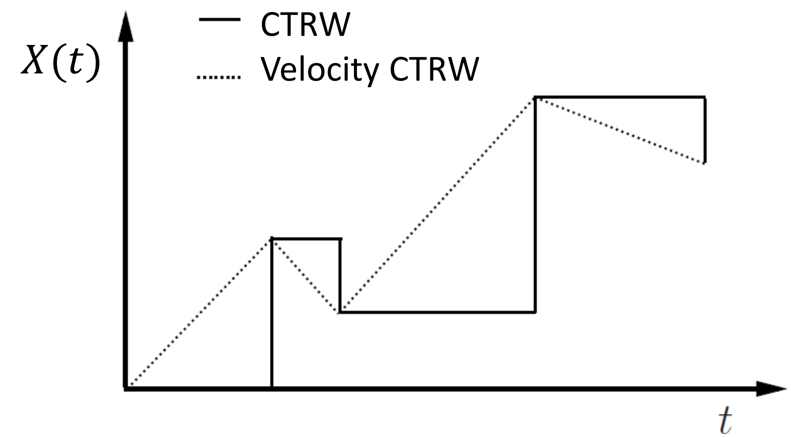
$\phi(x)$: Jump-length probability distribution function (*dispersal kernel*)
$\varphi(t)$: Waiting-time probability distribution function

(This is typically known as a Continuous-Time Random Walk –CTRW- and includes the **Lévy Flight** case as a particular case)

Types of motion (III): ‘Velocity’ model

We use the same definition as before $X_n = \sum_{i=1}^{n} Z_i$ $T_n = \sum_{i=1}^{n} \Theta_i$





…where now $\varphi(t)$ and $\phi(x)$ are not independent, but coupled through a velocity distribution $h(v)$ in the form

$$\phi(x) = \int_0^\infty dt \;\; \varphi(t) \int_{-\infty}^{\infty} dt \; \delta(x - vt) h(v)$$

(This is typically known as the “velocity version” of the CTRW, and includes the **Lévy** walk case, together with some other that ‘mimic’ the **Ornstein-Uhlenbeck** process in v)

Methods for finding $f(t)$ and/or $\langle T \rangle$

Direct resolution:
We formally define the problem as $L_{FP}[p(x,t)] = 0$ with boundary condition $p(\Omega, t) = 0$, being Ω the surface of the target, and computing the flux at Ω. For the Wiener process, for example, $L_{FP} = \frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}$.

Master equation approach:
For the Bernoulli random walk (with probability ½ to each side, jump size a and waiting time τ) we have

$$\langle T \rangle(x_0) = \frac{1}{2}[\langle T \rangle(x_0 + a) + \tau] + \frac{1}{2}[\langle T \rangle(x_0 - a) + \tau]$$

$$\frac{a^2}{2\tau}\frac{\partial^2 \langle T \rangle(x_0)}{\partial {x_0}^2} = -1$$

$$\langle T \rangle(x_0) = \frac{x_0(L - x_0)\tau}{a^2}$$

Methods for finding $f(t)$ and/or $\langle T\rangle$

The renewal approach

We assume one target located at $x = x_t$ and introduce $k_n(t; x_0)$ as the rate at which the n-th passage occurs. Using a renewal assumption we have

$$k(t; x_0)dt = k_1(t; x_0)dt + k_2(t; x_0)dt + k_3(t; x_0)dt + \cdots =$$
$$= [f(t; x_0) + f(t; x_0) * f(t; x_t) + f(t; x_0) * f(t; x_t) * f(t; x_t) + \cdots]dt$$

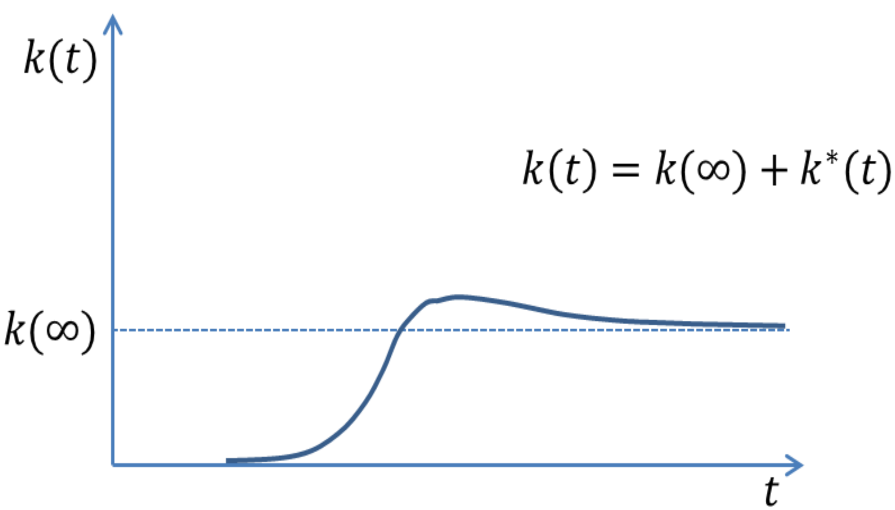
$$k(s; x_0) = f(s; x_0) \sum_{i=0}^{\infty} [f(s; x_t)]^i = \frac{f(s; x_0)}{1 - f(s; x_t)}$$

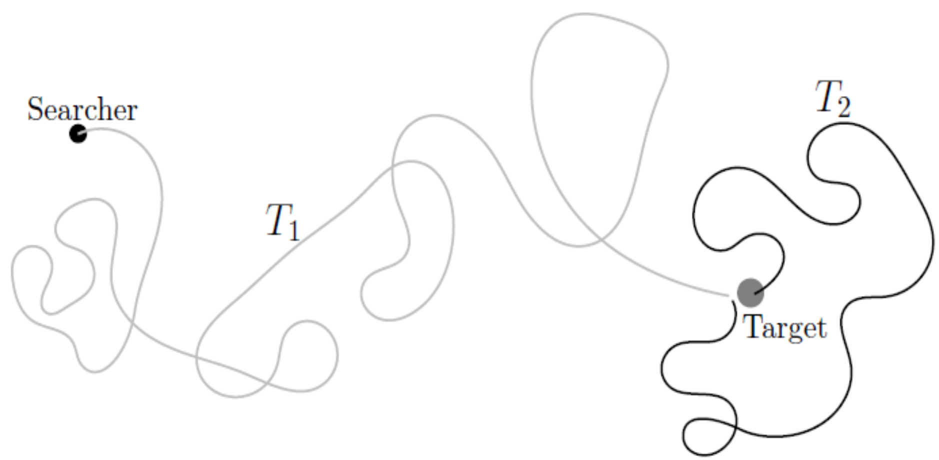
$$\boxed{f(s; x_0) = \frac{k(s; x_0)}{1 + k(s; x_t)}}$$

1. Introduction (Random search theory: applications and tools)

An essential advantage of this framework is that it allows a very general and intuitive understanding of the Mean First Passage Time (MFPT):

$$\langle T\rangle = \int_0^\infty dt\, t f(t;x_0) = \lim_{s\to 0}\frac{df(s;x_0)}{ds} = \underbrace{\lim_{s\to 0}\left(\frac{k^*(s;x_t)}{k(\infty)} - \frac{k^*(s;x_0)}{k(\infty)}\right)}_{\langle \boldsymbol{T_1}\rangle} + \underbrace{\frac{1}{k(\infty)}}_{\langle \boldsymbol{T_2}\rangle}$$

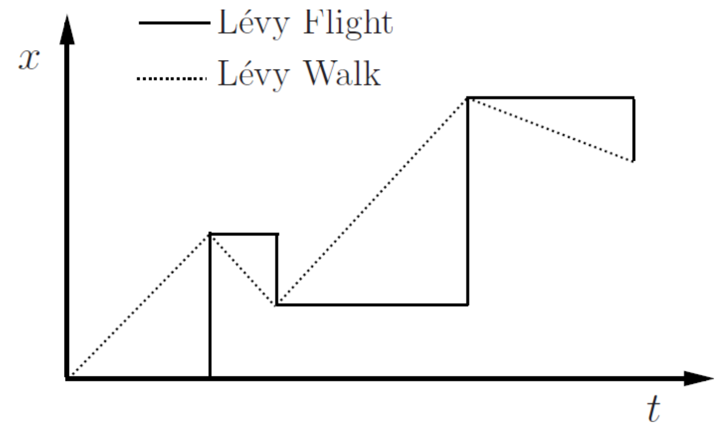




2. The optimal walk to the (Lévy) walk

WHAT ARE LEVY FLIGHTS AND LEVY WALKS?

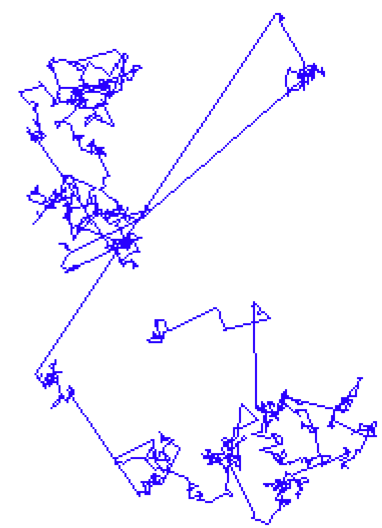
The Lévy Flight fits our 'jump' model scheme with $\phi(x)$ a jump length distribution which decays according to $\lim_{t\to\infty}\phi(x) \sim x^{-\mu}$, with $1 < \mu < 3$



The Lévy Walk fits our 'velocity' model scheme, with v fixed and $\varphi(t)$ a flight time distribution which decays according to $\lim_{t\to\infty}\varphi(t) \sim t^{-\mu}$, with $1 < \mu < 3$.

Note that this implies that $\langle x^q\rangle \equiv \int_{-\infty}^{\infty} dx\phi(x)x^q$ and $\langle t^q\rangle \equiv \int_0^{\infty} dt\varphi(t)t^q$, respectively, diverge for $q - \mu \geq -1$

In the Lévy Flight case these divergences extend to the overall behavior of the particle, so $\langle X^2\rangle$ also diverges. In contrast, for the Lévy Walk case, thanks to the coupling between flight durations and lengths through v:



$$\langle X^2\rangle \sim \begin{cases} t^2 & ,1 < \mu < 2 \\ t^{4-\mu} & ,2 < \mu < 3 \end{cases}$$

(Viswanathan et. al. Nature 401, 911 (1999)

THE LÉVY FLIGHT OPTIMAL SOLUTION

Define the search efficiency $\frac{1}{\langle l \rangle N}$, where $\langle l \rangle$ is the mean flight distance between targets and N the mean number of flights to cover the distance between targets

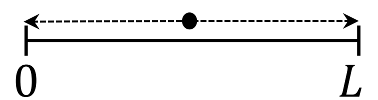
Given $\phi(x) \sim x^{-\mu}$ and a mean path between targets of β,

$$\langle l \rangle \approx \frac{\int_0^\beta dx\, x^{1-\mu} + \beta \int_\beta^\infty dx\, x^{-\mu}}{\int_0^\infty dx\, x^{-\mu}}$$

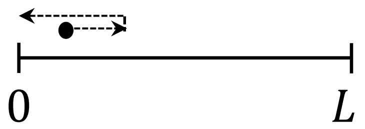
and the mean number of flights satisfies $N \sim \beta^{(\mu-1)/2}$ if the target is close enough.
All this leads to a search efficiency optimization for $\mu = 2$.

INTUITIVE MEANING

Optimal ballistic approach

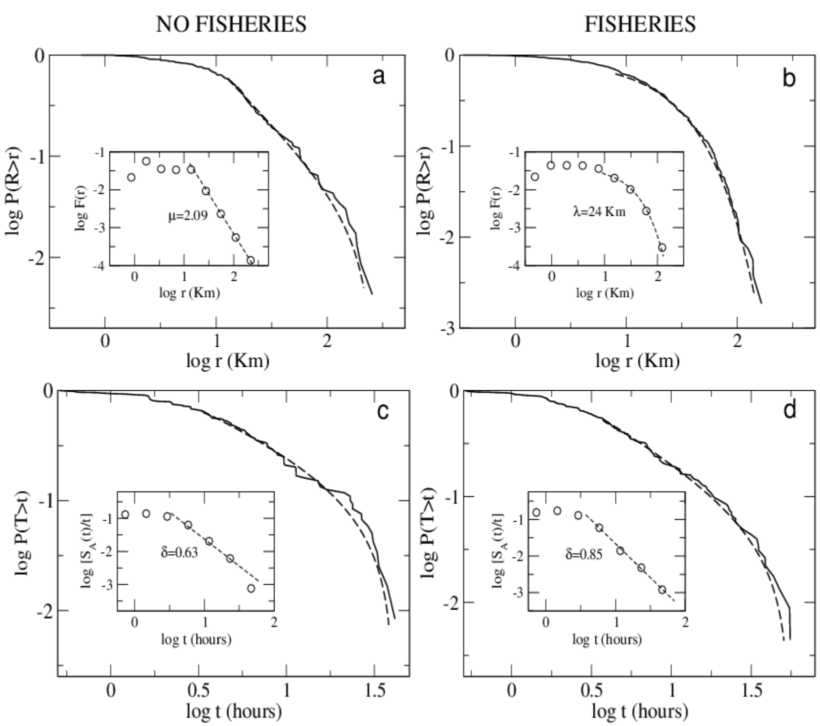


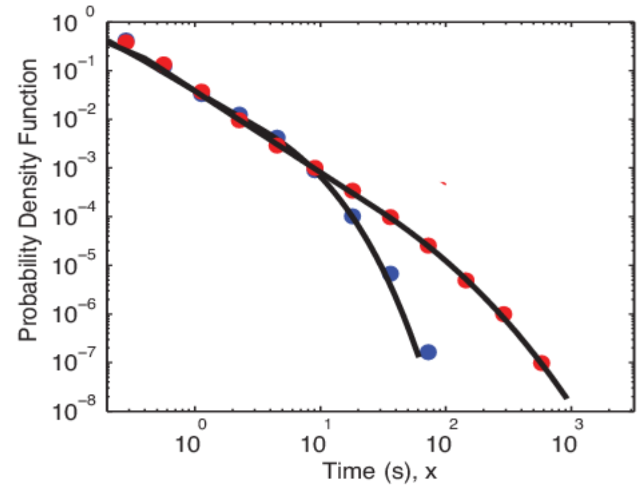
Optimal reorientation (‘correction’)

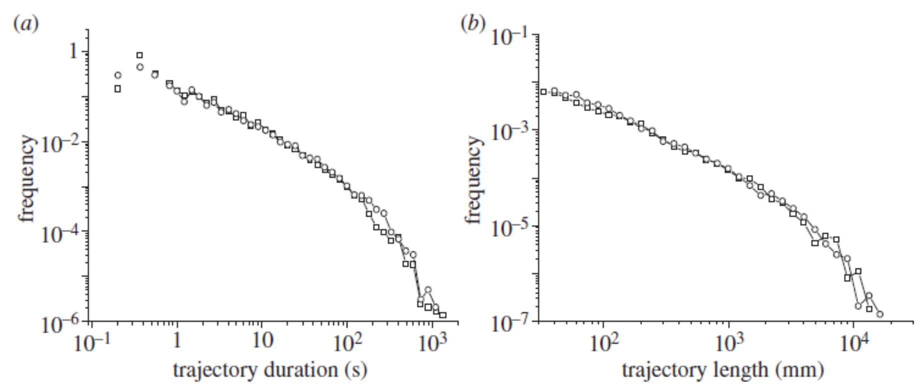


<u>THE LEVY FLIGHT FORAGING HYPOTHESIS</u>

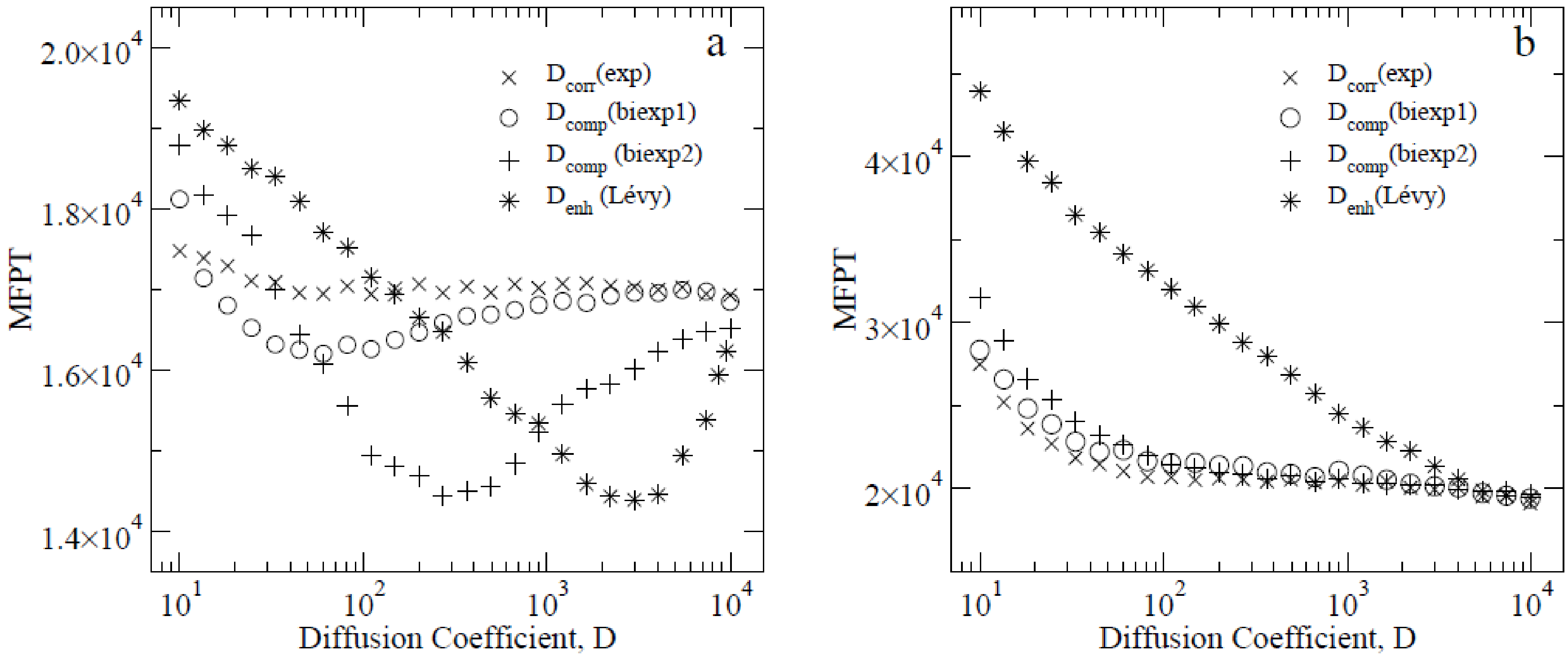
"Given Lévy Flight optimality, evolution should have favoured sensorymotor mechanisms that facilitate the emergence of motion patterns similar to the Lévy case in search situations of poor information."



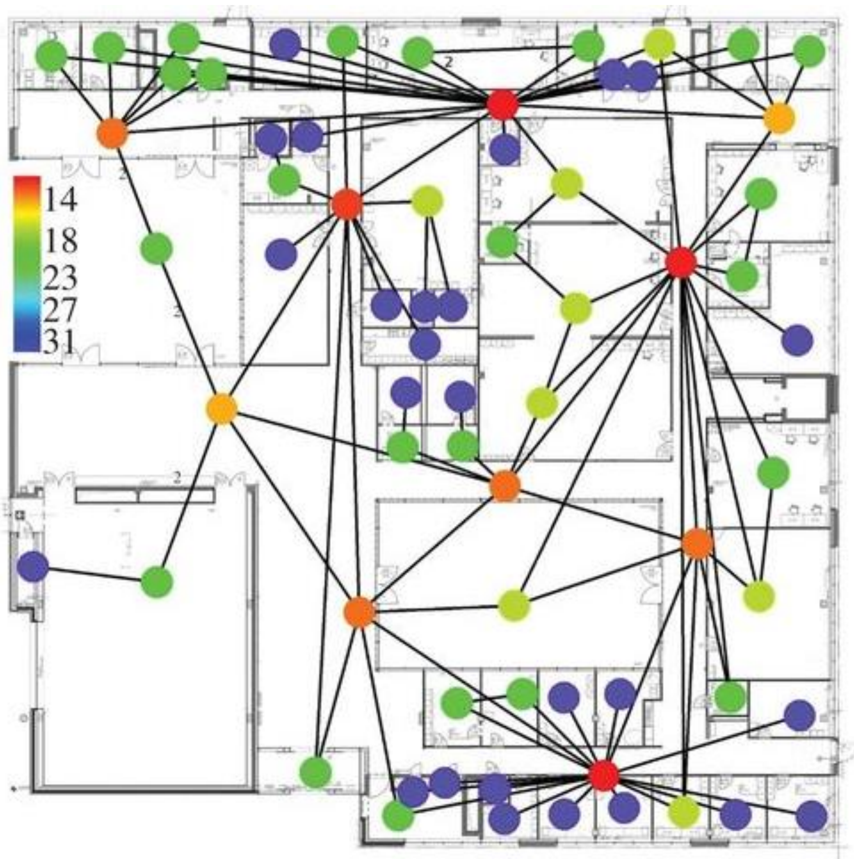




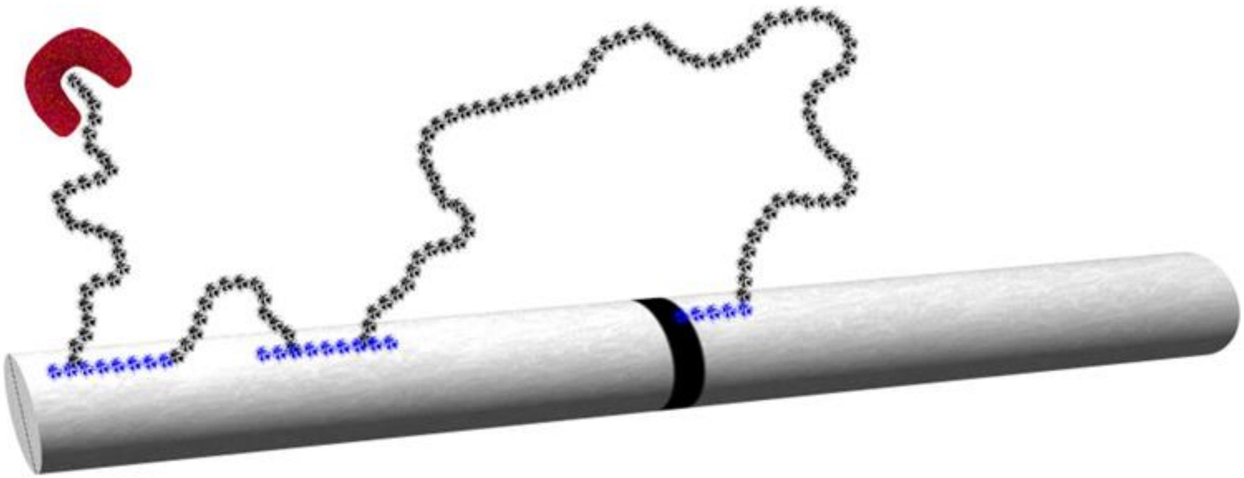
ARE LÉVY FLIGHTS SO SPECIAL?



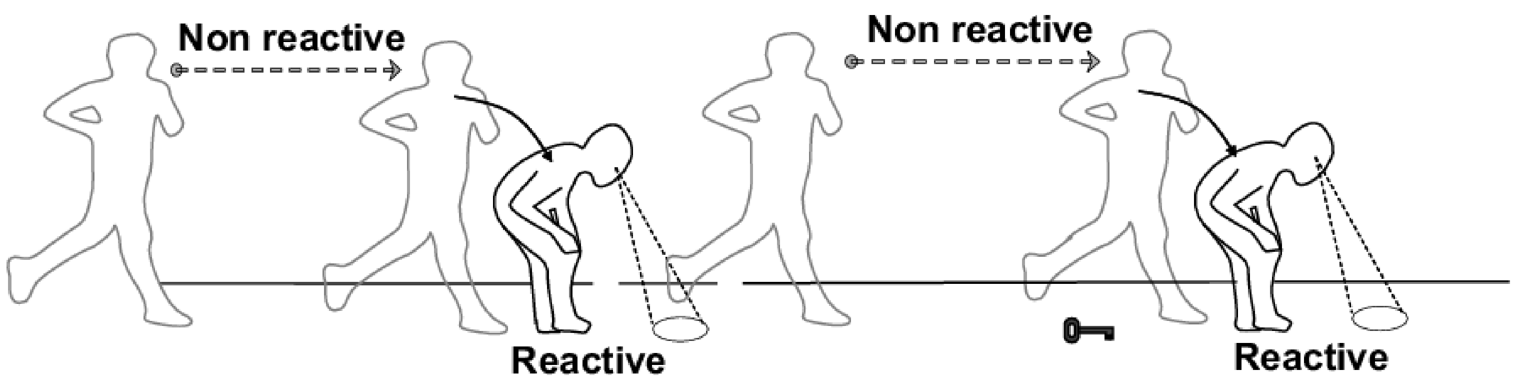
3. The optimal walk to the (intermittent) walk



Practical example 1: DNA facilitated target location (1D sliding + jumping)

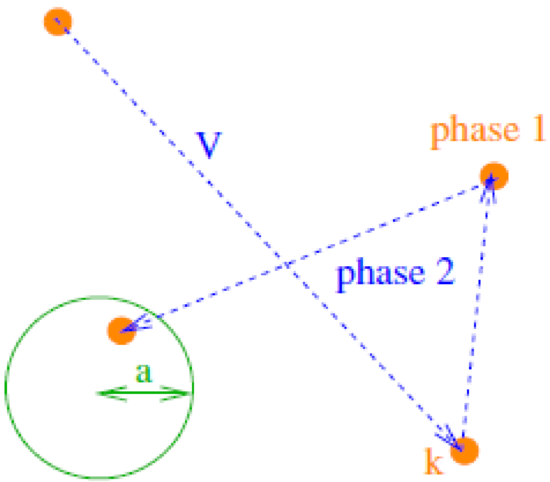


Practical example 2: Saltatory search strategies

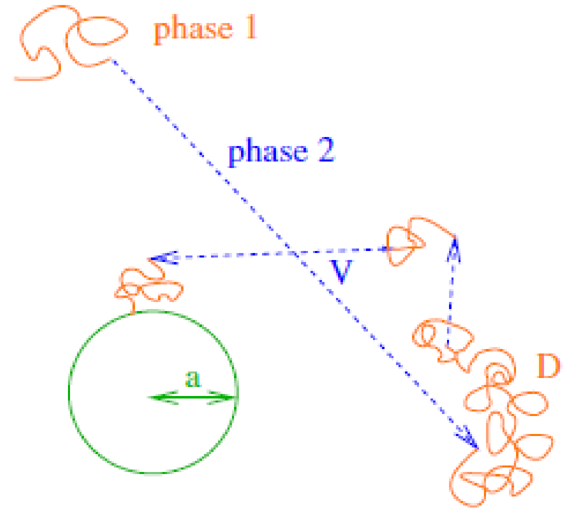


3. The optimal walk to the (intermittent) walk

O. Bénichou et. al. Rev. Mod. Phys. 83, 81 (2011)



Static mode. The slow reactive phase is static and detection takes place with finite rate *k*.



Diffusive mode.The slow reactive phase is diffusive and detection is infinitely effcient.

$$V\frac{dt_2^+}{dx} + \frac{1}{\tau_2}(t_1 - t_2^+) = -1$$

$$-V\frac{dt_2^-}{dx} + \frac{1}{\tau_2}(t_1 - t_2^-) = -1$$

$$\frac{1}{\tau_1}\left(\frac{t_2^+ + t_2^-}{2} - t_1\right) = -1$$

$$\frac{1}{\tau_1}t_2 - \left(\frac{1}{\tau_1} + k\right)t_1 = -1.$$

$$V\frac{dt_2^+}{dx} + \frac{1}{\tau_2}(t_1 - t_2^+) = -1$$

$$-V\frac{dt_2^-}{dx} + \frac{1}{\tau_2}(t_1 - t_2^-) = -1$$

$$D\frac{d^2t_1}{dx^2} + \frac{1}{\tau_1}\left(\frac{t_2^+}{2} + \frac{t_2^-}{2} - t_1\right) = -1,$$

$$\left\{ V\frac{dt_2^+}{dx} - \frac{1}{\tau_2}t_2^+ = -1 \qquad -V\frac{dt_2^-}{dx} - \frac{1}{\tau_2}t_2^- = -1 \right.$$

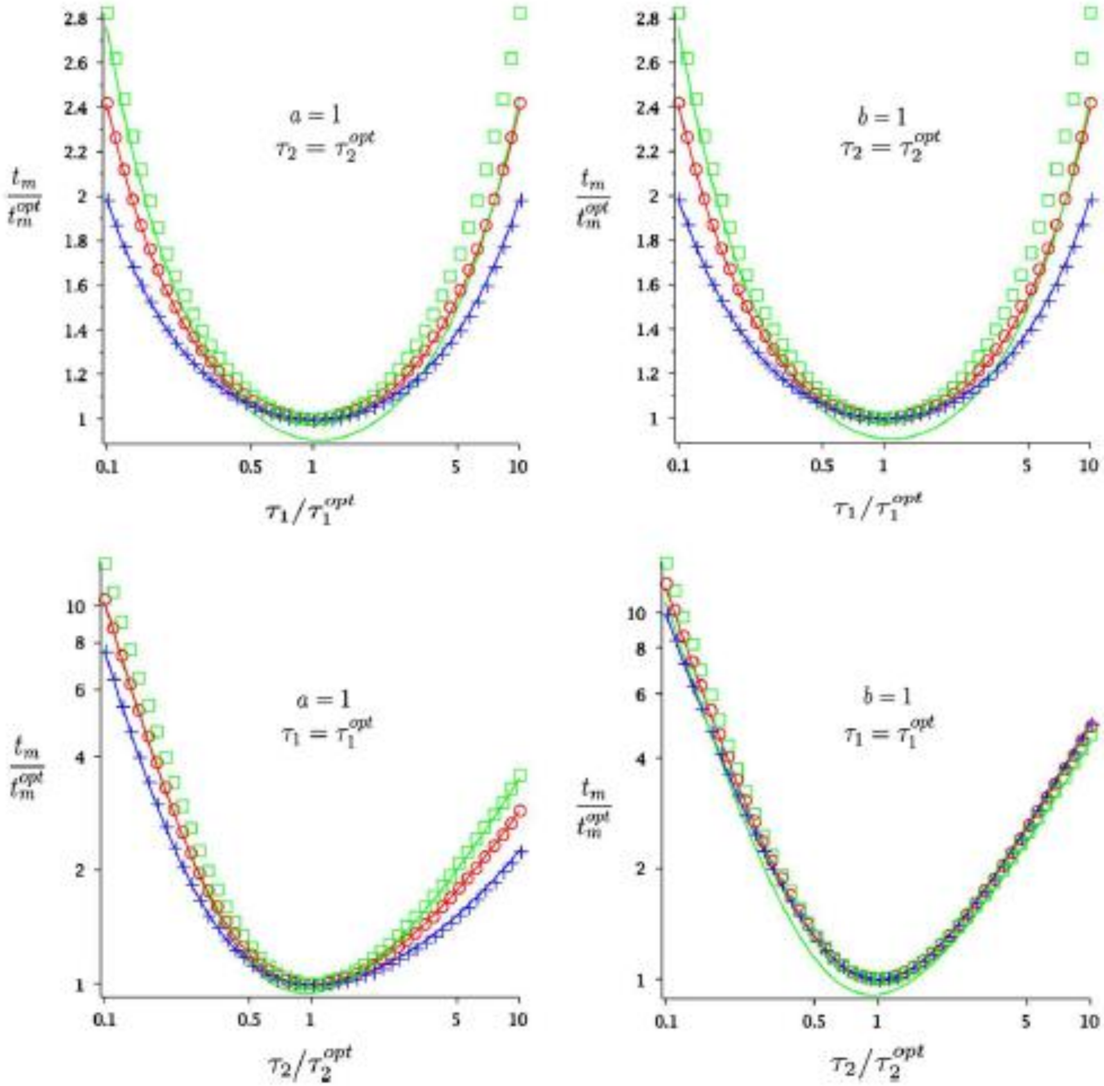
Static mode (1D)

For a domain of size b:

$$t_m = (\tau_1 + \tau_2)\left[\frac{b^2}{3V^2\tau_2^2} + \left(\frac{1}{k\tau_1} + 1\right)\frac{b}{a}\right]$$

$$\tau_1^{\rm opt} = \sqrt{\frac{a}{Vk}}\left(\frac{b}{12a}\right)^{1/4}$$

$$\tau_2^{\rm opt} = \frac{a}{V}\sqrt{\frac{b}{3a}}$$



3. The optimal walk to the (intermittent) walk

Diffusive mode (1D)

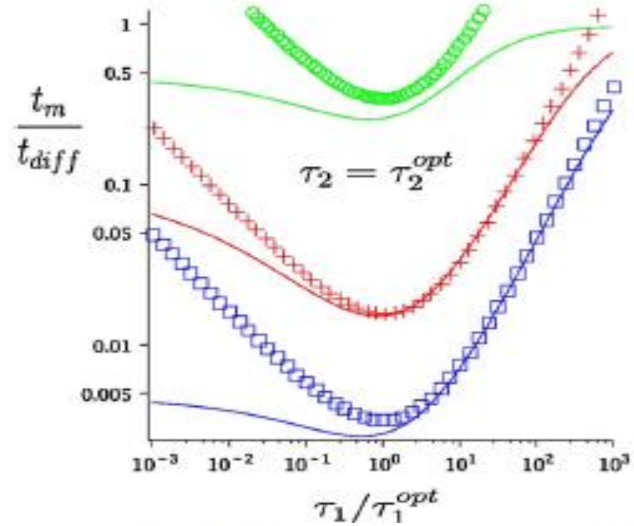
<u>$bD^2 < a^3V^2$</u>

$$t_m = (\tau_1 + \tau_2)b\left(\frac{b}{3V^2\tau_2^2} + \frac{1}{\sqrt{D\tau_1}}\right)$$

$$\tau_1^{\text{opt}} = \frac{1}{2}\sqrt[3]{\frac{2b^2D}{9V^4}}$$

$$\tau_2^{\text{opt}} = \sqrt[3]{\frac{2b^2D}{9V^4}},$$

$$t_m^{\text{opt}} \simeq \sqrt[3]{\frac{3^5}{2^4}\frac{b^4}{DV^2}}.$$



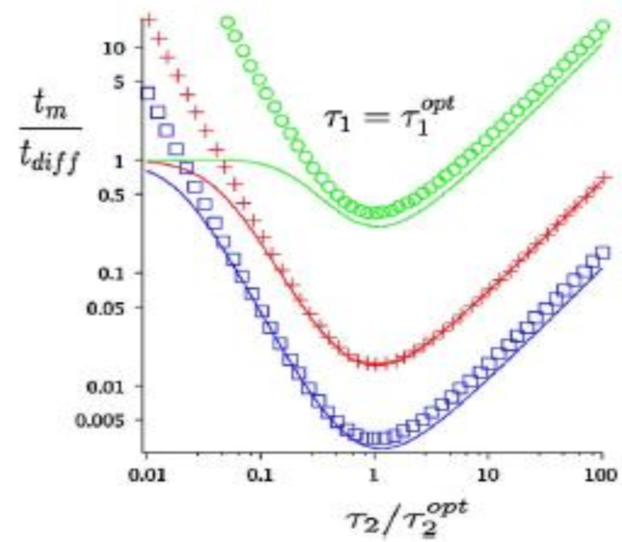
<u>$bD^2 > a^3V^2$</u>

$$t_m \simeq \frac{b}{a}(\tau_1 + \tau_2)\left(\frac{a}{a + \sqrt{D\tau_1}} + \frac{ab}{3V^2\tau_2^2}\right).$$

$$\tau_1^{\text{opt}} = \frac{Db}{48V^2a},$$

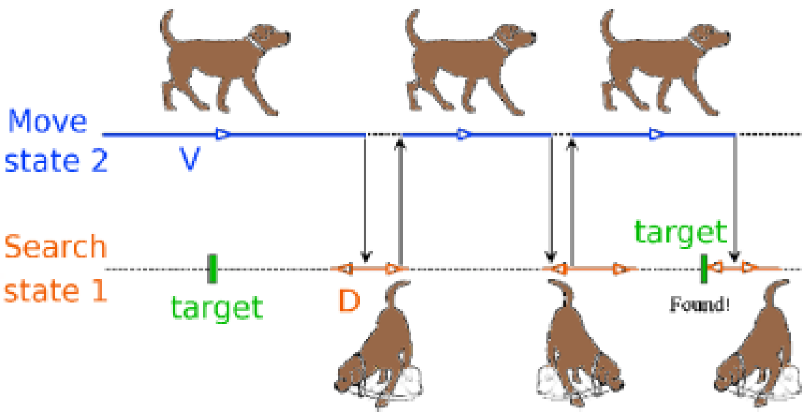
$$\tau_2^{\text{opt}} = \frac{a}{V}\sqrt{\frac{b}{3a}},$$

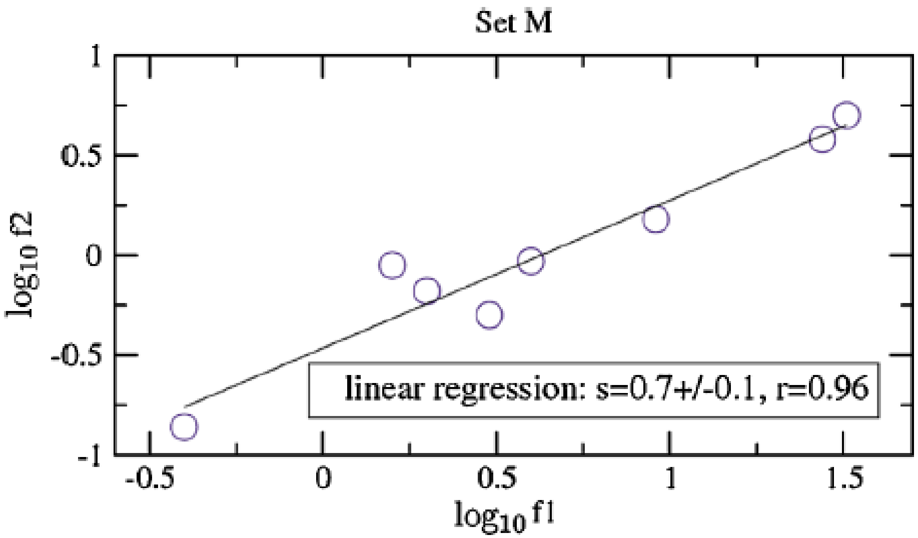
$$t_m^{\text{opt}} \simeq \frac{2a}{V\sqrt{3}}\left(\frac{b}{a}\right)^{3/2}$$



O. Bénichou et. al. Phys. Rev. Lett. 94, 198101 (2005)

Diffusive mode (1D)





$$\langle t \rangle \simeq \frac{L(\tau_2+\tau_1)(D\tau_1+2\tau_2^2V^2)}{2\tau_2 V\sqrt{D\tau_1}\sqrt{D\tau_1+4\tau_2^2V^2}}$$

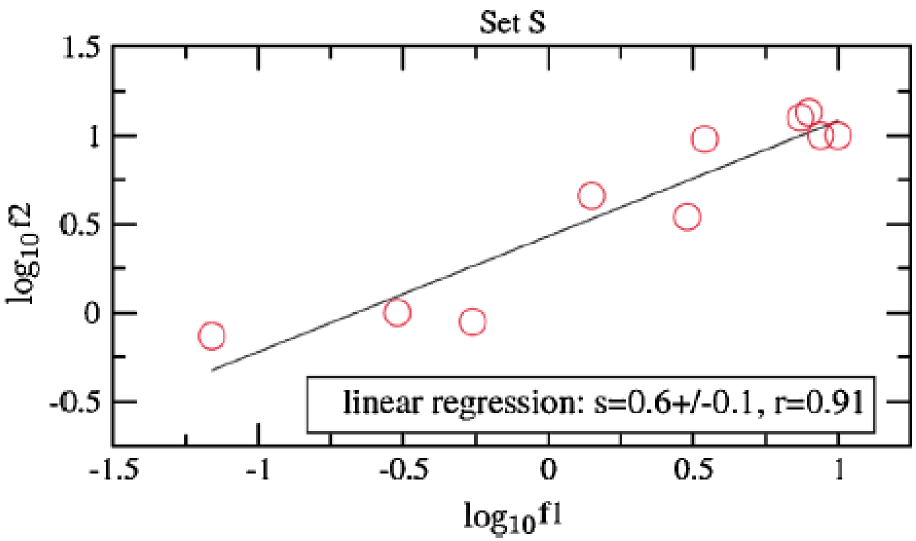
$$\tau_1^3 + 6\frac{\tau_1^2\tau_2^2}{\tau} - 8\frac{\tau_2^5}{\tau^2} = 0$$

$\underline{\tau_1 \gg D/V^2}$

$$\tau_2^{\text{opt}} = \left(\frac{3\tau\tau_1^2}{4}\right)^{1/3}$$

$\underline{\tau_1 \ll D/V^2}$

$$\tau_2^{\text{opt}} = \left(\frac{\tau^2\tau_1^3}{8}\right)^{1/5}$$



4. The optimal walk to the (myopic) walk

4. The optimal walk to the (myopic) walk

For a constant probability α of detection, $k(t;x_0) \rightarrow \alpha k(t;x_0)$

For a speed-dependent probability $\alpha = \alpha(v)$:

$$k(t;x_0)dt = \int_0^{\infty} dv \int_{0-vdt}^{0} dx\, \alpha(v) p(x,v,t;x_0) + \int_{-\infty}^{0} dv \int_0^{vdt} dx\, \alpha(v) p(x,v,t;x_0) \approx$$

$$\approx \int_0^{\infty} dv\, v\alpha(v) p(0,v,t;x_0)dt + \int_{-\infty}^{0} dv\, v\alpha(v) p(0,v,t;x_0)dt$$

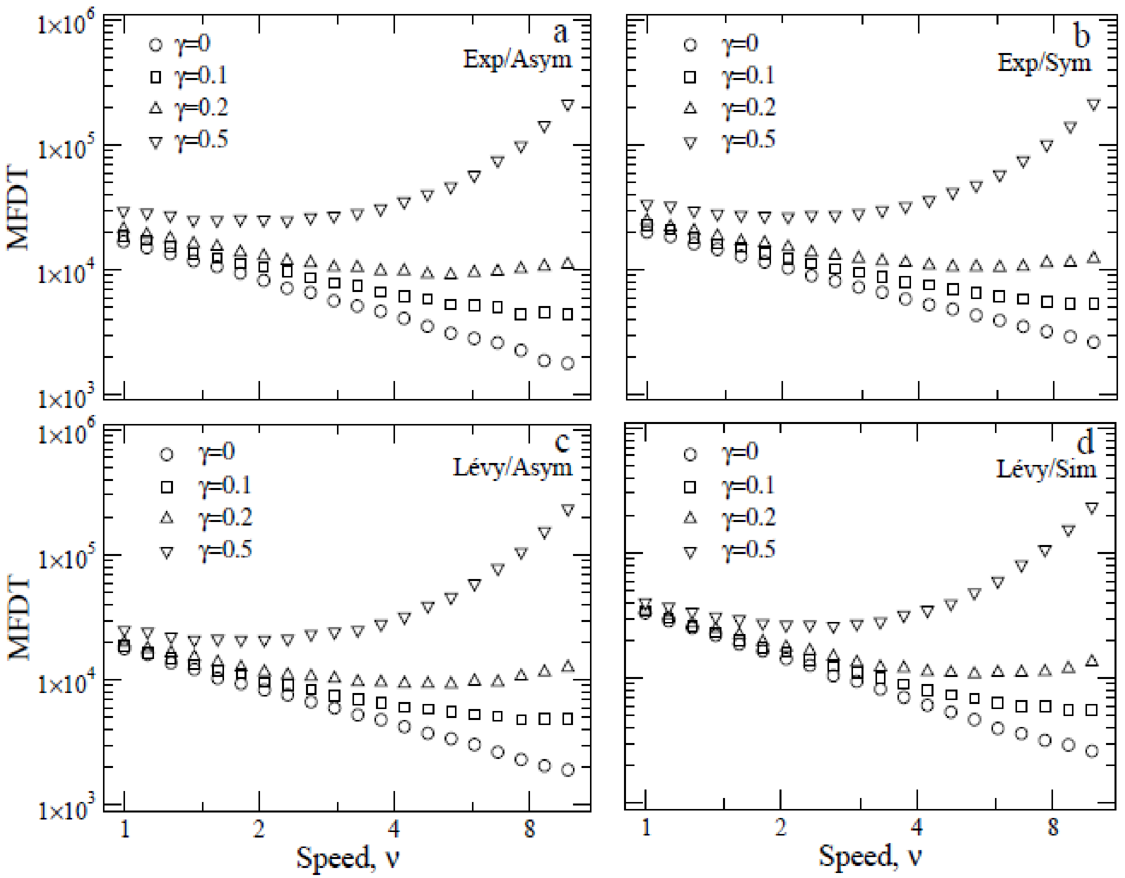
In general, $\alpha = \alpha(x,v,t)$:

$$k(t;x_0)dt \approx \int_0^{\infty} dv\, v\alpha(x,v,t) p(0,v,t;x_0)dt - \int_{-\infty}^{0} dv\, v\alpha(x,v,t) p(0,v,t;x_0)dt$$

4. The optimal walk to the (myopic) walk

Case 1D 'velocity' model with v fixed and $\varphi(t) = \lambda e^{-\lambda t}$ or $\varphi(s) = e^{-a|k|^{\mu}}$, $\alpha = e^{-\gamma v}$

$$\langle T \rangle = \underbrace{\frac{2x_0(L-x_0)\lambda}{v^2}}_{\langle \boldsymbol{T_1} \rangle} + \underbrace{\frac{L}{v\alpha(v)}}_{\langle \boldsymbol{T_2} \rangle}$$

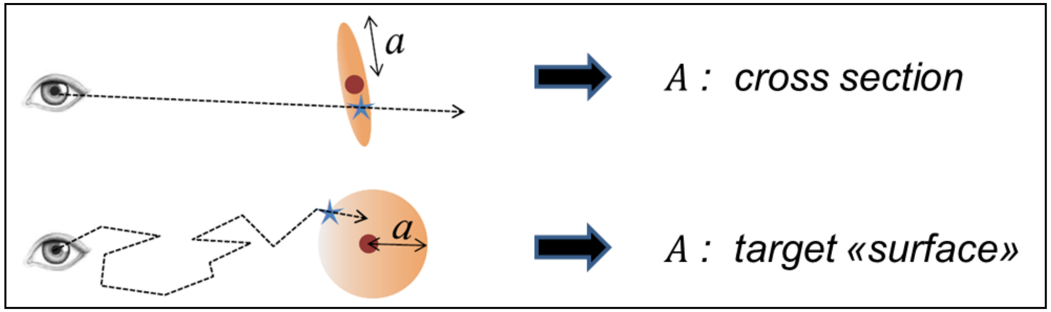


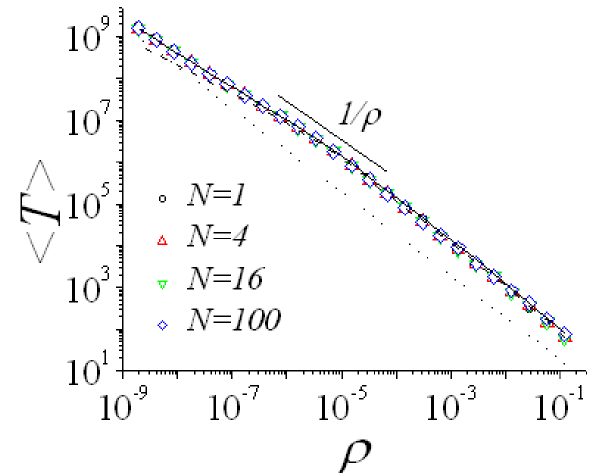
4. The optimal walk to the (myopic) walk

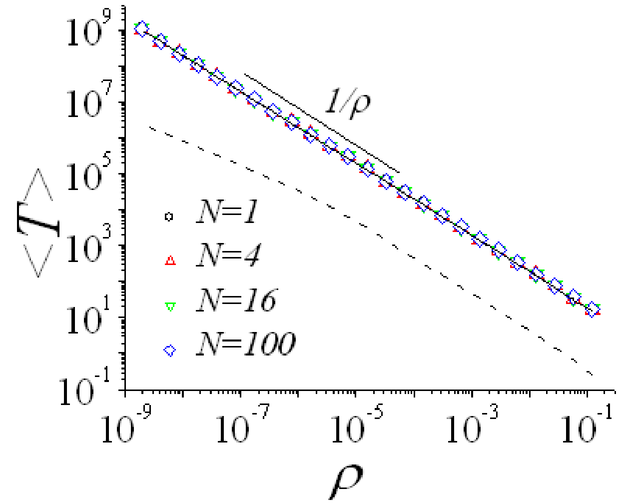
Case 2D or higher 'velocity' model with v fixed and $\varphi(t) = \lambda e^{-\lambda t}$, $\alpha = e^{-\gamma v}$

$$\langle T \rangle = \frac{L^2}{v^2} g_d(x_0) + \frac{L^d}{A v \alpha(v)} = \frac{L^2}{v^2} g_d(x_0) + \frac{1}{\rho} \frac{1}{A v \alpha(v)}$$

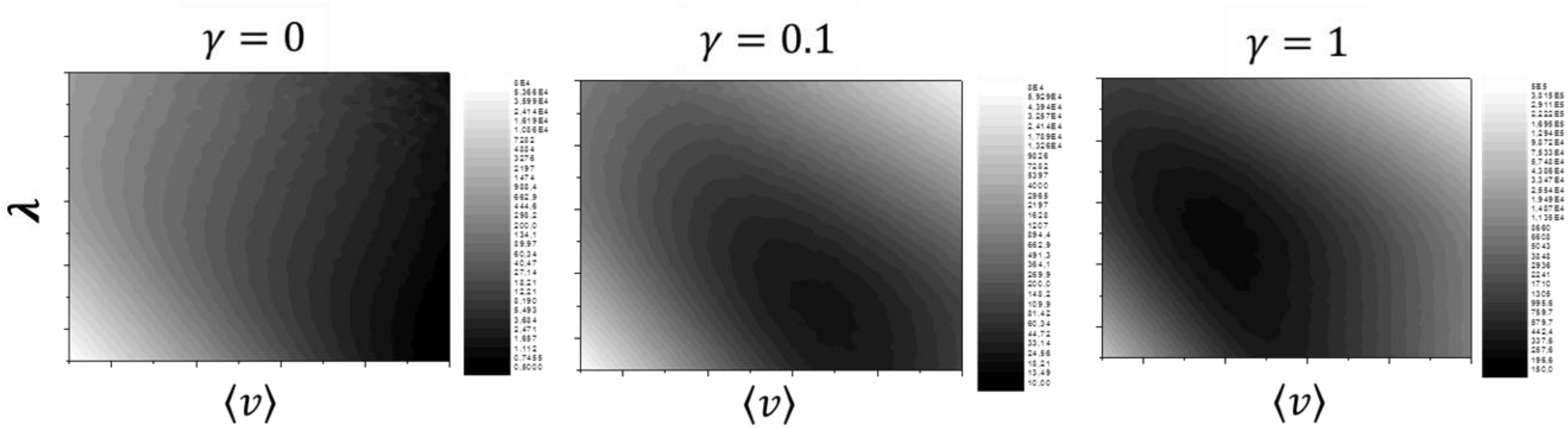
$\langle \boldsymbol{T_1} \rangle$ $\langle \boldsymbol{T_2} \rangle$

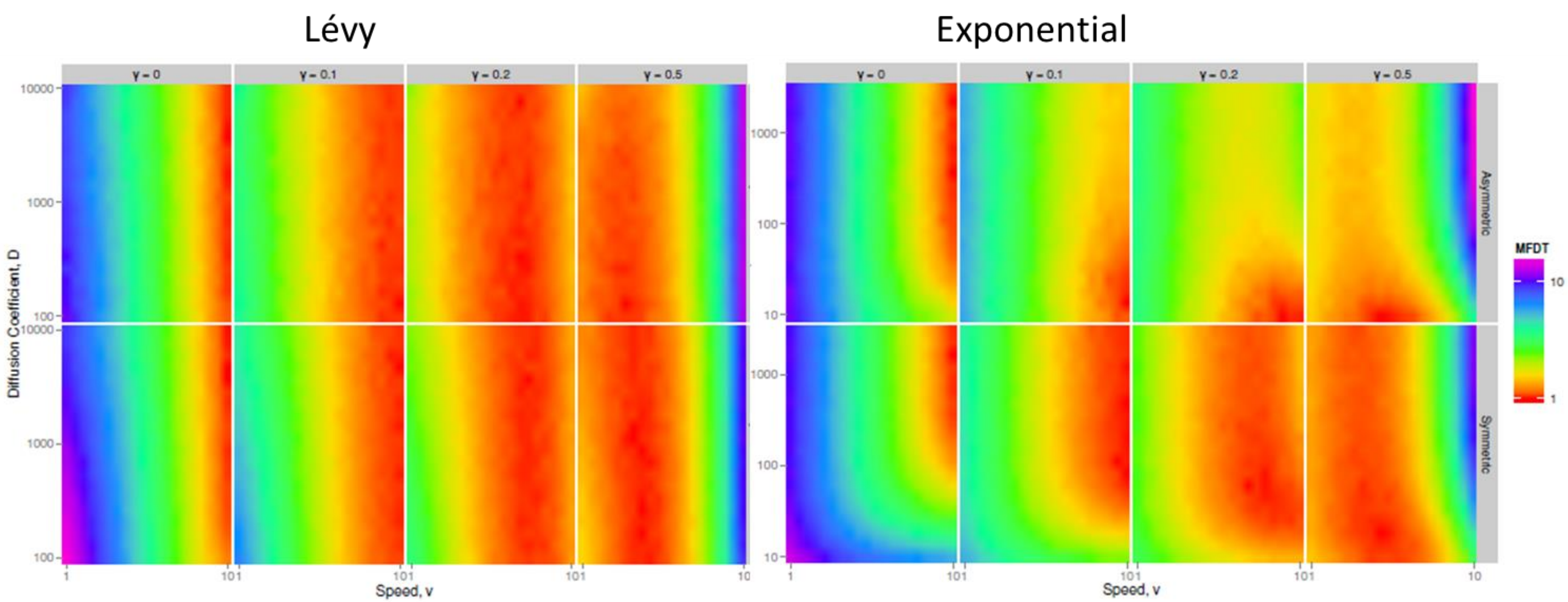




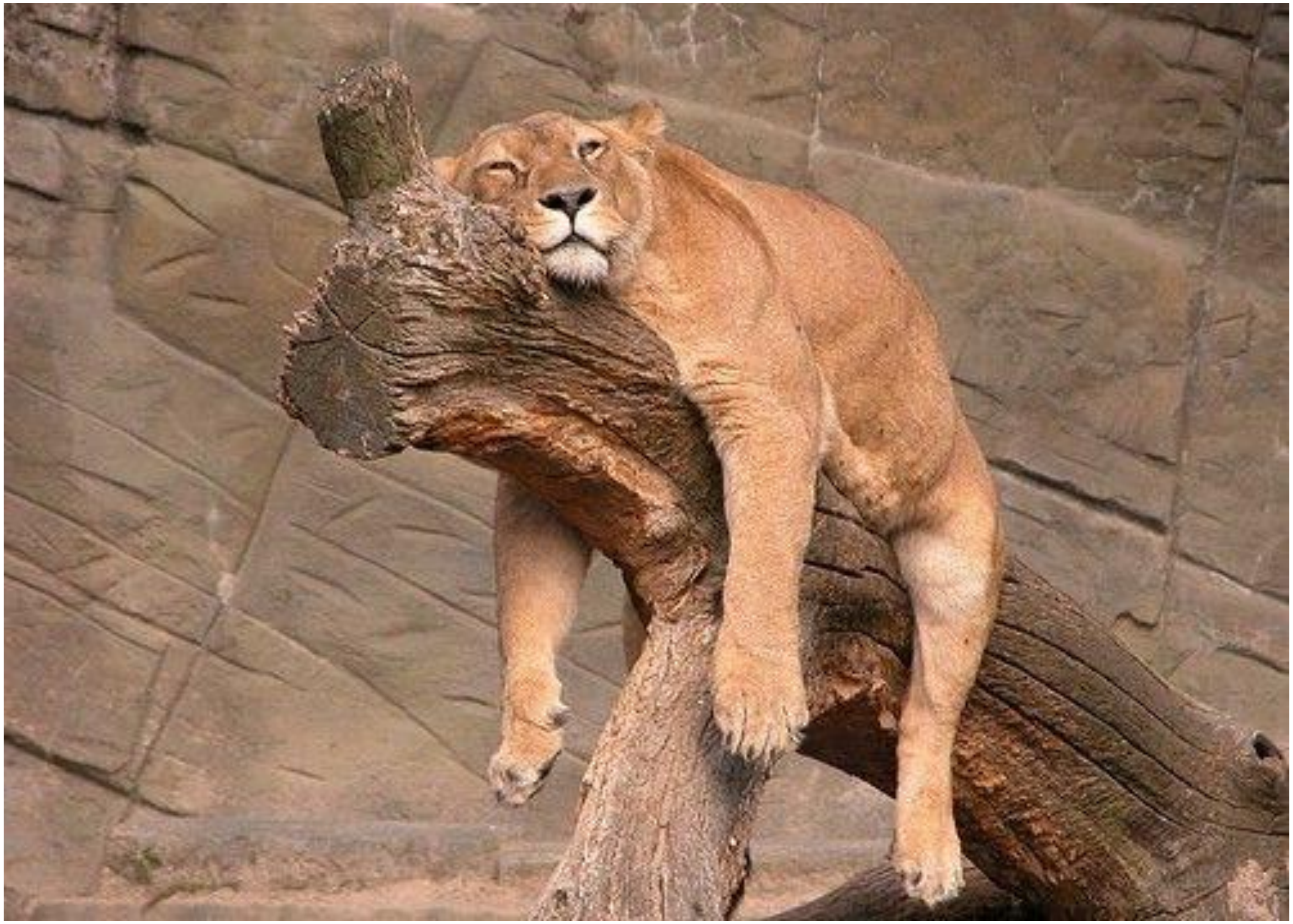


Case 2D with v fixed and $\varphi(t) = \lambda e^{-\lambda t}$ or $\varphi(s) = e^{-a|k|^{\mu}}$, $\alpha = e^{-\gamma v}$





5. The optimal walk to the (mortal) walk



5. The optimal walk to the (mortal) walk

For a constant mortality rate ω:

$$k(t;x_0) \to e^{-\omega t} k(t;x_0) \qquad \to \qquad k(s;x_0) \to k(s+\omega;x_0) \qquad \to \qquad f(s;x_0) = \frac{k(s+\omega;x_0)}{1+k(s+\omega;x_t)}$$

Here $S(\infty)$ is the most relevant parameter:

$$S(\infty) = \int_0^\infty dt\, f(t;x_0) = \lim_{s\to 0} \int_0^\infty dt\, e^{-st} f(t;x_0) = \lim_{s\to 0} f(s;x_0)$$

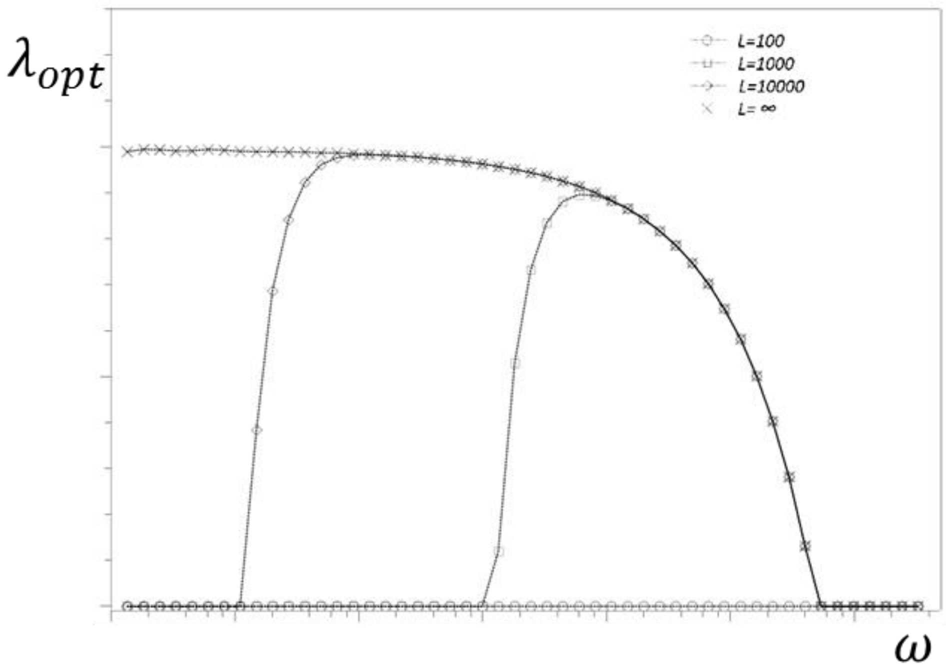
For the 'pure' diffusion model:

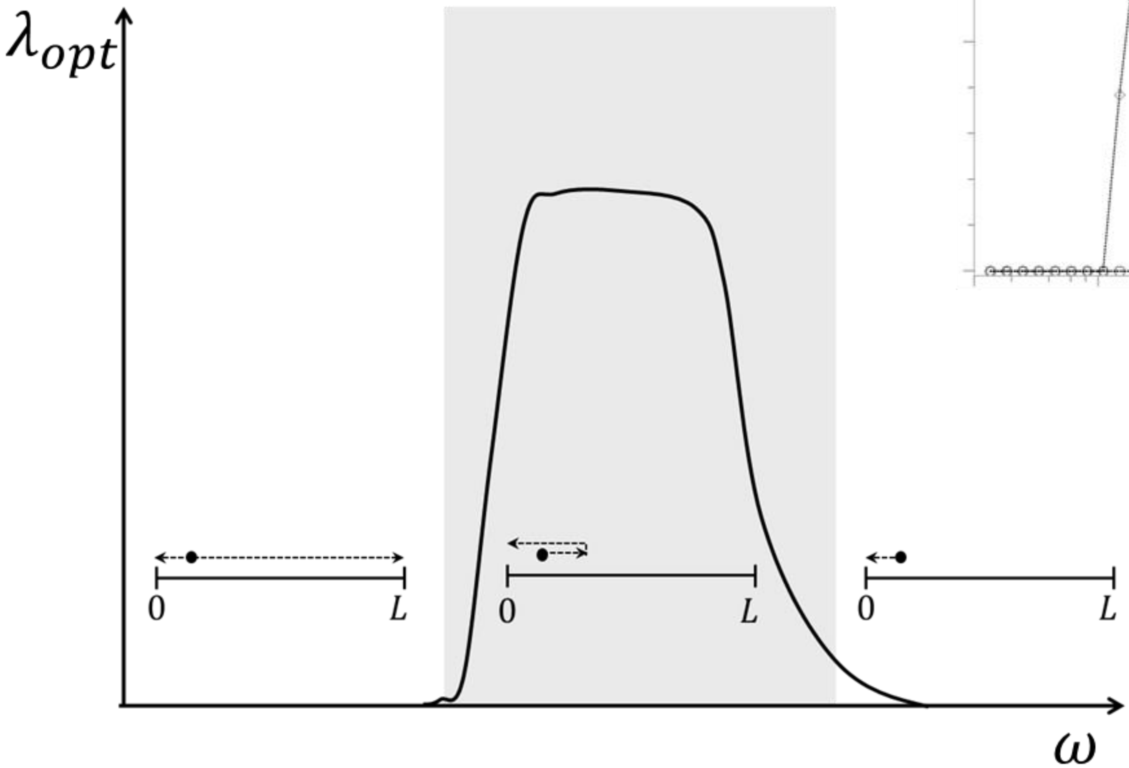
$$S(\infty) = 1 - \frac{\sqrt{\omega\lambda}\left(e^{-\sqrt{\omega\lambda}\,x_0/v} + e^{-\sqrt{\omega\lambda}\,(L-x_0)/v}\right)}{1 + e^{-\sqrt{\omega\lambda}\,L/v}}$$

For the 'velocity' model with v fixed and $\varphi(t) = \lambda e^{-\lambda t}$:

$$S(\infty) = 1 - \frac{\sqrt{\omega(\omega+\lambda)}\left(e^{-\sqrt{\omega(\omega+\lambda)}x_0/v} + e^{-\sqrt{\omega(\omega+\lambda)}(L-x_0)/v}\right)}{\omega\left(1 - e^{-\sqrt{\omega(\omega+\lambda)}L/v}\right) + \sqrt{\omega(\omega+\lambda)}\left(1 + e^{-\sqrt{\omega(\omega+\lambda)}L/v}\right)}$$

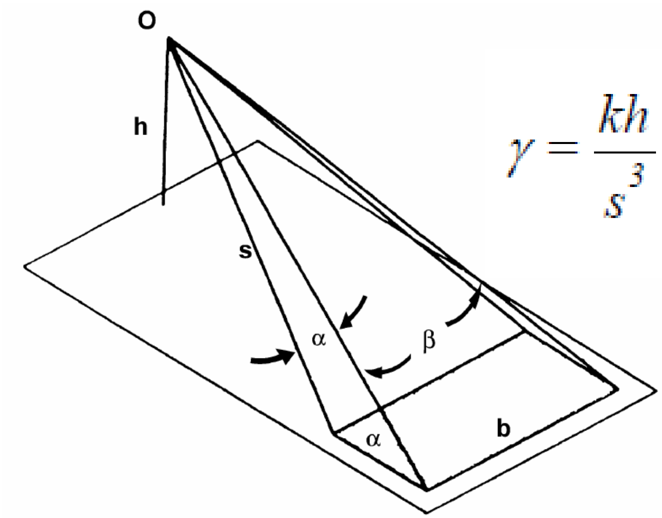
Implications for the Lévy flight paradigm:



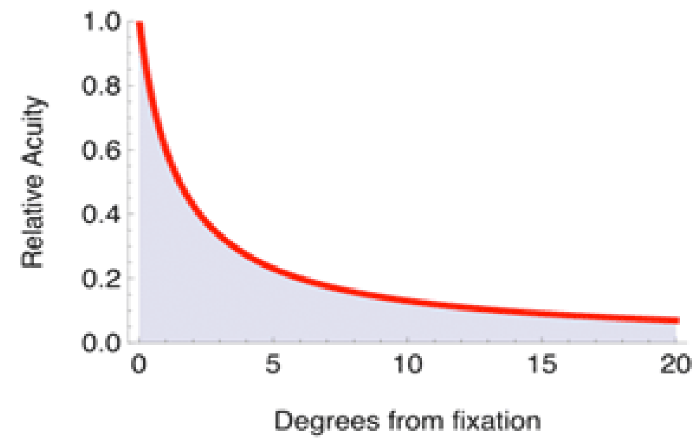


6. The optimal walk to the (systematic?) walk

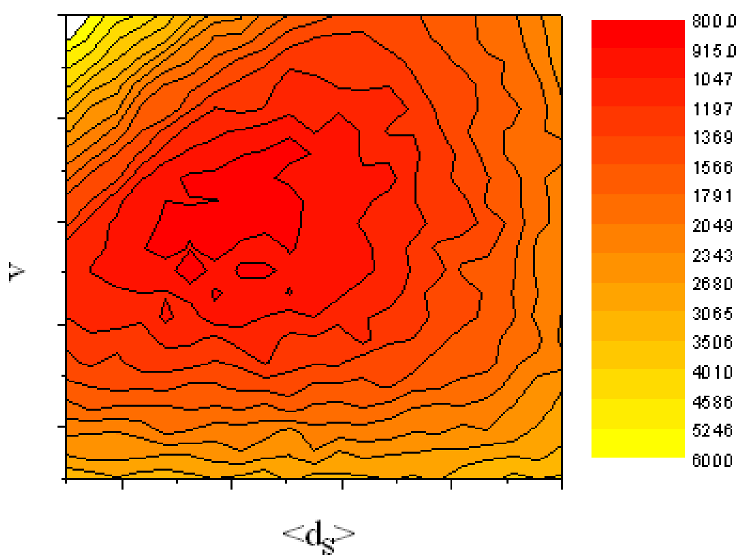
NON-PERFECT DETECTION: SACCADE-FIXATION MECHANISM



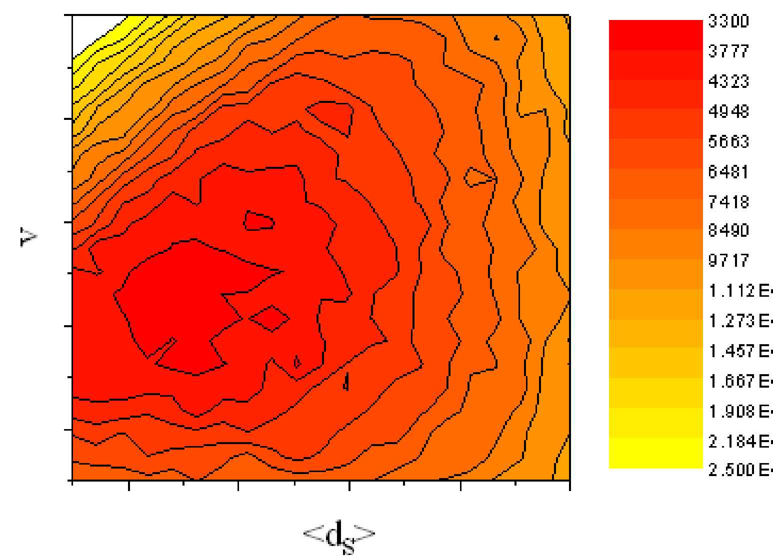
$$\gamma = \frac{kh}{s^3} = \frac{kh}{(h^2 + r^2)^{\frac{3}{2}}}$$



$c = 1$

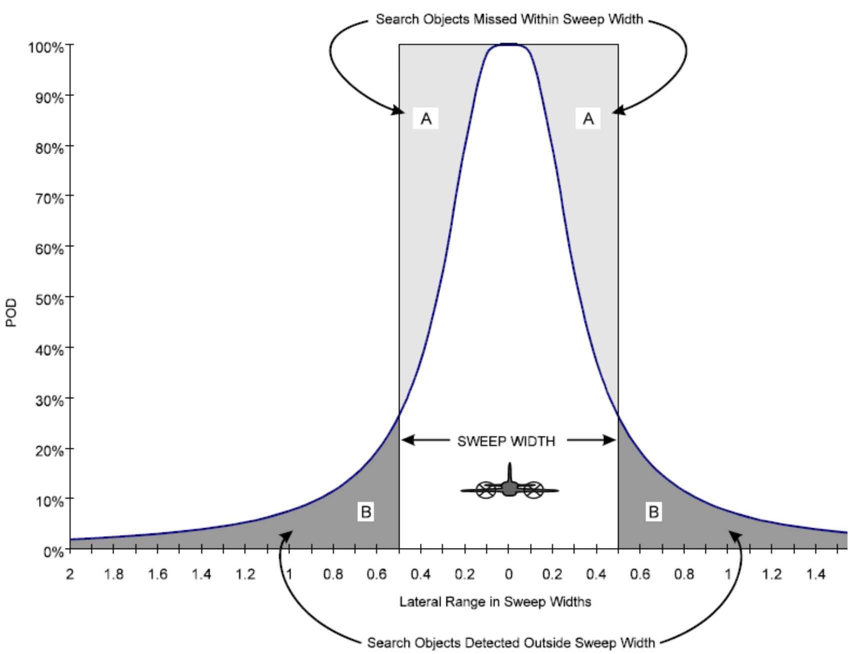


$c = 0.2$



6. The optimal walk to the (systematic?) walk

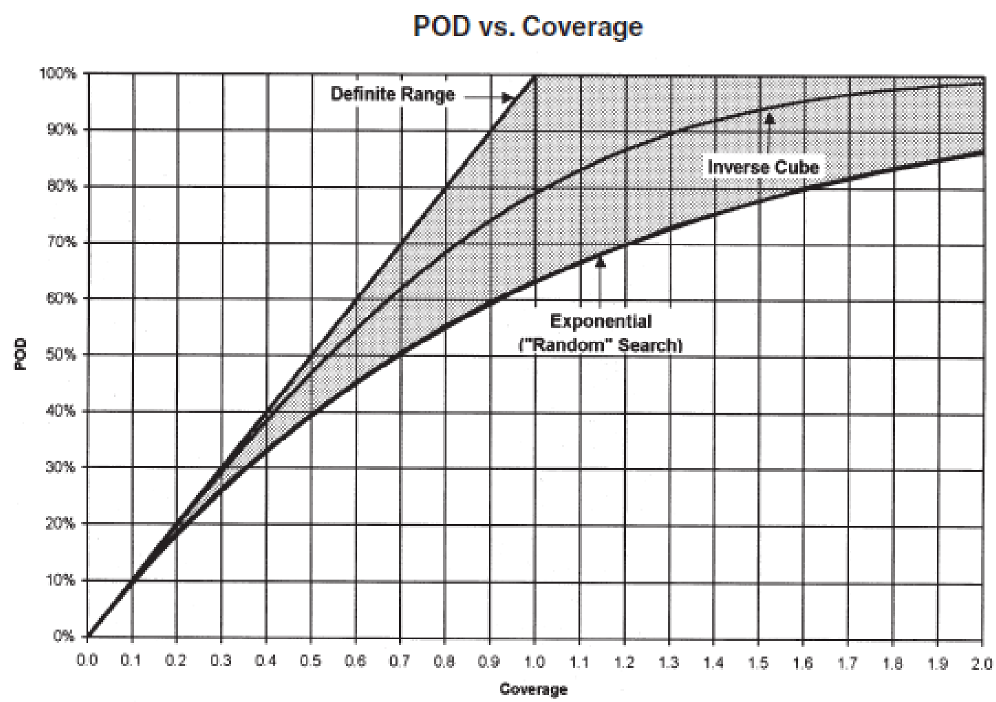
Optimal search theory:



$$Z = W x V x T \qquad C = \frac{Z}{A}$$

"Random" search

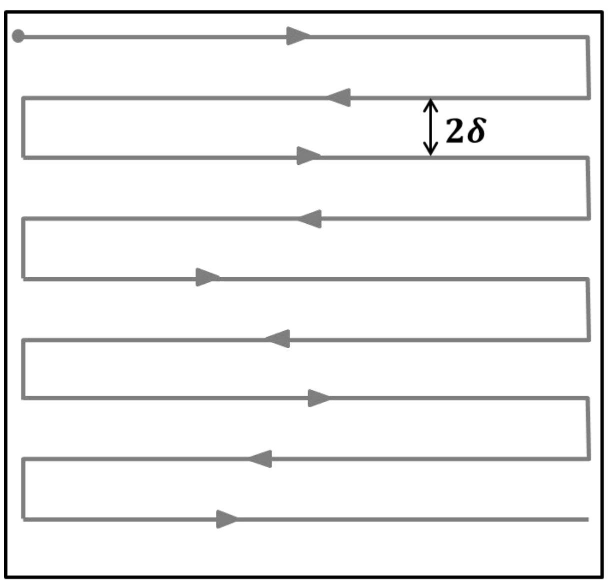
$$POD = 1 - e^{-C}$$



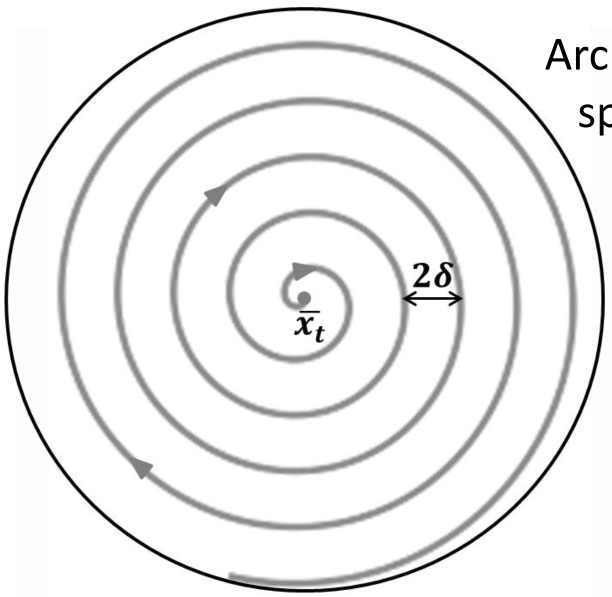
Optimal effort allocation *vs* optimal path planning

3.23%	3.23%	6.45%	6.45%	6.45%
3.23%	3.23%	6.45%	9.68%	12.90%
3.23%	3.23%	9.68%	9.68%	12.90%

Parallel Sweep (PS)

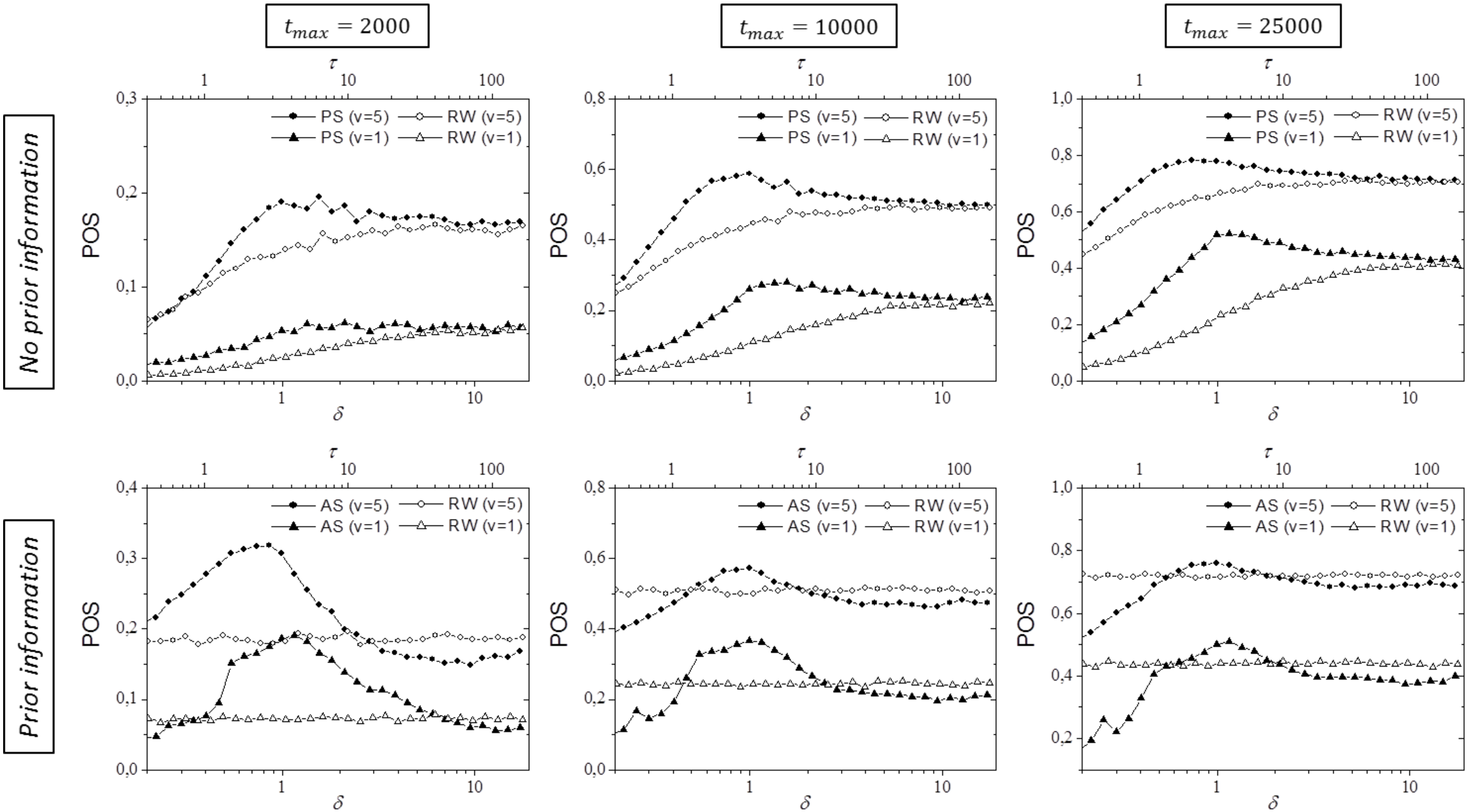


Archimedean spiral (AS)

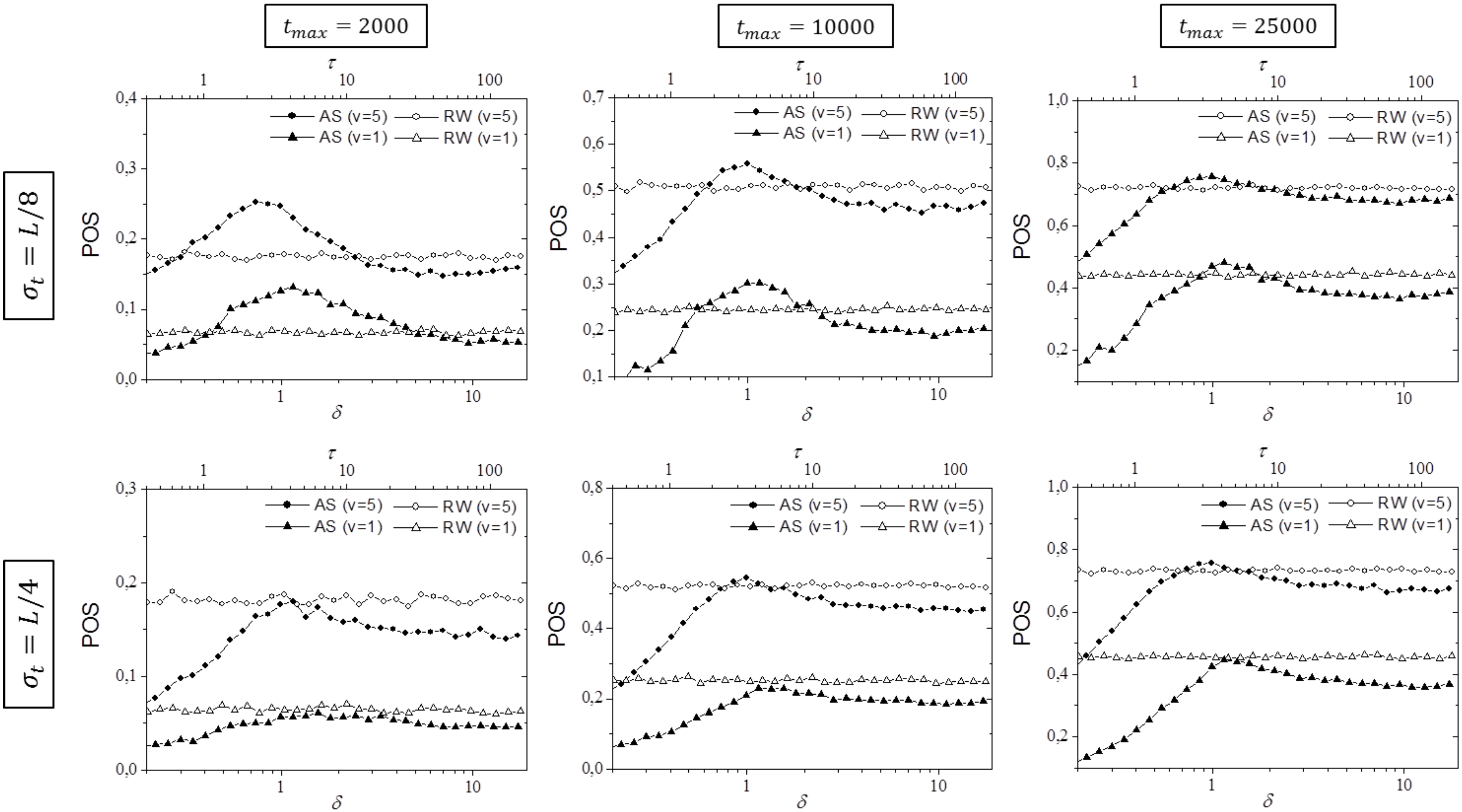


6. The optimal walk to the (systematic?) walk

Errors in pattern sizing



Errors in prior information

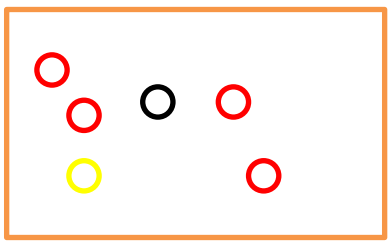


6. The optimal walk to the (systematic?) walk

Does it make any sense to foresee human errors?

According to psychologycal research during the last 50 years, yes

Example: Ellberg’s paradox



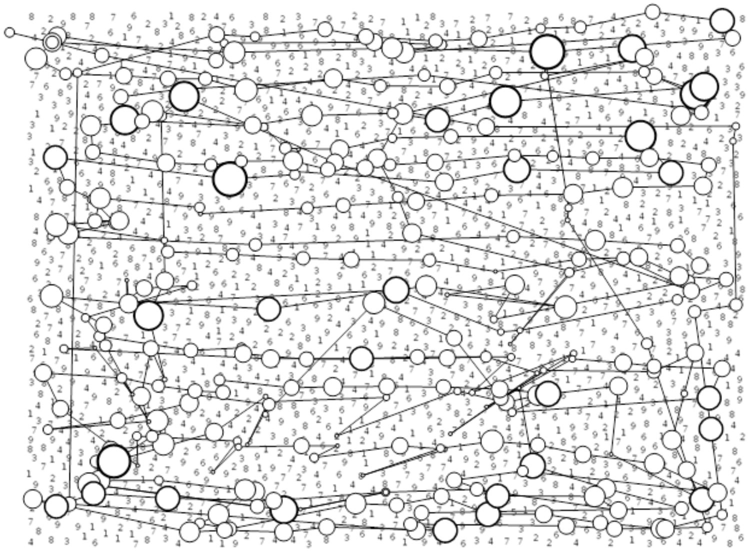
30 balls

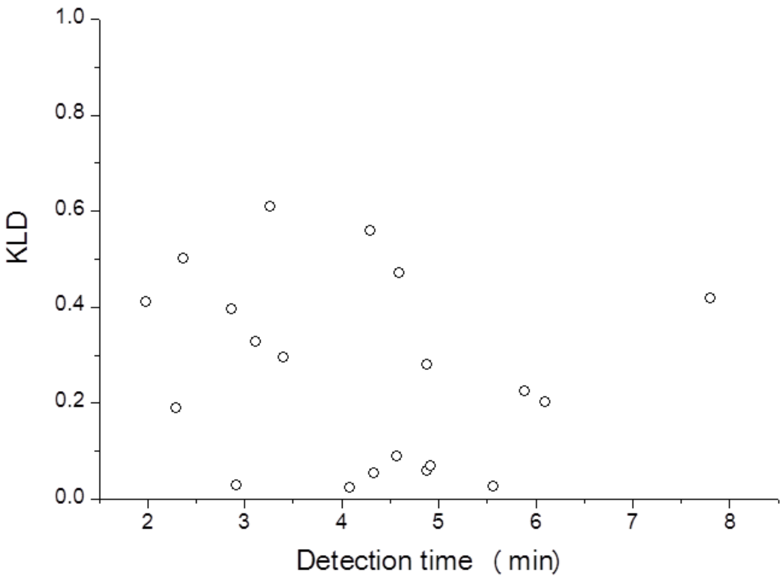
Red: 10
Yellow+Black: 20

gamble 1: receive $100 if red is drawn
gamble 2: receive $100 if black is drawn
gamble 3: receive $100 if red or yellow is drawn
gamble 4: receive $100 if black or yellow is drawn.

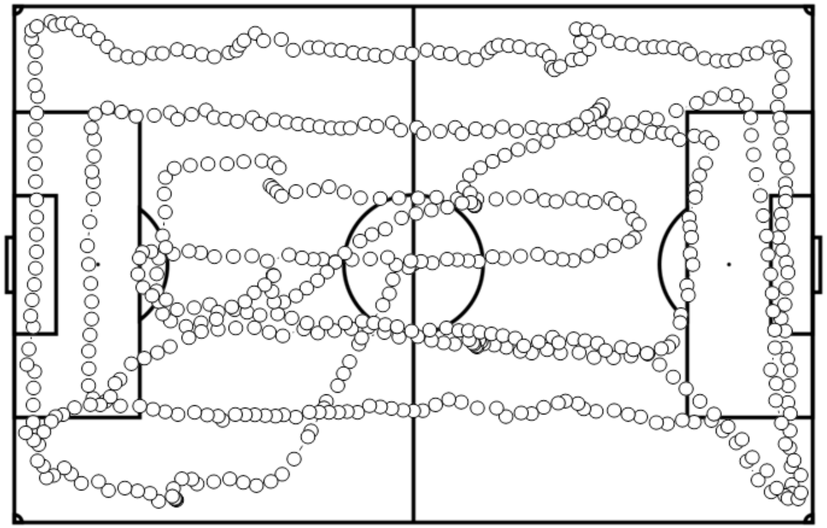
6. The optimal walk to the (systematic?) walk

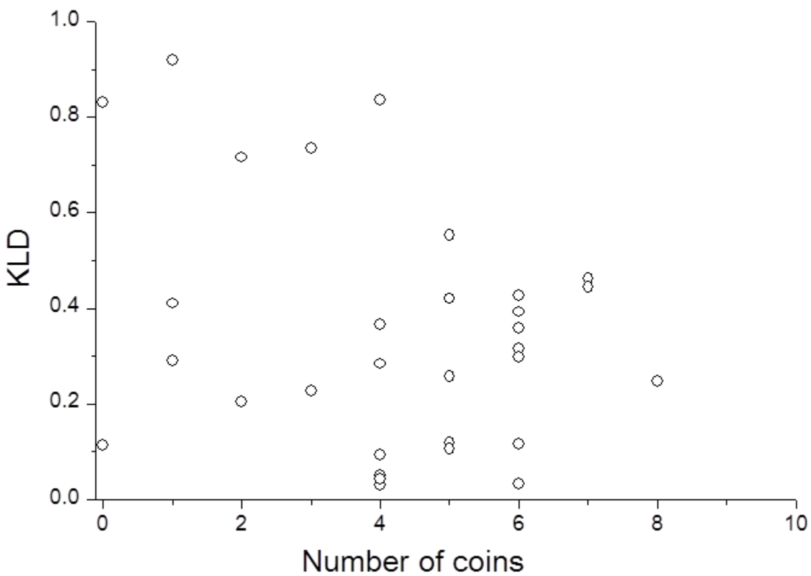
Eye-tracking





Search in an open field





Collaborators:

Vicenç Méndez
Universitat Autònoma de Barcelona

Frederic Bartumeus
Centre d'Estudis Avançats de Blanes, Girona

Xavier Espadaler
Jordi Catalán
John Palmer
Centre de Recerca Ecològica i Aplicacions Forestals, UAB

Enrique Abad
Santos Bravo Yuste
Universidad de Extremadura

Will Ryu
University of Toronto

José Soares de Andrade Jr
Universidade Federal do Ceará

Katja Lindenberg
University of California San Diego