Convergence and Regularity of Probability Laws using an interpolation inequality

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Abstract Integration by Parts Formulas : Let F be a random variable. There exists $H_\alpha(F) \in L^p$ such that

$$IP_{\alpha,p}(F) \qquad E(\partial_\alpha f(F)) = E(f(F)H_\alpha(F)) \qquad \forall f \in C_c^\infty$$

Basic example : $F \sim N(0, \sigma)$. Then

$$\begin{aligned} E(f(F)) &= \frac{1}{\sqrt{2\pi\sigma}} \int f'(x) e^{-\frac{x^2}{2\sigma}} dx \\ &= \frac{-1}{\sqrt{2\pi\sigma}} \int f(x) \frac{x}{\sigma} e^{-\frac{x^2}{2\sigma}} dx = -E(f(F)\frac{F}{\sigma}). \end{aligned}$$

Malliavin calculus : Simple functionals $\Delta_k = W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})$

$$F_n = \phi(\Delta_1, ..., \Delta_n)$$

. For F_n we define

$$D_s F_n = \frac{\partial}{\partial \Delta_s} \phi(\Delta_1, ..., \Delta_n), \qquad DF_n \in L^2(0,1).$$

Definition. $F \in D^{1,2}$ if there exists $F_n, n \in N$ simple functionals such that

$$F_n \to F \qquad L^2(dP)$$
$$DF_n \to U \qquad L^2(ds \times dP)$$

Then $DF = U$.

In this way one builts an infinite dimensional differential calculus and employes it in order to built $H_\alpha(F)$.

Regularity of the Law Fourier Approach. $\widehat{p}_F(\xi) = E(e^{i\xi F})$. Then

$$\int_{\{|\xi|>1\}} |\widehat{p}_F(\xi)|^2 \, d\xi < \infty \qquad \Longrightarrow \qquad P_F(dx) = p_F(x)dx$$

$$\int_{\{|\xi|>1\}} |\xi|^k \, |\widehat{p}_F(\xi)| \, d\xi < \infty \qquad \Longrightarrow \qquad p_F(x) \in C^k.$$

Link with IP. We have

$$\partial_x e^{i\xi x} = i\xi e^{i\xi x} \qquad \to \qquad e^{i\xi x} = \frac{1}{i\xi}\partial_x e^{i\xi x}$$

so

$$\widehat{p}_F(\xi) = \frac{1}{i\xi}E(\partial_x e^{i\xi F}) = \frac{1}{i\xi}E(e^{i\xi F}H_1(F))$$

Than

$$\int_{\{|\xi|>1\}} |\widehat{p}_F(\xi)|^2 \, d\xi \leq \int_{\{|\xi|>1\}} \frac{1}{\xi^2} \left| E(e^{i\xi F}H_1(F)) \right|^2 d\xi \leq E(|H_1(F)|^2) \int_{\{|\xi|>1\}} \frac{1}{\xi^2} d\xi < \infty.$$

Malliavin $\to IP \to$**Regularity.**

Alternative Approach

Step 1. We consider simple functionals

$$F_n = \phi(\Delta_1, ..., \Delta_n)$$

and we use Elementary Finite Dimensional integration by parts and we obtain

$$IP_{\alpha,p}(F_n) \qquad E(\partial_\alpha f(F_n)) = E(f(F_n)H_\alpha(F_n)) \qquad \forall f \in C_c^\infty$$

Step 2. We fix ξ and we write

$$\begin{aligned} |\widehat{p}_F(\xi)| &\leq \left|\widehat{p}_F(\xi) - \widehat{p}_{F_n}(\xi)\right| + \left|\widehat{p}_{F_n}(\xi)\right| \\ &\leq |\xi|\, E\,|F_n - F| + \frac{1}{|\xi|} E(|H_1(F_n)|). \end{aligned}$$

Passing to the limit

$$|\widehat{p}_F(\xi)| \leq \frac{1}{|\xi|} \sup_n E(|H_1(F_n)|) < \frac{1}{|\xi|} \times C.$$

Remark : We are close to Malliavin calculus.

Problem :

$$\sup_n E(|H_1(F_n)|^2) = \infty.$$

Solution

$$E\,|F_n - F| \downarrow 0 \qquad \color{red}{BALANCE} \qquad E(|H_1(F_n)|) \uparrow \infty$$

Examples : Diffusion processes.

$$X_t(x) = x + \sum_{j=1}^{d} \int_0^t \sigma_j(X_s(x))dW_s^j + \int_0^t b(X_s(x))dW_s \in R^n.$$

Semigroup :

$$P_t f(x) = E(f(X_t(x))).$$

Faymann Kac Formula :

$$\partial_t(P_t f) = L(P_t f) \qquad L = (\sigma\sigma^*)\Delta + b\nabla.$$

Example (Fournier-Printemps)

$$X_t = x + \sum_{j=1}^{d} \int_0^t \sigma_j(X_s) dW_s^j + \int_0^t b(X_s) dW_s \in R^n.$$

Theorem. If $n = 1$ and σ_j Hölder continuous of order $h > \frac{1}{2}$ and $\sigma\sigma^* > c$ then

$$P_{X_T}(dx) = p_{X_T}(x)dx.$$

Proof. We fix $\delta > 0$ and construct

$$X_T^\delta = X_{T-\delta} + \sum_{j=1}^{d} \sigma_j(X_{T-\delta})(W_T^j - W_{T-\delta}^j) + b(X_{T-\delta})\delta.$$

Then

$$E\left|X_T - X_T^\delta\right| \leq C\delta^{\frac{1}{2}(1+h)}$$

and (**IP for the Gaussian law**)

$$E\left|H_k(X_T^\delta)\right| \leq \frac{C}{\delta^{k/2}}.$$

Balance

$$\left|\widehat{p}_{X_T}(\xi)\right| \leq |\xi|\,\delta^{\frac{1}{2}(1+h)} + \frac{1}{|\xi|^k\,\delta^{k/2}}.$$

Optimization :

$$\delta = |\xi|^{-\frac{2(k+1)}{k+1+h}}$$

And obtain

$$\left|\widehat{p}_{X_T}(\xi)\right| \leq C\,|\xi|^{-\frac{kh}{k+1+h}} \sim C\,|\xi|^{-h}$$

Problem : Dimension.

$$\int_{R^n} \left|\widehat{p}_{X_T}(\xi)\right|^2 d\xi = \int_{R^n} |\xi|^{-2h} d\xi < \infty \qquad \Longleftrightarrow \qquad 2h > n.$$

Theorem. $\forall\, n \in N$ and σ_j Hölder continuous of order $h > 0$ and $\sigma\sigma^* > c$ then

$$P_{X_T}(dx) = p_{X_T}(x)dx.$$

Problem : How to eliminate the Dimension ?

Property (False for $p > 2$) For every $p \in N$ we have

$$\int_{R^n} \left|\widehat{p}_{X_T}(\xi)\right|^p d\xi < \infty \qquad \Longrightarrow \qquad \text{Abs Continuity.}$$

Criterion : **Some notation**

1. **(Sobolev)** Let F such that $IP_{\alpha,p}(F)$ holds for $|\alpha| \leq k$. We define

$$\|F\|_{k,p} = \|F\|_p + \sum_{|\alpha|\leq k} \|H_\alpha(F)\|_p$$

2. Fortet-Mourier (Wasserstein) distance

$$d_1(F,G) = \sup\{|E(\phi(F)) - E(\phi(G))|\,;\, \|\phi\|_\infty + \|\nabla\phi\|_\infty \leq 1\}.$$

3. The interpolation fuctional : Given $F \in R^d$ and a sequence $F_n \in R^d, n \in N$

$$\pi_p(F,(F_n)_n) := \sum_n 2^{n(1+d/p_*)} d_1(F,F_n) + \sum_n \frac{1}{2^{2mn}} \|F_n\|_{2m,p}\,.$$

Lemma. Suppose that $F \sim f(x)dx$. Then for every sequence $F_n \in R^d, n \in N$ and every $p > 1$

$$\|f\|_p \leq \pi_p(F,(F_n)_n).$$

Corollary Let F. Suppose that there exists $p > 1$ and a sequence $F_n \in R^d, n \in N$ such that $\pi_p(F, (F_n)_n) < \infty$. Then $F \sim f(x)dx$.

Proof. Put

$$F^\varepsilon = \varepsilon\mathbf{\Delta} + F, \qquad F_n^\varepsilon = \varepsilon\mathbf{\Delta} + F_n.$$

Then

$$d_1(F^\varepsilon, F_n^\varepsilon) \leq d_1(F, F_n) \qquad \|F_n^\varepsilon\|_{2m,p} \leq \|F_n\|_{2m,p}$$

so that

$$\|f^\varepsilon\|_p \leq \pi_p(F^\varepsilon, (F_n^\varepsilon)_n) \leq \pi_p(F, (F_n)_n) < \infty, \qquad \forall\varepsilon > 0.$$

Relative compacity $f^\varepsilon \to f \in L^p$. $\square$

Corollary Suppose that

$$d_1(F, F_n) \times \|F_n\|_{2m,p}^\alpha \leq C, \qquad \alpha > \frac{1 + d/p_*}{2m}.$$

Then $F \sim f(x)dx$.

Diffusion :

$$d_1(X_T, X_T^\delta) \leq E\left|X_T - X_T^\delta\right| \leq C\delta^{\frac{1}{2}(1+h)}$$

and

$$(E\left|H_{2m}(X_T^\delta)\right|^p)^{1/p} \leq \frac{C_p}{\delta^m}$$

We need

$$d_1(X_T, X_T^\delta) \times \left\|X_T^\delta\right\|_{2m,p}^\alpha = \delta^{\frac{1}{2}(1+h)} \times \frac{1}{\delta^{m\alpha}} \leq C, \qquad \alpha > \frac{1+d/p_*}{2m}.$$

This amounts to

$$\frac{1}{2}(1+h) > \frac{1+d/p_*}{2}$$

with is true for $p \to 1(p_* \to \infty)$.

Convergence. Suppose that

$$d_1(F, F_n) \times \|F_n\|_{2m,p}^{\beta} \leq C, \qquad \beta > \frac{1 + d/p_*}{m}.$$

Then $F \sim f(x)dx$, $F_n \sim f_n(x)dx$ and

$$d_{TV}(F, F_n) \leq \|f - f_n\|_p \leq C d_1^{\theta}(F, F_n) \qquad \theta = \min\{\frac{1}{\beta}, 1 - \frac{1}{m\beta}\}.$$

Example. X diffusin process with C_b^{∞} coefficients and $\sigma\sigma^* \geq c > 0$ (unifform ellipticity).

X^n Euler scheme. Then for every $m \in N, p > 1$

$$\sup_n \|X_T^n\|_{m,p} \leq C < \infty \qquad d_1(X_T, X_T^n) \leq \frac{C}{n}.$$

We mau take **any** β. We optimize

$$\beta = \frac{m+1}{m} \qquad \rightarrow \qquad \theta = \frac{1}{\beta} = 1 - \frac{1}{m+1}$$

Corollary (Sub-optimal) For every $\varepsilon > 0$

$$d_{TV}(X_T, X_T^n) \leq \frac{C}{n^{1-\varepsilon}}$$

Link with Interpolation Spaces.

$$\pi_p(F,(F_n)_n) := \sum_n 2^{n(1+d/p_*)} d_1(F,F_n) + \sum_n \frac{1}{2^{2mn}} \|F_n\|_{2m,p}\,.$$

If $F \sim f(x)dx$, and $F_n \sim f_n(x)dx$ the above functional **"is"**

$$\widehat{\pi}_p(f,(f_n)_n) := \sum_n 2^{n(1+d/p_*)} \|f-f_n\|_{W_*^{1,\infty}} + \sum_n \frac{1}{2^{2mn}} \|f_n\|_{W^{2m,p}}$$

Then, with

$$X = W^{2m,p} \qquad \subset \qquad Y = W_*^{1,\infty}$$

we have

$$\widehat{\pi}_p(f,(f_n)_n) < \infty \qquad \Leftrightarrow \qquad f \in (X,Y)_\rho, \qquad \rho = \frac{1+d/p_*}{2m}.$$

We have proved that

$$\widehat{\pi}_p(f,(f_n)_n) < \infty \qquad f \in L^p$$

so

$$(X,Y)_\rho \subset L^p$$

Idea of the proof.

$$h_\alpha = \text{Hermite functions on } R^n$$

$$W_k = Vect\{h_\alpha : |\alpha| = k\} \qquad L^2(R^n, dx) = \oplus_{k=0}^{\infty} W_k.$$

Operator

$$L = \triangle - |x|^2$$

$$Lf = (2k+n)f \qquad f \in W_k.$$

Decomposition : $f, f_k \in L^2$ such that $f_k \to f$. We write

$$f = \sum_k J_k f = \sum_k (J_k(f - f_k) + J_k f_k) = \sum_k \left(J_k(f - f_k) + \frac{1}{(2k+n)^m} L^m J_k f_k \right)$$

Analogy

$$\begin{array}{ccc} \text{Fourier Transform} & \longrightarrow & \text{Projection } J_k \\ \partial_\xi & \longrightarrow & L \end{array}$$

$$\|f\|_p \leq \sum_k \left(\|J_k(f - f_k)\|_p + \frac{1}{(2k+n)^m} \|L^m J_k f_k\|_p \right) \leq \pi_{m,p}(f, (f_n)_n)$$

Projections

$$H_k(x,y) = \sum_{\|\alpha\|=k} h_\alpha(x)h_\alpha(y) \qquad \to \qquad J_k f(x) = \int H_k(x,y)f(y)dy$$

Mixed Projections

We take a function $a : R_+ \to R_+$ with the support included in $[\frac{1}{4}, 4]$ such that

$$a(t) + a(4t) = 1, \qquad t \in (0,1)$$

and we define

$$H_N^a(x,y) = \sum_{j=4^{N-1}}^{4^{N+1}} a(\frac{j}{4^N})H_j(x,y)$$

and

$$J_N^a f(x) = \int H_N^a(x,y)f(y)dy$$

Decomposition : $f, f_k \in L^2$

$$f = \sum_k J_k^a f = \sum_k (J_k^a(f - f_k) + J_k^a f_k) = \sum_k \left(J_k^a(f - f_k) + \frac{1}{(2k+n)^m} L^m J_k^a f_k \right)$$

Crucial estimate For each $m \in N$

$$\left| \frac{\partial}{\partial x^\alpha} H_k^a(x,y) \right| \leq C_m \frac{2^{k(|\alpha|+n)}}{(1 + 2^k |x-y|)^m}$$

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