An HJM approach to equity derivatives

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26 November 2013

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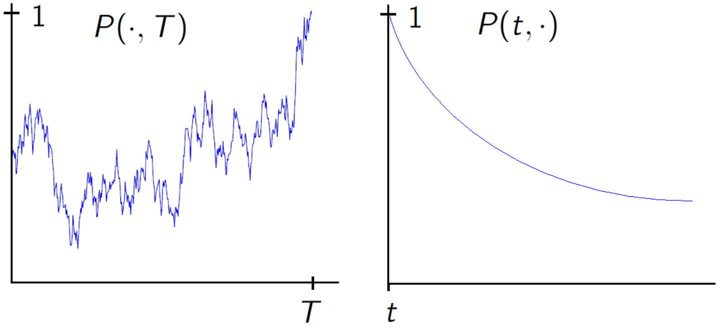
Zero-coupon bonds

- $P(t, T)$ = price at time t for maturity T
- no arbitrage: $P(\cdot, T)$ is a semimartingale for each T
- boundary condition $P(T, T) = 1$



Short rate modelling

- $dr = a(r)dt + b(r)dW$
- Let $V(t, r; T)$ solve

$$\frac{\partial V}{\partial t} + a\frac{\partial V}{\partial r} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial r^2} = rV$$
$$V(T, r; T) = 1$$

- bond price: $P(t, T) = V(t, r_t; T)$
- $\exp\left(-\int_0^t r_s ds\right) P(t, T)$ is a local martingale
- no arbitrage
- Vasicek 1977, Cox–Ingersoll–Ross 1985, Hull–White 1990...

Finding the functions $a(r)$ and $b(r)$:

- estimation: given $r_t^{\text{obs}} : t < 0$, use statistics
- problem: W is a Brownian motion under risk-neutral measure $\neq$ objective measure in general
- calibration: instead make $V(0, r_0; T) \approx P^{\text{obs}}(0, T)$
- shortcoming: market data incorporated indirectly

Classical HJM(M)

- $df_t(x) = \left(\frac{\partial f_t}{\partial x} + A_t(x)\right) dt + B_t(x) dW_t$
- drift condition: $A_t(x) = B_t(x) \cdot \int_0^x B_t(y) dy$
- short rate: $r_t = f_t(0)$
- bond price: $P(t,T) = \exp\left(-\int_0^{T-t} f_t(x) dx\right)$
- $\exp\left(-\int_0^t r_s ds\right) P(t,T)$ local martingale
- no arbitrage
- Heath–Jarrow–Morton 1992, Musiela 1993...

- Shortcoming avoided by taking

$$f_0(T) = -\frac{\partial}{\partial T} \log P^{\mathrm{obs}}(0, T)$$

- Market data incorporated directly
- How to find $B_t(x)$?
- Calibrate with other derivatives...

Main features

- reparametrise from prices to forward rates: boundary condition $P(T, T) = 1$ automatic
- drift condition ensures prices are local martingales
- initial condition $f_0(x)$ and dynamic parameters $B_t(x)$ decouple

How about other markets?

Call options

- S_t = price of stock at time t
- $C(t, T, K)$ price of European call option strike K
- boundary condition $C(T, T, K) = (S_T - K)^+$
- simplify: zero interest rates

Stochastic volatility modelling

- stock price: $dS = S\sigma dW$
- volatility $d\sigma = a(\sigma)dt + b(\sigma)dW$
- Let $V(t, S, \sigma; T, K)$ solve

$$\frac{\partial V}{\partial t} + a\frac{\partial V}{\partial \sigma} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} + S\sigma b\frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial \sigma^2} = 0$$
$$V(T, S, \sigma; T, K) = (S - K)^+$$

- $C(t, T, K) = V(t, S_t, \sigma_t; T, K)$ local martingale
- no arbitrage
- Black–Scholes 1973, Stein–Stein 1991, Heston 1993...

HJM approaches

- Schönbucher 1999
- Davis 2004
- Jacod–Protter 2006
- Schweizer–Wissel 2006, 2008
- Carmona–Nadtochiy 2009, 2012
- Kallsen–Krühner
- ...

Theorem

Suppose F is evolves as

$$dF_t(x,m) = \left(\frac{\partial F_t}{\partial x} - \frac{1}{2} m^2 \frac{\partial^2 F_t}{\partial m^2} \sigma_t^2 + m \frac{\partial B_t}{\partial m} \sigma_t - B_t \sigma_t \right) dt + B_t(x,m) dW_t$$

$$F_t(0,m) = (1-m)^+$$

and suppose that

$$dS_t = S_t \sigma_t dW_t.$$

Let

$$C(t,T,K) = S_t F_t(T-t, K/S_t).$$

Then $C(t,T,K)$ is a local martingale such that $C(T,T,K) = (S_T - K)^+$.

- Set
$$F_0(x, m) = \frac{C^{\rm obs}(0, x, S_0 m)}{S_0}$$
to match prices.
- End of story?

PROBLEMS

- Boundary condition (Durrleman 2007)

$$\sigma_t = \sqrt{2\pi} \lim_{x \downarrow 0} \frac{F_t(x, 1)}{\sqrt{x}}$$

- The SPDE is ill-posed since the drift

$$\frac{\partial F}{\partial x} - \frac{1}{2} m^2 \frac{\partial^2 F}{\partial m^2} \sigma^2 + m \frac{\partial B}{\partial m} \sigma - B\sigma$$

is very badly behaved.

- $F_0(x, m)$ and $B_t(x, m)$ are *coupled*!
- By the Clark–Ocone formula

$$B_t(x, m) = \left(m \frac{\partial F_t}{\partial m} - F_t \right) \sigma_t + \mathbb{E}\left[\frac{D_t S_{t+x}}{S_t} \mathbb{1}_{\{S_{t+x}/S_t > m\}} \middle| \mathcal{F}_t \right]$$

where D is the Malliavin derivative

The martingale

$$dS_t = S_t \sigma_t dW_t$$

is a fake geometric Brownian motion if

$$\log(S_t/S_0) \sim N(-\sigma_0^2 t/2, \sigma_0^2 t)$$

for some $\sigma_0 > 0$. Equivalently,

$$F_0(x,m) = \Phi\left(-\frac{\log m}{\sigma_0\sqrt{x}} + \frac{\sigma_0\sqrt{x}}{2}\right) - m\Phi\left(-\frac{\log m}{\sigma_0\sqrt{x}} - \frac{\sigma_0\sqrt{x}}{2}\right)$$

Theorem (MT 2009)

Suppose S is a fake geometric Brownian motion and the put-call symmetry

$$F_t(x,m) = mF_t(x,1/m) + 1 - m$$

holds for all $t \geq 0$. Then $F_t = F_0$ for all $t \geq 0$, that is S comes from the Black–Scholes model.

Theorem (MT 2013)

Suppose S is fake geometric Brownian motion. Then

$$\frac{1}{t}\int_0^t \sigma_t^2 \to \sigma_0^2.$$

Furthermore, if $S = \mathcal{E}(\sigma_0 X)$ for a local martingale X, then $\sqrt{\varepsilon} X_{t/\epsilon} \to$ Brownian motion.

Theorem

Let $G_t(x,\lambda)$ evolve as

$$dG_t(x,\lambda) = \left(\frac{\partial G_t}{\partial x} + \frac{\left| \frac{\partial G_t}{\partial \lambda}\sigma_t - \frac{\partial \beta_t}{\partial \lambda} \right|^2}{2\frac{\partial^2 G_t}{\partial \lambda^2}} - \beta_t \sigma_t \right) dt$$
$$+ \beta_t(\tau,\lambda) dW_t$$
$$G(0,\lambda) = \lambda, \ \textit{all } 0 \le \lambda \le 1$$

and $dS_t = S_t \sigma_t dW_t$. Then
$C(t,T,K) = \sup_{\lambda\in[0,1]} S_t G_t(T-t,\lambda) - \lambda K$. is a local martingale.
(Compare with Musiela–Zariphopoulou SPDE)

Note that for the Black–Scholes model

$$G_{\mathrm{BS}}(x, \lambda) = \Phi(\Phi^{-1}(\lambda) + \sigma_0 \sqrt{x}).$$

In fact, **Theorem.** Let

$$G(x, \lambda) = \Psi(\Psi^{-1}(\lambda) + h(x))$$

with h increasing, $h(0) = 0$, $h(\infty) = \infty$. Then

$$F(x, m) = \sup_{\lambda \in [0,1]} G(x, \lambda) - \lambda m$$

is a call price if and only if Ψ is the distribution function of a log-concave density.

- Power option payout: S_T^θ for $0 < \theta < 1$
- Replication by covered calls:

$$S^\theta = \theta(1-\theta)\int_0^\infty (S \wedge K) K^{\theta-2} dK$$

- Replication of covered calls:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S^{\theta+\mathrm{i}x} \frac{K^{1-\theta-\mathrm{i}x}}{(x-\mathrm{i}\theta)(x+\mathrm{i}(1-\theta))} dx = S \wedge K$$

Theorem (MT, Cheng 2013)

Let $f_t(x,\theta)$ *evolve as*

$$df_t(x,\theta) = \left(\frac{\partial f_t}{\partial x} + A_t(x,\theta) \right) dt + B_t(x,\theta) \cdot dW_t$$

where

$$A_t(x,\theta) = B_t(x,\theta) \cdot \left(\frac{1}{2}\theta(1-\theta) \int_0^x B_t(y,\theta)dy - \theta\sigma_t \right),$$

and

$$f_t(0,\theta) = \|\sigma_t\|^2 \text{ for all } 0 < \theta < 1.$$

Let $dS_t = S_t \sigma_t \cdot dW_t$. *Then* $P(t,T,\theta) = S_t^\theta \exp\left(-\frac{1}{2}\theta(1-\theta) \int_0^{T-t} f_t(x,\theta)dx \right)$ *is a local martingale.*

- Parametrisation chosen so that for the Black–Scholes model $f_t(x, \theta) = \sigma_0^2$
- Poorly behaved differential operator replaced by a coupling at $x = 0$

Time-dependent Black Scholes

- Let $f_0(x,\theta) = Q_0(x)$ for all $0 < \theta < 1$
- and $B_t(x,\theta) = 0$
- then $f_t(x,\theta) = Q_0(t + x, \theta)$

Time-dependent Heston

- Let $f_0(x,\theta) = Q_1(x,\theta)v_0 + Q_0(x,\theta)$ constant $v_0 \geq 0$ and $Q_0(x,\theta)$ and $Q_1(x,\theta)$ which solve

$$\frac{\partial Q_1}{\partial x} = [\theta c\rho - b]Q_1 - \frac{1}{2}\theta(1-\theta)cQ_1 \int_0^x c(y)Q_1(y,\theta)dy$$

$$\frac{\partial Q_0}{\partial x} = aQ_1$$

$$Q_1(0,\theta) = 1 \text{ and } Q_0(0,\theta) = 0$$

given functions a, b, c, ρ

- Let $B_t(x,\theta) = c(x)Q_1(x,\theta)\sqrt{v_t}\binom{1}{0}$ where $dv_t = [a(t) - b(t)v_t]dt + c(t)\sqrt{v_t}dW_t^1$.
- Then $f_t(x,\theta) = Q_1(x,\theta)v_t + Q_0(x,\theta)$ and $\sigma_t = \sqrt{v_t}\binom{\rho(t)}{\sqrt{1-\rho(t)^2}}$.

Time-dependent Stein–Stein

- Let $f_0(x,\theta) = \frac{1}{2}Q_2(x,\theta)v_0^2 + Q_1(x,\theta)v_0 + Q_0(x,\theta)$ constant v_0 functions Q_0, Q_1 and Q_2 which verify

$$\frac{\partial Q_2}{\partial x} = [\theta c\rho - b] - \frac{1}{2}\theta(1-\theta)cQ_2 \int_0^x c(y)Q_2(y,\theta)dy$$

$$\begin{aligned}\frac{\partial Q_1}{\partial x} =& [\theta c\rho - b]Q_1 + aQ_2 \\ & - \frac{1}{2}\theta(1-\theta)\left[cQ_1 \int_0^x c(y)Q_2(y,\theta)dy\right. \\ & + \left.\left[cQ_2 \int_0^x c(y)Q_1(y,\theta)dy\right]\right.\end{aligned}$$

$$\frac{\partial Q_0}{\partial x} = aQ_1 - \frac{1}{2}\theta(1-\theta)cQ_1 \int_0^x c(y)Q_1(y,\theta)dy + \frac{1}{2}c^2Q_2$$

- $Q_2(0,\theta) = 1$ and $Q_1(0,\theta) = Q_0(0,\theta) = 0$ for all $0 < \theta < 1$ given functions a, b, c and ρ

Stein–Stein continued

- and $dv_t = [a(t) - b(t)v_t]dt + c(t)dW_t$.
- then $f_t(x,\theta) = \frac{1}{2}Q_2(x,\theta)v_t^2 + Q_1(x,\theta)v_t + Q_0(x,\theta)$ with $B_t(x,\theta) = c(x)[Q_2(x,\theta)v_t + Q_1(x,\theta)]\binom{1}{0}$ and $\sigma_t = v_t\big(\frac{\rho(t)}{\sqrt{1-\rho(t)^2}}\big)$.

General case

- Let $V(Z, x, \theta)$ solve

$$\frac{\partial V}{\partial x} = \frac{1}{2}\|b\|^2 \left(\frac{\partial^2 V}{\partial Z^2} - \theta(1-\theta)\frac{\partial V}{\partial Z} \int_0^x \frac{\partial V}{\partial Z}(Z, y, \theta) dy \right) + (a + \theta b \cdot U)\frac{\partial V}{\partial Z}$$

$$V(Z, 0, \theta) = \|U(Z)\|^2$$

- $dZ = a(Z)dt + b(Z) \cdot dW$
- $f_t(x, \theta) = V(Z_t, x, \theta)$ solves equation with $\sigma_t = V_0(Z_t)$ and

$$B_t(x, \theta) = b(Z_t)\frac{\partial V}{\partial Z}(Z_t, x, \theta)$$

- Let H be a suitable Hilbert space of functions on $[0, \infty) \times [0, 1]$
- Let $(S_t h)(x, \theta) = h(t + x, \theta)$
- Generator of semigroup $(S_t)_{t \geq 0}$ is $\frac{\partial}{\partial x}$.
- Rewrite SPDE in mild form

$$f_t = S_t f_0 + \int_0^t S_{t-s} A_s dt + \int_0^t S_{t-s} B_s dW_s$$

PROBLEM

- Let
$$H_0 = \{ h \in H : \theta \mapsto h(0,\theta) \text{ is constant } \}$$
- Even if $f_0 \in H_0$, no guarantee that $S_t f_0 \in H_0$!!

- Let
$$(T_t h)(x,\theta) = h(x+t,\theta) - h(t,\theta) + h(t,0)$$
- $(T_t)_{t\geq 0}$ semigroup on H_0 with generator
$$\mathcal{L} = \frac{\partial}{\partial x} - \frac{\partial}{\partial x}\Big|_{x=0} + \frac{\partial}{\partial x}\Big|_{x=0,\theta=0}$$
- *Question:* When do the solutions of
$$df = (\mathcal{L}f + A)dt + BdW$$
also satisfy
$$df = \left(\frac{\partial f}{\partial x} + A\right) dt + BdW?$$