BSDEs driven by time-changed Lévy noises and optimal control

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Based on a joint work with Giulia Di Nunno.

Agenda

- Let L be a mean-zero Lévy process defined on $[0, T]$.
- Let λ be stochastic process, *independent* of L.
- λ is positive $dt \times d\mathbb{P}$ a.e., continuous in probability and $\mathbb{E}\left[\int_0^T \lambda_s \, ds\right] < \infty$.
- Let $\tilde{\Lambda}_t := \int_0^t \lambda_s \, ds$

Then

$$\eta_t := L_{\tilde{\Lambda}_t}$$

is a process with conditionally independent increments (CII). We have

$$L_{\tilde{\Lambda}_t} \stackrel{d}{=} \underbrace{B_t}_{\text{Time changed Brownian}} + \int_0^t \int_{\mathbb{R}_0} z \underbrace{\tilde{H}(ds, dz)}_{\text{Doubly stochastic Poisson}}$$

Define the random measure Λ on $[0, T] \times \mathbb{R}$ by

$$\Lambda(\Delta) := \int_0^T \mathbf{1}_{\{(t,0)\in\Delta\}}(t)\,\lambda_t\,dt + \int_0^T \int_{\mathbb{R}_0} \mathbf{1}_{\Delta}(t,z)\,\nu(dz)\lambda_t\,dt,$$

Let $\mathcal{F}^\lambda$ be the σ-algebra generated by Λ.

The time changed Brownian motion

$\mathbb{P}\Big(B(\Delta) \le x \,\Big|\mathcal{F}^\Lambda\Big) = \mathbb{P}\Big(B(\Delta) \le x\,\Big|\Lambda(\Delta)\Big) = \Phi\big(\frac{x}{\sqrt{\Lambda(\Delta)}}\big)$,

$x \in \mathbb{R}$, $\Delta \subseteq [0, T] \times \{0\}$. Here Φ is the CDF for the standard normal r.v.

The doubly stochastic Poisson

$\mathbb{P}\Big(H(\Delta) = k\,\Big|\mathcal{F}^\Lambda\Big) = \mathbb{P}\Big(H(\Delta) = k\,\Big|\Lambda(\Delta)\Big) = \frac{\Lambda(\Delta)^k}{k!}\,e^{-\Lambda(\Delta)}$,

$k \in \mathbb{N}$, $\Delta \subseteq [0, T] \times \mathbb{R}_0$. Set $\tilde{H}(dt, dz) = H(dt, dz) - \nu(dz)\lambda_t\,dt$, where ν is a deterministic on $\mathbb{R}_0$ satisfying $\int_{\mathbb{R}_0} z^2\,\nu(dz) < \infty$.

Definition

The signed random measure μ is given by

$$\mu(\Delta) := B\Big(\Delta \cap [0,T] \times \{0\}\Big) + \tilde{H}\Big(\Delta \cap [0,T] \times \mathbb{R}_0\Big), \quad \Delta \subseteq [0,T] \times \mathbb{R}.$$

- $\mathbb{F} := \{\mathcal{F}_t, t \in [0,T]\}$ is the filtration generated by μ.
- $\mathbb{G} := \{\mathcal{G}_t, t \in [0,T]\}$ with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}^\Lambda$.
- μ is a martingale random field with respect to $\mathbb{G}$ and $\mathbb{F}$.

Some properties:

- $\mathbb{E}\big[\mu(\Delta) \,\big|\, \mathcal{F}^\Lambda\big] = 0$.
- $\mathbb{E}\big[\mu(\Delta_1)\mu(\Delta_2) \,\big|\, \mathcal{F}^\Lambda\big] = 0$ for $\Delta_1 \cap \Delta_2 = \emptyset$.
- $\mathbb{E}\big[\mu(\Delta)^2 \,\big|\, \mathcal{F}^\Lambda\big] = \Lambda(\Delta)$.

Relevance to mathematical finance

The time-changed Lévy processes occur in mathematical finance in the modeling of asset prices as follows:

$$
\begin{aligned}
dS_t &= S_{t\text{-}}\Big(\alpha_t\,dt + \int_{\mathbb{R}} \psi_t(z)\,\mu(dt,dz)\Big) \\
&= S_{t\text{-}}\Big(\alpha_t\,dt + \psi_t(0)\,dB_t + \int_{\mathbb{R}_0} \psi_t(z)\,\tilde{H}(dt,dz)\Big) \quad S_0 > 0. \quad (1)
\end{aligned}
$$

- Here S would be $\mathbb{F}$-adapted.
- Examples include stochastic volatility models like [1, Carr et al '03], [2, Stein et al '91].
- When the independence between L and λ is satisfied, also [3, Heston '93].
- Models of this type is also used in credit risk [4, Lando '98]

The integral representation

$\mathcal{I}$ is the $\mathbb{G}$-predictable random fields satisfying $\mathbb{E}[\int_0^T \int_{\mathbb{R}} \phi_s(z)^2 \Lambda(ds, dz)] < \infty$.

Theorem (Jacod and Shiryaev (2003) [5])

Assume $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$. Then there exists a unique $\phi \in \mathcal{I}$ such that

$$\xi = \mathbb{E}[\xi \,|\mathcal{F}^\Lambda] + \int_0^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz). \tag{2}$$

- The two summands in (2) are orthogonal.
- The representation in (2) is not possible with ϕ $\mathbb{F}$-adapted unless Λ is deterministic.

The BSDE

Let $\xi \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$. (Remark that $\mathcal{F}_T = \mathcal{G}_T$.)
We are interested in

$$\begin{aligned} Y_t &= \xi + \int_t^T g_s(\lambda_s, Y_s, \phi_s)\, ds - \int_t^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz) \\ &= \xi + \int_t^T g_s(\lambda_s, Y_s, \phi_s)\, ds - \int_t^T \phi_s(0)\, dB_s - \int_t^T \int_{\mathbb{R}_0} \phi_s(z)\, \tilde{H}(ds, dz) \end{aligned}$$

with $\phi \in \mathcal{I}$ and suitable conditions on g.

Definition

We say that (ξ, g) are *standard parameters* when $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ and $g : [0, T] \times [0, \infty)^2 \times \mathbb{R} \times \Phi \times \Omega \to \mathbb{R}$ such that g satisfies (for some $K_g > 0$)

$$g_\cdot(\lambda, Y, \phi, \cdot)\text{is } \mathbb{G}\text{-adapted},$$

$$\mathbb{E}[\int_0^T g_s(\lambda, 0, 0)^2 \, ds] < \infty,$$

$$|g_t((\lambda), y_1, \phi^{(1)}) - g_t(\lambda), y_2, \phi^{(2)})| \leq K_g\Big(|y_1 - y_2|$$

$$+ |\phi^{(1)}(0) - \phi^{(2)}(0)|\sqrt{\lambda} + \sqrt{\int_{\mathbb{R}_0} |\phi^{(1)}(z) - \phi^{(2)}(z)|^2 \, \nu(dz)}\sqrt{\lambda}\Big),$$

for all $(\lambda) \in [0, \infty)^2, y_1, y_2 \in \mathbb{R}$, and

$$|\phi^{(1)}(0)|^2 + |\phi^{(2)}(0)|^2 + \int_{\mathbb{R}_0} |\phi^{(1)}(z)|^2 + |\phi^{(2)}(z)|^2 \, \nu(dz) < \infty \quad dt \times d\mathbb{P} \text{ a.e}$$

Theorem

Let (g,ξ) be standard parameters. Then there exists a unique couple $(Y,\phi)\in S\times\mathcal{I}$ such that

$$
\begin{aligned}
Y_t &= \xi + \int_t^T g_s(\lambda_s,\, Y_s, \phi_s)\, ds - \int_t^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds,\, dz) \\
&= \xi + \int_t^T g_s(\lambda_s,\, Y_s, \phi_s)\, ds - \int_t^T \phi_s(0)\, dB_s - \int_t^T \int_{\mathbb{R}_0} \phi_s(z)\, \tilde{H}(ds,\, dz).
\end{aligned}
$$

Remark. From the construction in the proof and properties of integral representation we have

$$
Y_0 = \mathbb{E}\Big[\xi + \int_0^T g_s(\lambda_s,\, Y_s, \phi_s)\, ds \,\Big|\, \mathcal{F}^\Lambda\Big].
$$

Proof.

Let (g, ξ) be standard parameters. S is the space of $\mathbb{G}$-adapted stochastic processes such that
$\| Y \|_S := \mathbb{E}[\sup_{0 \leq t \leq T} | Y_t|^2]^{1/2} < \infty$.
Define the mapping

$$\Theta : S \times \mathcal{I} \to S \times \mathcal{I}, \quad \Theta(U, \psi) := (Y, \phi)$$

as follows. The component ϕ is given by martingale representation as the unique element in $\mathcal{I}$ so that

$$M_t = M_0 + \int_0^t \int_{\mathbb{R}} \phi_s(z) \, \mu(ds, dz), \quad t \in [0, T]$$

of the martingale $M_t = \mathbb{E}[\xi + \int_0^T g_s(\lambda_s, U_s, \psi_s) \, ds \,|\mathcal{G}_t]$.
The component Y in the mapping Θ is defined by

$$Y_t = \mathbb{E}\Big[\xi + \int_t^T g_s(\lambda_s, U_s, \psi_s) \, ds \,\Big|\mathcal{G}_t\Big], \quad t \in [0, T].$$

proof cont.

- From the inequalities in the conditions on the standard parameters we can show that Θ is well defined.
- Θ is a contraction on the interval $[t_1, T]$ for some $0 \leq t_1 < T$.
- We can prove an unique solution $\tilde{Y}$, $\tilde{\phi}$ exist on $[t_1, T]$ in a suitable sense.
- Consider a new BSDE on the interval $[0, t_1]$ with terminal condition $\tilde{Y}_{t_1}$ and driver g.
- There exist $0 \leq t_2 < t_1$ so that Θ is a contraction on the interval $[t_1, T]$ (in a suitable sense).
- Repeating the procedure and combining the results from different intervals yields the result.

□

Optimization

The performance functional:

$$J(u) = \mathbb{E}\Big[\int_0^T f_t(\lambda_t, u_t, X_{t\text{-}})\, dt + l(X_T) \Big],$$

with state process X_t, $X_0 \in \mathbb{R}$,

$$dX_t = b_t(\lambda_t, u_t, X_{t\text{-}})\, dt + \int_{\mathbb{R}} \kappa_t(z, \lambda_t, u_t, X_{t\text{-}})\, \mu(dt, dz),$$

- $l(x, \omega)$, $x \in \mathbb{R}$, $\omega \in \Omega$ differentiable in x.
- $f_t(\lambda, u, x, \omega)$, $t \in [0, T]$, $\lambda \in [0, \infty)^2$, $u \in \mathcal{U}$, $x \in \mathbb{R}$, $\omega \in \Omega$ differentiable in x
- $\mathcal{U} \subseteq \mathbb{R}$ is a closed, convex set.

Definition

The admissible controls are càglàd stochastic processes $u : [0,T] \times \Omega \to \mathcal{U}$, such that X has a unique strong solution,

$$\mathbb{E}\Big[\int_0^T |f_t(\lambda_t, u_t, X_{t\text{-}})|^2 \, dt + |l(X_T)| + |\partial_x l(X_T)|^2\Big] < \infty,$$

and for some $K_1 > 0$ we have

$$\begin{aligned}
\Big|\partial_x \kappa_t(0, \lambda_t, u_t, X_{t\text{-}})\Big|\sqrt{\lambda_t} &\leq K_1 \quad dt \times d\mathbb{P}\text{-a.e,}\\
\int_{\mathbb{R}_0} \big(\partial_x \kappa_t(z, \lambda_t, u_t, X_{t\text{-}})\big)^2 \nu(dz)\sqrt{\lambda_t} &\leq K_1 \quad dt \times d\mathbb{P}\text{-a.e,}\\
\big|\partial_x b_t(\lambda_t, u_t, X_{t\text{-}})\big| &\leq K_1 \quad dt \times d\mathbb{P}\text{-a.e.}
\end{aligned}$$

The admissible controls are either $\mathbb{G}$-predictable or $\mathbb{F}$-predictable and we denote these sets as $\mathcal{A}^{\mathbb{G}}$ and $\mathcal{A}^{\mathbb{F}}$ respectively.

We define the Hamiltonian,
$\mathcal{H} : [0, T] \times [0, \infty)^2 \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \Phi \times \Omega \to \mathbb{R}$ by

$$\mathcal{H}_t(\lambda_t, u_t, X_t, Y_t, \phi_t) = f_t(\lambda_t, u_t, X_t) + b_t(\lambda_t, u_t, X_t)\, Y_t \\ + \kappa_t(0, \lambda_t, u_t, X_t)\phi_t(0)\lambda_t + \int_{\mathbb{R}_0} \kappa_t(z, \lambda, u, x)\phi(z)\, \lambda_t\, \nu(dz).$$

Corresponding to the admissible pair (u, X) is the couple (Y, ϕ), which is the solution to the BSDE

$$Y_t = \partial_x l(X_T) + \int_t^T \partial_x \mathcal{H}_s(\lambda, u_s, X_{s\text{-}}, Y_{s\text{-}}, \phi_s)\, ds - \int_t^T \int_{\mathbb{R}} \phi_s(z)\, \mu(ds, dz),$$

Theorem

Let $\hat{u} \in \mathcal{A}^{\mathbb{G}}$*. Assume that*

$$\mathbb{E}\Big[\int_0 \int_{\mathbb{R}} \big|\,\hat{Y}_{s\text{-}}(\hat{\kappa}_s(z) - \kappa_s(z))\big|^2 + \big|(\hat{X}_{s\text{-}} - X_{s\text{-}})\hat{\phi}_s(z)\big|^2 \Lambda(ds, dz)\Big] < \infty$$

for all $u \in \mathcal{A}^{\mathbb{G}}$*. If*

$$h_t(x) = \max_{u \in \mathcal{U}} \mathcal{H}_t(\lambda_t, u, x, \hat{Y}_{t\text{-}}, \hat{\phi}_t)$$

exists and is a concave function in x *for all* $t \in [0, T]$ $\mathbb{P}$*-a.s., and*

$$\mathcal{H}_t(\lambda_t, \hat{u}_t, \hat{X}_{t\text{-}}, \hat{Y}_{t\text{-}}, \hat{\phi}_t) = h_t(\hat{X}_t)$$

for all $t \in [0, T]$*, then* $\hat{u}$ *is optimal for* $J(u)$*.*

Proof.

We use a technique from [6, Framstad et al 2004]

$$\mathbb{E}\Big[\ l(\hat{X}_T) - l(X_T)\Big] \geq \mathbb{E}\Big[\partial_x\ l(\hat{X}_T)(\hat{X}_T - X_T)\Big] = \mathbb{E}\Big[\ \hat{Y}_T(\hat{X}_T - X_T)\Big]$$

Combining the above with

$$\begin{aligned}\hat{f}_s - f_s = \hat{\mathcal{H}}_s(\hat{u}_s, \hat{X}_{s\text{-}}) - \hat{\mathcal{H}}_s(u_s, X_{s\text{-}}) - \big(\hat{b}_s - b_s\big)\,\hat{Y}_{s\text{-}} \\ - \big(\hat{\kappa}_s(0) - \kappa_s(0)\big)\hat{\phi}_s(0)\lambda_s - \int_{\mathbb{R}_0} \big(\hat{\kappa}_s(z) - \kappa_s(z)\big)\hat{\phi}_s(z)\,\lambda_t\nu(dz),\end{aligned}$$

gives

$$\begin{aligned}J(\hat{u}) - J(u) \geq \mathbb{E}\Big[\int_0^T \Big\{\hat{\mathcal{H}}_s(\hat{u}_s, \hat{X}_{s\text{-}}) \\ - \hat{\mathcal{H}}_s(u_s, X_{s\text{-}}) - (\hat{X}_{s\text{-}} - X_{s\text{-}})\partial_x\hat{\mathcal{H}}_s(\hat{u}_s, \hat{X}_{s\text{-}})\Big\}\, ds\Big].\end{aligned}$$

The result then follows from a concavity argument. □

Theorem

Let $\hat{u} \in \mathcal{A}^{\mathbb{F}}$. *Assume the integrability conditions holds. Denote*

$$\mathcal{H}_t^{\mathbb{F}}(\lambda_t, u, x, \hat{Y}_{t\text{-}}, \hat{\phi}_t) := \mathbb{E}\Big[\mathcal{H}_t(\lambda_t, u, x, \hat{Y}_{t\text{-}}, \hat{\phi}_t)\,\big|\mathcal{F}_t\Big]$$
$$= f_t(\lambda_t, u, x) + b_t(\lambda_t, u, x)\mathbb{E}\big[\hat{Y}_{t\text{-}}\,\big|\mathcal{F}_t\big] + \kappa_t(0, \lambda_t, u, x)\mathbb{E}\big[\hat{\phi}_t(0)\,\big|\mathcal{F}_t\big]$$
$$+ \int_{\mathbb{R}_0} \kappa_t(z, \lambda_t, u, x)\mathbb{E}\big[\phi_t(z)\,\big|\mathcal{F}_t\big]\,\lambda_t\,\nu(dz)$$

for all $t \in [0, T]$. *If*

$$h_t^{\mathbb{F}}(x) = \max_{u \in \mathcal{U}} \mathcal{H}_t^{\mathbb{F}}(\lambda_t, u, x, \hat{Y}_{t\text{-}}, \hat{\phi}_t)$$

exists and is a concave function in x *for all* $t \in [0, T]$, *and*

$$\mathcal{H}_t^{\mathbb{F}}(\lambda_t, \hat{u}_t, \hat{X}_t, \hat{Y}_{t\text{-}}, \hat{\phi}_t) = h_t^{\mathbb{F}}(\hat{X}_t),$$

then $(\hat{u}, \hat{X})$ *is the optimal* $\mathbb{F}$*-adapted solution.*

Proof.

We can use the same arguments as in the $\mathbb{G}$-predictable case to get

$$J(\hat{u}) - J(u) \geq \mathbb{E}\Big[\int_0^T \hat{\mathcal{H}}_s(\hat{u}_s, \hat{X}_{s\text{-}}) - \hat{\mathcal{H}}_s(u_s, X_{s\text{-}}) - (\hat{X}_{s\text{-}} - X_{s\text{-}})\partial_x \hat{\mathcal{H}}_s(\hat{u}_s, \hat{X}_{s\text{-}})\, ds\Big].$$

By the linearity of the strictly $\mathbb{G}$-measurable terms

$$\mathbb{E}\Big[\int_0^T \hat{\mathcal{H}}_s(\hat{u}_s, \hat{X}_{s\text{-}}) - \hat{\mathcal{H}}_s(u_s, X_{s\text{-}}) - \partial_x \hat{\mathcal{H}}_s(\hat{u}_s, \hat{X}_{s\text{-}})(\hat{X}_{s\text{-}} - X_{s\text{-}})\, ds\Big]$$
$$= \mathbb{E}\Big[\int_0^T \hat{\mathcal{H}}^{\mathbb{F}}_s(\hat{u}_s, \hat{X}_{s\text{-}}) - \hat{\mathcal{H}}^{\mathbb{F}}_s(u_s, X_{s\text{-}}) - \partial_x \hat{\mathcal{H}}^{\mathbb{F}}_s(\hat{u}_s, \hat{X}_{s\text{-}})(\hat{X}_{s\text{-}} - X_{s\text{-}})\, ds\Big].$$

Which is exactly the expression $\hat{u} \in \mathcal{A}^{\mathbb{F}}$ maximizises. □

A necessary maximum principle

Assume the following

- For all $t, r \in [0, T]$, $t < r \leq T$, and $\mathcal{F}_t$-measurable random variables α satisfying $\alpha(\omega) \in \mathcal{U}$ a.s., the control $\beta(s) = \alpha(\omega)\mathbf{1}_{(t,r]}(s)$ belongs to $\mathcal{A}^{\mathbb{F}}$.
- For all $u, \beta \in \mathcal{A}^{\mathbb{F}}$ with β bounded there exists $\delta > 0$ such that $u + y\beta \in \mathcal{A}^{\mathbb{F}}$ for $y < \delta$.
- For $y < \delta$
  $\frac{d}{dy}\partial f_t(\lambda_t, u_t + y\beta_t, X_t^{u+y\beta})$ is uniformly $dt \times d\mathbb{P}$-integrable.
  $\frac{d}{dy} l'(X_T^{u+y\beta})$ is uniformly $\mathbb{P}$-integrable
- The process $\zeta_t^{(u,\beta)} = \frac{\partial}{\partial y} X_t^{u+y\beta}|_{y=0}$ exists as an element of $L_2(\Omega, \mathcal{G}, \mathbb{P})$.

Theorem

Let $\hat{u} \in \mathcal{A}^{\mathbb{F}}$*. Suppose*

$$\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}} \big|\hat{Y}_s\big(\frac{\partial \hat{\kappa}_s}{\partial x}(z)\zeta_s^{\hat{u},\beta} + \frac{\partial \hat{\kappa}_s}{\partial u}(z)\big)\big|^2 + |\zeta_s^{\hat{u},\beta}\hat{\phi}_s(z)|^2 \, \Lambda(ds,dz)\Big] < \infty,$$

holds for all bounded, admissible controls $\beta \in \mathcal{A}^{\mathbb{F}}$*.*
If $\hat{u}$ *is a critical point for* $J(u)$*, in the sense that*

$$\frac{\partial}{\partial y} J(\hat{u} + y\beta)\big|_{y=0} = 0 \quad \textit{for all bounded } \beta \in \mathcal{A}^{\mathbb{F}},$$

then

$$\mathbb{E}\Big[\frac{\partial \mathcal{H}_t}{\partial u}(\lambda_t, \hat{u}_t, \hat{X}_t, \hat{Y}_t, \hat{\phi}_t)\Big|\mathcal{F}_t\Big] = 0, \quad dt \times d\mathbb{P}\text{-a.e.}$$

Proof.

- Assume $\hat{u}$ is a critical point
- Compute $\frac{d}{dy}J(\hat{u}+y\beta)$ evaluated at $y=0$.
- Let $\beta_s(\omega)=\alpha(\omega)\mathbf{1}_{t,t+h}(s)$ where α is a $\mathcal{F}_t$-measurable random variable and $0\le t<t+h\le T$.

This leads to

$$0=\mathbb{E}\Big[\int_t^{t+h}\Big(\frac{\partial\hat{f}_s}{\partial u}+\hat{Y}_{t\text{-}}\frac{\partial\hat{b}_s}{\partial u}\Big)\alpha\,ds+\int_t^{t+h}\int_{\mathbb{R}}\frac{\partial\hat{\kappa}_s}{\partial u}(z)\hat{\phi}_s(z)\alpha\,\Lambda(ds,dz)\Big].$$

Taking the derivative of h at $h=0$ gives the desired result. □

Application: Optimal mean-variance portfolio selection

Two assets, a risk free asset R and a risky asset S defined by

$$dR_t = \rho_t R_{t\text{-}}\, dt, \qquad R_0 = 1,$$
$$dS_t = \alpha_t S_{t\text{-}}\, dt + S_{t\text{-}} \int_{\mathbb{R}} \psi_s(z)\, \mu(ds, dz), \qquad S_0 > 0.$$

Let u denote the amount of wealth invested in the risky asset S and assume the portfolio is self-financing. The wealth equation is

$$dX_t = \big[\rho_t X_t + (\alpha_t - \rho_t) u_t\big]\, dt + u_t \int_{\mathbb{R}} \psi_s(z)\, \mu(ds, dz).$$

Want to solve the mean-variance portfolio problem

$$\sup_{u \in \mathcal{A}^{\mathbb{F}}} J(u) = \sup_{u \in \mathcal{A}^{\mathbb{F}}} \mathbb{E}\Big[-\frac{1}{2}\big(X_T - k\big)^2 \Big], \quad k \in \mathbb{R}$$

Theorem

Consider the feedback control $\hat{u}^{\mathbb{F}} \in \mathcal{A}^{\mathbb{F}}$ *given by*

$$\hat{u}_t^{\mathbb{F}} = -\frac{1}{\mathbb{E}[A_t|\mathcal{F}_t]} \frac{(\alpha_t - \rho_t)\Big(\mathbb{E}[A_t|\mathcal{F}_t]\hat{X}_t(\hat{u}_t^{\mathbb{F}}) + \mathbb{E}[C_t|\mathcal{F}_t]\Big)}{|\psi_t(0)|^2\lambda_t^B + \int_{\mathbb{R}_0} |\psi_t(z)|^2 \, \lambda_t^H \, \nu(dz)},$$

where A *and* C *is given by*

$$A_t = -\exp\Big\{-\int_t^T \frac{(\alpha_s - \rho_s)^2}{|\psi_s(0)|^2\lambda_s^B + \int_{\mathbb{R}_0} |\psi_s(z)|^2 \, \lambda_s^H \, \nu(dz)} - 2\rho_s \, ds\Big\}$$

$$C_t = k\exp\Big\{-\int_t^T \frac{(\alpha_s - \rho_s)^2}{|\psi_s(0)|^2\lambda_s^B + \int_{\mathbb{R}_0} |\psi_s(z)|^2 \, \lambda_s^H \, \nu(dz)} - \rho_s \, ds\Big\}.$$

If certain integrability assumptions holds then $J(\hat{u}^{\mathbb{F}}) = \sup_{u \in \mathcal{A}^{\mathbb{F}}} J(u)$.

Proof.

- Studying the linear representation, we guess that $Y_t = \mathbb{E}[A_t|\mathcal{F}_t]\hat{X}_t + \mathbb{E}[C_t|\mathcal{F}_t]$
- Study Y_t via the product rule from Itôs formula. This requires the computation of $d\mathbb{E}[A_t|\mathcal{F}_t]$.
- Show that with $\hat{u}^{\mathbb{F}}$ as in the Theorem, Y is indeed a solution of the adjoint equation.
- Show that the conditions from the sufficient maximum principle are satisfied.

$\square$

Linear BSDEs

Theorem

$$-dY_t = \left[A_t Y_t + C_t + E_t(0)\phi_t(0)\sqrt{\lambda_t} + \int_{\mathbb{R}_0} E_t(z)\phi_t(z)\,\nu(dz)\sqrt{\lambda_t} \right] dt$$
$$- \phi_t(0)\,dB_t - \int_{\mathbb{R}_0} \phi_t(z)\,\tilde{H}(dt,dz), \quad Y_T = \xi, \tag{3}$$

where the coefficients satisify (for some $K_E > 0$)

- A *is a bounded stochastic process* $\mathbb{P}$*-a.s.,*
- $\mathbb{E}[\int_0^T |c_s|^2\, ds] < \infty$,
- $E \in \mathcal{I}$,
- $0 \le E_t(z) < K_E z$ *for* $z \in \mathbb{R}_0$*, and* $|E_t(0)| < K_E \quad dt \times d\mathbb{P}$*-a.e.*

Then (3) has an unique solution (Y,ϕ) *in* $S \times \mathcal{I}$*.*

Linear BSDEs continued

... furthermore Y has representation

$$Y_t = \mathbb{E}\Big[\xi \Gamma_T(t) + \int_t^T \Gamma_s(t) C_s \, ds \Big| \mathcal{G}_t\Big], \quad t \in [0, T],$$

where

$$d\Gamma_t(s) = \Gamma_{t\text{-}}(s)\Big(A_t \, dt + E_t(0) \frac{\mathbf{1}_{\{\lambda_t \neq 0\}}}{\sqrt{\lambda_t}} \, dB_t + \int_{\mathbb{R}_0} E_t(z) \frac{\mathbf{1}_{\{\lambda_t \neq 0\}}}{\sqrt{\lambda_t}} \, \tilde{H}(dt, dz)\Big).$$

for $s < t \leq T$ and we have $\Gamma_t(t) = 1$. (The equation on the previous page was:

$$-dY_t = \Big[A_t Y_t + C_t + E_t(0)\phi_t(0)\sqrt{\lambda_t} + \int_{\mathbb{R}_0} E_t(z)\phi_t(z)\,\nu(dz)\sqrt{\lambda_t}\Big] \, dt$$
$$- \phi_t(0) \, dB_t - \int_{\mathbb{R}_0} \phi_t(z) \, \tilde{H}(dt, dz), \quad Y_T = \xi \text{).}$$

The term $\frac{\mathbf{1}_{\{\lambda_t \neq 0\}}}{\sqrt{\lambda_t}}$ occurs because we need

$$\int_{\mathbb{R}_0} |E_t(z)\phi_t(z)|\,\nu(dz)\sqrt{\lambda_t} \leq \sqrt{\int_{\mathbb{R}_0} E_t^2(z)\,\nu(dz)}\sqrt{\int_{\mathbb{R}_0} \phi_t^2(z)\,\nu(dz)}\sqrt{\lambda_t}$$
$$\leq K_E\sqrt{\int_{\mathbb{R}_0} z^2\,\nu(dz)}\sqrt{\int_{\mathbb{R}_0} \phi_t^2(z)\,\nu(dz)}\sqrt{\lambda_t},$$

to have standard parameters. But we also need that $E_t(z)\frac{\mathbf{1}_{\{\lambda_t \neq 0\}}}{\sqrt{\lambda_t}}$ is square integrable with respect to $\Lambda \times \mathbb{P}$.

Proof.

- The conditions on the coefficients ensure that it is a BSDE with standard parameters.
- with $\Gamma_t = \Gamma_t(0)$

$$d(Y_t\Gamma_t) =$$

$$-\Gamma_{t\text{-}} C_t \, dt + \Big[\Gamma_{t\text{-}}\phi_t(0) + Y_{t\text{-}} E_t(0)\frac{\mathbf{1}_{\{\lambda_t \neq 0\}}}{\sqrt{\lambda_t}}\Big] \, dB_t + \int_{\mathbb{R}_0} \Big[\phi_t(z)\Gamma_{t\text{-}}$$

$$+ Y_{t\text{-}}\Gamma_{t\text{-}} E_t(z)\frac{\mathbf{1}_{\{\lambda_t \neq 0\}}}{\sqrt{\lambda_t}} + \Gamma_{t\text{-}}\phi_t(z)E_t(z)\frac{\mathbf{1}_{\{\lambda_t \neq 0\}}}{\sqrt{\lambda_t}}\Big] \, \tilde{H}(dt, dz).$$

- Hence $Y_t\Gamma_t + \int_0^t \Gamma_s C_s \, ds$, $t \in [0, T]$, is a $\mathbb{G}$-martingale so that

$$Y_t\Gamma_t + \int_0^t \Gamma_s C_s \, ds = \mathbb{E}\Big[Y_T\Gamma_T + \int_0^T \Gamma_s C_s \, ds \Big| \mathcal{G}_t \Big].$$

□

Theorem

Let $(g^{(1)}, \xi^{(1)})$ *and* $(g^{(2)}, \xi^{(2)})$ *be two sets of standard parameters for the BSDE's with solutions* $(Y^{(1)}, \phi^{(1)})$, $(Y^{(2)}, \phi^{(2)}) \in S \times \mathcal{I}$. *Assume that*

$$g_t^{(2)}(\lambda, y, \phi, \omega) = f_t\Big(y, \phi(0)\kappa_t(0)\sqrt{\lambda^B}, \int_{\mathbb{R}_0} \phi(z)\kappa_t(z)\,\nu(dz)\sqrt{\lambda}, \omega\Big)$$

where $\kappa \in \mathcal{I}$ *satisfies certain boundedness conditions and* f *satisfies, for some* $K_f > 0$,

$$|f_t(y, b, h) - f_t(y', b', h')| \leq K_f\Big(|y - y'| + |b - b'| + |h - h'|\Big), \; dt \times d\mathbb{P} \text{ a}$$

$$\mathbb{E}\Big[\int_0^T |f_t(0,0,0)|^2\, dt\Big] < \infty.$$

If $\xi^{(1)} \leq \xi^{(2)}$ $\mathbb{P}$*-a.s. and* $g_s^{(1)}(\lambda_s, Y_s^{(1)}, \phi_s^{(1)}) \leq g_s^{(2)}(\lambda_s, Y_s^{(1)}, \phi_s^{(1)})$ $dt \times d\mathbb{P}$*-a.e., then*

$$Y_t^{(1)} \leq Y_t^{(2)} \quad dt \times d\mathbb{P}\text{-a.e.}$$

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