Cylindrical Lévy processes

in Banach spaces and Hilbert spaces

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Wiener processes

Definition Let U be a Hilbert space.

A stochastic process $(W(t) : t \geqslant 0)$ with values in U is called **Wiener process**, if

(1) $W(0) = 0$;

(2) W has independent, stationary increments;

(3) $W(t) - W(s) \stackrel{\mathscr{D}}{=} N(0, (t-s)Q)$ for all $0 \leqslant s \leqslant t$,

where $Q : U \to U$ is a linear operator with the following properties:

symmetric: $\langle Qu, v \rangle_U = \langle u, Qv \rangle_U$ for all $u, v \in U$;

non-negative: $\langle Qu, u \rangle_U \geqslant 0$ for all $u \in U$;

nuclear: $\sum_{k=1}^{\infty} \langle Qe_k, e_k \rangle_U < \infty$ for an orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$.

Cylindrical random variables

and

cylindrical measures

Cylindrical processes

Let U be a Banach space with dual space U^* and dual pairing $\langle \cdot, \cdot \rangle$ and let $(\Omega, \mathscr{A}, P)$ denote a probability space.

Definition: A cylindrical random variable X in U is a mapping

$$X : U^* \to L^0_P(\Omega; \mathbb{R}) \quad \text{linear and continuous.}$$

A cylindrical process in U is a family $(X(t) : \; t \geqslant 0)$ of cylindrical random variables.

- I. E. Segal, 1954
- I. M. Gel'fand 1956: Generalized Functions
- L. Schwartz 1969: seminaire rouge, radonifying operators

Cylindrical measures

Let $X : U^* \to L^0_P(\Omega; \mathbb{R})$ be a cylindrical random variable.

For $a_1, \dots, a_n \in U^*$, $B \in \mathfrak{B}(\mathbb{R}^n)$ and $n \in \mathbb{N}$ the relation

$$\mu\Big(\{u \in U : (\langle u, a_1\rangle, \dots, \langle u, a_n\rangle) \in B\} \Big) := P\Big((Xa_1, \dots, Xa_n) \in B\Big)$$

defines the cylindrical measure

$$\mu : \{\text{all cylindrical sets}\} \to [0, 1].$$

Cylindrical measures

Let $X : U^* \to L_P^0(\Omega;\mathbb{R})$ be a cylindrical random variable.

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defines the cylindrical measure

$$\mu : \{\text{all cylindrical sets}\} \to [0,1].$$

- for fixed $a_1,\dots,a_n \in U^*$ the mapping

  $$B \mapsto \mu\Big(\{u \in U : (\langle u, a_1\rangle, \dots, \langle u, a_n\rangle) \in B\} \Big)$$

  is a probability measure on $\mathfrak{B}(\mathbb{R}^n)$;
- finitely additive on the sets of cylindrical sets;
- not defined on the Borel σ-algebra $\mathfrak{B}(U)$.

cylindrical measures: characteristic function

For a cylindrical measure μ the mapping

$$\varphi_\mu : U^* \to \mathbb{C}, \qquad \varphi_\mu(a) := \int_U e^{i\langle u,a\rangle}\,\mu(du)$$

is called characteristic function of μ.

Theorem (Uniqueness)

For cylindrical measures μ and ν the following are equivalent:

(1) $\mu = \nu$;

(2) $\varphi_\mu = \varphi_\nu$.

Example: induced cylindrical random variable

Example: Let $X : \Omega \to U$ be a (classical) random variable. Then

$$Z : U^* \to L^0_P(\Omega; \mathbb{R}), \qquad Za := \langle X, a \rangle$$

defines a cylindrical random variable.

Example: induced cylindrical random variable

Example: Let $X : \Omega \to U$ be a (classical) random variable. Then

$$Z : U^* \to L_P^0(\Omega; \mathbb{R}), \qquad Za := \langle X, a \rangle$$

defines a cylindrical random variable.

But: not for every cylindrical random variable $Z : U^* \to L_P^0(\Omega; \mathbb{R})$ there exists a classical random variable $X : \Omega \to U$ satisfying

$$Za = \langle X, a \rangle \qquad \text{for all } a \in U^*.$$

Example: cylindrical Wiener process

Definition:

A cylindrical process $(W(t)\,:\;\; t \geqslant 0)$ is called a *cylindrical Wiener process*, if for all $a_1, \ldots, a_n \in U^*$ and $n \in \mathbb{N}$ the stochastic process :

$$\Big((W(t)a_1, \ldots, W(t)a_n) : \; t \geqslant 0\Big)$$

is a centralised Wiener process in $\mathbb{R}^n$.

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“Theorem”

Every object which satisfies one of the definitions of a cylindrical Wiener process in the literature satisfies (in a certain sense) the definition above.

Cylindrical Lévy processes

Definition: cylindrical Lévy process

Definition: (Applebaum, Riedle (2010))
A cylindrical process $(L(t) : \ t \geqslant 0)$ is called a *cylindrical Lévy process*, if for all $a_1, \dots, a_n \in U^*$ and $n \in \mathbb{N}$ the stochastic process :

$$\Big((L(t)a_1, \dots, L(t)a_n) : \ t \geqslant 0 \Big)$$

is a Lévy process in $\mathbb{R}^n$.

Infinitely divisible cylindrical measure

Definition

A cylindrical measure μ is called infinitely divisible if for each $k \in \mathbb{N}$ there exists a cylindrical measure μ_k such that

$$\varphi_\mu(a) = (\varphi_{\mu_k}(a))^k \qquad \text{for all } a \in U^*.$$

Example: if $(L(t) : t \geqslant 0)$ is a cylindrical Lévy process then the cylindrical distribution of $L(1)$ is infinitely divisible.

Lévy-Khintchine formula

Theorem: For a cylindrical measure μ the following are equivalent:

(1) μ is infinitely divisible;

(2) the characteristic function of μ is of the form

$$\varphi_\mu(a) = \exp\left(i\, p(a) - \tfrac{1}{2} q(a) + \int_U \left(e^{i\langle u,a\rangle} - 1 - i\langle u,a\rangle\, \mathbb{1}_{B_1}(\langle u,a\rangle) \right)\, \nu(du) \right)$$
$$=: \exp\left(\mathscr{S}_{p,q,\nu}(a) \right)$$

where
- $p : U^* \to \mathbb{R}$ is (non-linear) continuous and $p(0) = 0$;
- $q : U^* \to \mathbb{R}$ is a quadratic form;
- ν cylindrical measure, $\displaystyle\int_U \left(\langle u,a\rangle^2 \wedge 1 \right), \nu(du) < \infty$ for all $a \in U^*$;
- $\displaystyle a \mapsto \left(ip(a) + \int_U \left(e^{i\langle u,a\rangle} - 1 - i\langle u,a\rangle\, \mathbb{1}_{B_1}(\langle u,a\rangle) \right)\, \nu(du) \right)$

  is negative definite.

Example: series approach

Theorem Let U be a Hilbert space with ONB $(e_k)_{k\in\mathbb{N}}$ and $(\sigma_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$;

$(h_k)_{k\in\mathbb{N}}$ be a sequence of independent, real-valued Lévy processes.

If for all $u^* \in U^*$ and $t \geqslant 0$ the sum

$$L(t)u^* := \sum_{k=1}^{\infty} \langle e_k, u^* \rangle \sigma_k h_k(t)$$

converges P-a.s. then it defines a cylindrical Lévy process $(L(t) : t \geqslant 0)$.

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Example 0: for h_k standard, real-valued Brownian motion:

$(\sigma_k)_{k\in\mathbb{N}}\in\ell^\infty \iff$ cylindrical (Wiener) Lévy process

$(\sigma_k)_{k\in\mathbb{N}}\in\ell^2 \iff$ honest (Wiener) Lévy process

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converges P-a.s. then it defines a cylindrical Lévy process $(L(t) : t \geqslant 0)$.

Example 1: for h_k Poisson process with intensity 1:

$(\sigma_k)_{k\in\mathbb{N}} \in \ell^2 \iff$ cylindrical Lévy process

$(\sigma_k)_{k\in\mathbb{N}} \in \ell^1 \iff$ honest Lévy process

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converges P-a.s. then it defines a cylindrical Lévy process $(L(t):\, t \geqslant 0)$.

Example 2: for h_k compensated Poisson process with intensity 1:

$$(\sigma_k)_{k\in\mathbb{N}} \in \ell^\infty \quad \Longleftrightarrow \quad \text{cylindrical Lévy process}$$

$$(\sigma_k)_{k\in\mathbb{N}} \in \ell^2 \quad \Longleftrightarrow \quad \text{honest Lévy process}$$

Example: series approach

Theorem Let U be a Hilbert space with ONB $(e_k)_{k\in\mathbb{N}}$ and $(\sigma_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$;

$(h_k)_{k\in\mathbb{N}}$ be a sequence of independent, real-valued Lévy processes.

If for all $u^* \in U^*$ and $t \geqslant 0$ the sum

$$L(t)u^* := \sum_{k=1}^{\infty} \langle e_k, u^* \rangle \sigma_k h_k(t)$$

converges P-a.s. then it defines a cylindrical Lévy process $(L(t) : t \geqslant 0)$.

Example 3: for h_k symmetric, standardised, α-stable:

$$(\sigma_k)_{k\in\mathbb{N}} \in \ell^{(2\alpha)/(2-\alpha)} \iff \text{cylindrical Lévy process}$$

$$(\sigma_k)_{k\in\mathbb{N}} \in \ell^{\alpha} \iff \text{honest Lévy process}$$

Example: subordination

Theorem

Let W be a cylindrical Wiener process in a Banach space U,
ℓ be a real-valued Lévy subordinator, independent of W.

Then, for each $t \geqslant 0$,

$$L(t) : U^* \to L^0_P(\Omega; \mathbb{R}), \qquad L(t)u^* = W(\ell(t))u^*$$

defines a cylindrical Lévy process $(L(t) : \, t \geqslant 0)$ in U.

Stochastic integration

Stochastic integration w.r.t cylindrical semi-martingales

- M. Métivier, J. Pellaumail, 1980
- G. Kallianpur, J. Xiong, 1996
- R. Mikulevicius, B.L. Rozovskii, 1998.

Stochastic integral: motivation

Assume: Y classical Lévy process in a Hilbert space H

$$\Psi(s) := \sum_{k=0}^{n-1} \mathbb{1}_{(t_k,t_{k+1}]}(s)\Phi_k \quad \text{for} \quad \Phi_k : \Omega \to \mathscr{L}_2(H,H).$$

Then
$$\left\langle \int_0^T \Psi(s)\, dY(s), h \right\rangle = \sum \langle \Phi_k\big(Y(t_{k+1}) - Y(t_k)\big), h\rangle$$
$$= \sum \langle Y(t_{k+1}) - Y(t_k), \Phi_k^* h\rangle$$

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Then
$$\begin{aligned}\langle \int_0^T \Psi(s)\,dY(s), h\rangle &= \sum \langle \Phi_k\big(Y(t_{k+1}) - Y(t_k)\big), h\rangle \\ &= \sum \langle Y(t_{k+1}) - Y(t_k), \Phi_k^* h\rangle \\ &= \sum \Big(L(t_{k+1}) - L(t_k)\Big)\big(\Phi_k^* h\big)\end{aligned}$$

if $(L(t) : t \geqslant 0)$ is a cylindrical Lévy process in H.

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if $(L(t) : t \geqslant 0)$ is a cylindrical Lévy process in H.

Two problems:

- does there exists a random variable $J_k : \Omega \to H$ such that:

$$\langle J_k, h\rangle = \big(L(t_{k+1}) - L(t_k)\big)(\Phi_k^* h) \quad \text{for all } h \in H.$$

- Is the mapping $\Psi \mapsto \int_0^T \Psi(s)\,dL(s)$ continuous?

Radonifying the increments

Consider for fixed $0 \leqslant t_k \leqslant t_{k+1}$ a simple random variable

$$\Phi : \Omega \to \mathscr{L}_2(H,H) \qquad \Phi(\omega) := \sum_{i=1}^{n} \mathbb{1}_{A_i}(\omega)\varphi_i,$$

where $\varphi_i \in \mathscr{L}_2(H,H)$

$$A_i \in \mathscr{F}_{t_k} := \sigma\big(L(s)h : \; s \in [0,t_k], \; h \in H\big).$$

Since φ_i is Hilbert-Schmidt there exists $Z_i : \Omega \to H$ such that

$$\big(L(t_{k+1}) - L(t_k)\big)(\varphi_i^* h) = \langle Z_i, h\rangle \qquad \text{for all } h \in H.$$

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$$\big(L(t_{k+1}) - L(t_k)\big)(\varphi_i^* h) = \langle Z_i, h\rangle \qquad \text{for all } h \in H.$$

Define the H-valued random variable

$$\Phi\big(L(t_{k+1}) - L(t_k)\big) := \sum_{i=1}^{n} \mathbb{1}_{A_i} Z_i.$$

It satisfies for each $h \in H$:

$$\big\langle\, \Phi\big(L(t_{k+1}) - L(t_k)\big) \,\big\rangle\big\langle\, h \,\big\rangle = \sum_{i=1}^{n} \mathbb{1}_{A_i} \big(L(t_{k+1}) - L(t_k)\big)(\varphi_i^* h)$$

Radonifying the increments

Theorem: (with A. Jakubowski)
Let $0 \leqslant t_k \leqslant t_{k+1}$ be fixed. For each $\mathscr{F}_{t_k}$-measurable random variable

$$\Phi : \Omega \to \mathscr{L}_2(H, H),$$

there exists a random variable $Y : \Omega \to H$ and a sequence $\{\Phi_n\}_{n\in\mathbb{N}}$ of simple random variables such that $\Phi_n \to \Phi$ P-a.s. and

$$Y = \lim_{n\to\infty} \Phi_n\big(L(t_{k+1}) - L(t_k)\big) \qquad \text{in probability.}$$

Define: $\Phi\big(L(t_{k+1}) - L(t_k)\big) := Y.$

Defining the stochastic integral

For a simple stochastic process of the form

$$\Psi : [0,T] \times \Omega \to \mathscr{L}_2(H,H), \qquad \Psi(t) = \sum_{j=0}^{N-1} \mathbb{1}_{(t_j,t_{j+1}]}(t)\Phi_j,$$

where $0 = t_0 < t_1 < \cdots < t_N = T$,

$\Phi_j : \Omega \to \mathscr{L}_2(H,H)$ is $\mathscr{F}_{t_j}$-measurable,

define the H-valued stochastic integral

$$I(\Psi) := \sum_{j=0}^{N-1} \Phi_j\big(L(t_{j+1}) - L(t_j)\big)$$

Defining the stochastic integral

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$$I(\Psi) := \sum_{j=0}^{N-1} \Phi_j\big(L(t_{j+1}) - L(t_j)\big)$$

Simple stochastic processes are dense in

$$\mathscr{H}(\mathscr{L}_2) := \big\{\Psi : \Omega \to D_-\big([0,T], \mathscr{L}_2(H,H)\big) : \text{predictable}\big\},$$

where $D_-\big([0,T], \mathscr{L}_2(H,H)\big) := \{f : [0,T] \to \mathscr{L}_2(H,H) : \text{càglàd}\}$,

equipped with the Skorokhod J_1-topology.

Defining the stochastic integral

Theorem: (with A. Jakubowski)
For every $\Psi \in \mathscr{H}(\mathscr{L}_2)$ there exists an H-valued random variable $I(\Psi)$ and a sequence $\{\Psi_n\}_{n\in\mathbb{N}}$ of simple stochastic processes such that $\Psi_n \to \Psi$ P-a.s. in J_1 and

$$\int_0^T \Psi(s)\,dL(s) := \lim_{n\to\infty} I(\Psi_n) \qquad \text{in probability.}$$

Defining the stochastic integral

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For every $\Psi \in \mathscr{H}(\mathscr{L}_2)$ there exists an H-valued random variable $I(\Psi)$ and a sequence $\{\Psi_n\}_{n\in\mathbb{N}}$ of simple stochastic processes such that $\Psi_n \to \Psi$ P-a.s. in J_1 and

$$\int_0^T \Psi(s)\,dL(s) := \lim_{n\to\infty} I(\Psi_n) \qquad \text{in probability.}$$

Proof: Show that

(1) $\{I(\Psi_n) : n \in \mathbb{N}\}$ is tight

(2) for every $h \in H$ there exists a real-valued random variable Y_h such

$$\langle I(\Psi_n), h\rangle \to Y_h \text{ in probability}$$

Special case: deterministic integrands

Let U, V be separable Banach spaces

$\Psi := \psi$ for deterministic $\psi : [0,T] \to \mathscr{L}(U,V)$

Theorem: Let L be a cylindrical Lévy process with cylindrical characteristic $\mathscr{S} : U^* \to \mathbb{C}$. Then the following are equivalent:

(1) ψ is integrable w.r.t. L;

(2) The function $\varphi : V^* \to \mathbb{C}$,

$$\varphi(v^*) := \exp\left(\int_0^T \mathscr{S}(\psi^*(s)v^*)\, ds \right)$$

is the characteristic function of a Radon measure on $\mathfrak{B}(V)$.

Ornstein-Uhlenbeck process

Stochastic evolution equations

$$dX(t) = AX(t)\,dt + G\,dL(t) \qquad \text{for all } t \in [0,T]$$

- A generator of C_0-semigroup $(S(t))_{t\geqslant 0}$ in V;
- $G : U \to \mathscr{L}(U,V)$;
- $(L(t) : t \geqslant 0)$ cylindrical Lévy process in U.

Definition: A stochastic process $(X(t) : t \in [0,T])$ in V is called a **weak solution** if it satisfies for all $v^* \in \mathrm{D}(A^*)$ and $t \in [0,T]$ that

$$\langle X(t), v^*\rangle = \langle X(0), v^*\rangle + \int_0^t \langle X(s), A^*v^*\rangle\,ds + L(s)(G^*v^*).$$

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Theorem: The following are equivalent:

(a) $t \mapsto S(t)G$ is stochastically integrable;

(b) there exists a weak solution $(X(t) : t \in [0,T])$.

In this case, the weak solution is given by

$$X(t) = S(0)X(0) + \int_0^t S(t-s)G\,dL(s) \qquad \text{for all } t \in [0,T].$$

Spatial regularity

$$dX(t) = AX(t)\,dt + G\,dL(t) \qquad \text{for all } t \in [0,T]$$

- A generator of C_0-semigroup $(S(t))_{t\geqslant 0}$ in V;
- $G : U \to \mathscr{L}(U,V)$;
- $(L(t) : t \geqslant 0)$ cylindrical Lévy process in U.

Corollary:

Assume that $S(t)(V) \subseteq W$ for all $t > 0$ for a Banach space $W \subseteq V$. Then the solution X is W-valued iff

$$f : [0,T] \to \mathscr{L}(V,W), \qquad f(t) = S(t)G$$

is stochastically integrable.

Temporal regularity

$$dX(t) = AX(t)\,dt + dL(t) \qquad \text{for all } t \in [0,T]$$

- A generator of C_0-semigroup $(S(t))_{t\geqslant 0}$ in V;
- V Hilbert space with ONB $(e_k)_{k\in\mathbb{N}}$;
- $(L(t) : t \geqslant 0)$ cylindrical Lévy process in V.

Theorem: Let ν be the cylindrical Levy measure of L. If there exists a constant $K > 0$ such that

$$\lim_{n\to\infty} \nu\left(\left\{v \in V : \sum_{k=1}^{n} \langle v, e_k\rangle^2 > K\right\}\right) = \infty,$$

then the solution does not have a (weak) càdlàg modification.

Temporal regularity

$$dX(t) = AX(t)\,dt + dL(t) \qquad \text{for all } t \in [0,T]$$

- A generator of C_0-semigroup $(S(t))_{t\geqslant 0}$ in V;
- V Hilbert space with ONB $(e_k)_{k\in\mathbb{N}}$;
- $(L(t) : t \geqslant 0)$ cylindrical Lévy process in V.

Example: (Peszat, Zabczyk, Imkeller,....Liu, Zhai)
Let $(L(t) : t \geqslant 0)$ be of the form

$$L(t)v^* = \sum_{k=1}^{\infty} \langle e_k, v^* \rangle \sigma_k h_k \qquad \text{for all } v^* \in V^*,$$

where h_k are real-valued, α-stable processes and $(\sigma_k) \in \ell^{(2\alpha)/(2-\alpha)} \setminus \ell^{\alpha}$.
Then the solution does not have a **càdlàg modification**.

Temporal regularity

$$dX(t) = AX(t)\,dt + dL(t) \qquad \text{for all } t \in [0,T]$$

- A generator of C_0-semigroup $(S(t))_{t\geqslant 0}$ in V;
- V Hilbert space with ONB $(e_k)_{k\in\mathbb{N}}$;
- $(L(t) : t \geqslant 0)$ cylindrical Lévy process in V.

Example: (Brzezniak, Zabczyk)
Let $(L(t) : t \geqslant 0)$ be of the form

$$L(t)v^* = W(\ell(t))v^* \qquad \text{for all } v^* \in V^*,$$

where W is a cylindrical but not a classical Wiener process in V and ℓ a real-valued Lévy subordinator. Then the solution has not a **càdlàg** modification.

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