A pricing measure to explain the risk premium in power markets

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Outline

1. Brief introduction to electricity markets
2. The mathematical model
3. The risk premium in the arithmetic spot model
4. The risk premium in the geometric spot model

Outline

Electricity markets

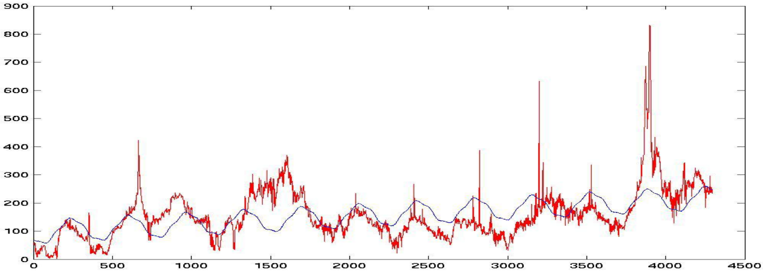
- Liberalisation of energy markets in the early 90's.
- Typically, power markets organize trade in:
  - Hourly spot electricity, next-day delivery.
  - Forward and futures contracts on the spot.
  - European options on forwards.
- The markets are local: One "name" but may different assets in practice.
- Interconnection between markets is difficult and not always possible.

Spot price

- An hourly market with physical delivery of electricity.
- Participants hand over bids before noon the day ahead.
  - Volume and price bids for each of the 24 hours next day.
  - Maximum amount of bids within technical volume and price limits.
- The exchange creates demand and production curves for each hour of the next day.
- The spot price is the equilibrium:
  - Price for delivery of electricity at a specific hour next day.
  - The daily spot price is the average of the 24 hourly prices.
- Reference price for the forward market.

Stylized facts of the electricity spot prices

- Seasonal behaviour in yearly, weekly and daily cycles.
- Approximate stationary behaviour: Mean reversion.
- Non-Gaussianity and extreme spikes.
- Historical spot price at NordPool from the beginning in 1992 (NOK/MWh).



Factor models for the spot price

- We will consider two kind of models:
  - The arithmetic spot price model, defined by

$$S(t) = \Lambda_a(t) + X(t) + Y(t), \quad t \in [0, T^*].$$

  - The geometric spot price model, defined by

$$S(t) = \Lambda_g(t) \exp\left(X(t) + Y(t)\right), \quad t \in [0, T^*].$$

- $\Lambda_a(t)$ and $\Lambda_g(t)$ are assumed to be deterministic and they account for seasonalities in the prices.
- $X(t)$ has continuous paths and explains *normal variations*.
- $Y(t)$ has jumps and accounts for the *spikes.*
- $X(t)$ and $Y(t)$ are mean reverting stochastic processes.
- **Lucia and Schwartz** (2002), **Cartea and Figueroa** (2005) and **Benth et al.** (2008).

The (instantaneous) forward price

- In practice, electricity is a non-storable commodity.
- There is no buy and hold strategies $\Longrightarrow$ classical non-arbitrage arguments break down.
- **Incomplete market**: any probability measure Q equivalent to the historical measure P is valid.
- The forward price with time to delivery $0 < T < T^*$ at time $0 < t < T$ is given by

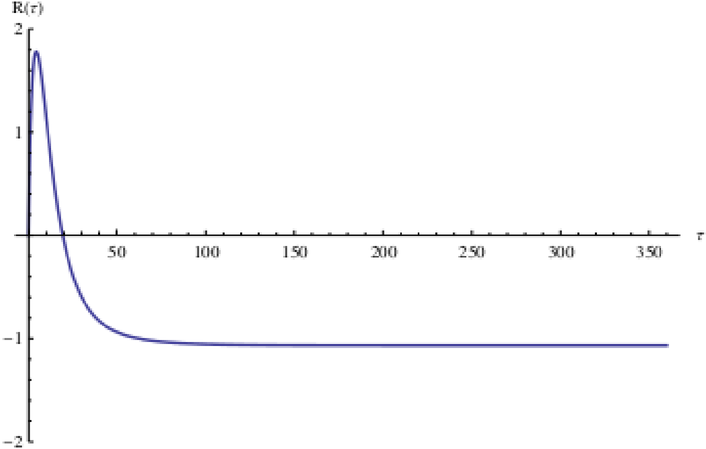
$$F_Q(t, T) = \mathbb{E}_Q[S(T)|\mathcal{F}_t]$$

  where $\mathcal{F}_t$ is the information in the market up to time t.
- The risk premium for forward prices is defined by the following expression

$$R_Q^F(t, T) \triangleq \mathbb{E}_Q[S(T)|\mathcal{F}_t] - \mathbb{E}_P[S(T)|\mathcal{F}_t].$$

Risk premium profile

- If $R_Q^F(t, T) > 0$, the market is in "contango".
- If $R_Q^F(t, T) < 0$, the market is in "normal backwardation".
- **Goal**: To reproduce the following risk premium profile.



The swap price

- The delivery of the underlying takes place over a period of time $[T_1, T_2]$, where $0 < T_1 < T_2 < T^*$.
- So we are interested in what we call swap prices, given by

$$\begin{aligned} F_Q(t, T_1, T_2) &\triangleq \mathbb{E}_Q\left[\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} S(T)dT \middle| \mathcal{F}_t\right] \\ &= \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} F_Q(t, T)dT. \end{aligned}$$

- The risk premium for forward prices is defined by the following expression

$$R_Q^F(t, T) \triangleq \mathbb{E}_Q[S(T)|\mathcal{F}_t] - \mathbb{E}_P[S(T)|\mathcal{F}_t],$$

  and, hence, for swap prices is given by

$$R_Q^S(t, T_1, T_2) \triangleq \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} R_Q^F(t, T)dT.$$

Outline

Mathematical modeling of the factors

- Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a complete filtered probability space, where $T > 0$ is a fixed finite time horizon.
- Consider a standard Brownian motion W and a pure jump Lévy subordinator

$$L(t) = \int_0^t \int_0^\infty z N^L(ds, dz), t \in [0, T],$$

  where $N^L(ds, dz)$ is a Poisson random measure with Lévy measure ℓ satisfying $\int_0^\infty z\ell(dz) < \infty$.
- Let $\kappa_L(\theta) \triangleq \log \mathbb{E}_P[e^{\theta L(1)}]$ and

$$\Theta_L \triangleq \sup\{\theta \in \mathbb{R}_+ : \mathbb{E}[e^{\theta L(1)}] < \infty\}.$$

- A minimal assumption is that $\Theta_L > 0$.
- In the geometric spot model we also need $\Theta_L > 1$.

Mathematical modeling of the factors

- Consider the Ornstein-Uhlenbeck processes

$$X(t) = X(0) - \alpha_X \int_0^t X(s)ds + \sigma_X W(t),$$
$$Y(t) = Y(0) + \int_0^t (\kappa_L'(0) - \alpha_Y Y(s))ds + \int_0^t \int_0^\infty z\tilde{N}^L(ds, dz),$$

with $t \in [0, T], \alpha_X, \sigma_X, \alpha_Y > 0, X(0) \in \mathbb{R}, Y(0) \geq 0.$

- Using integration by parts, one gets the following explicit representation

$$X(T) = X(t)e^{-\alpha_X(T-t)} + \sigma_X \int_t^T e^{-\alpha_X(T-s)} dW(s),$$
$$Y(T) = Y(t)e^{-\alpha_Y(T-t)} + \frac{\kappa_L'(0)}{\alpha_Y}(1 - e^{-\alpha_Y(T-t)})$$
$$+ \int_t^T \int_0^\infty e^{-\alpha_Y(T-s)} z\tilde{N}^L(ds, dz),$$

where $0 \leq t \leq T$.

The change of measure

- We will consider a parametrized family of measure changes.
- To this end, consider the following family of kernels

$$G_{\theta_1,\beta_1}(t) \triangleq \sigma_X^{-1}\left(\theta_1 + \alpha_X \beta_1 X(t)\right), \quad t \in [0, T],$$
$$H_{\theta_2,\beta_2}(t,z) \triangleq e^{\theta_2 z}\left(1 + \frac{\alpha_Y \beta_2}{\kappa_L''(\theta_2)} z Y(t-)\right), \quad t \in [0, T], z \in \mathbb{R},$$

where $\bar{\beta} \in [0,1]^2, \bar{\theta} \in \bar{D}_L \triangleq \mathbb{R} \times D_L$ and $D_L \triangleq (-\infty, \Theta_L/2)$.

- Next, define the following family of Wiener and Poisson integrals

$$\tilde{G}_{\theta_1,\beta_1}(t) \triangleq \int_0^t G_{\theta_1,\beta_1}(s) dW(s), \quad t \in [0, T],$$
$$\tilde{H}_{\theta_2,\beta_2}(t) \triangleq \int_0^t \int_0^\infty \left(H_{\theta_2,\beta_2}(s,z) - 1\right) \tilde{N}^L(ds,dz), \quad t \in [0, T],$$

associated to the kernels G_{θ_1,β_1} and H_{θ_2,β_2}, respectively.

The change of measure

- The desired family of measure changes is given by $Q_{\bar{\theta},\bar{\beta}} \sim P, \bar{\beta} \in [0,1]^2, \bar{\theta} \in \bar{D}_L$, with

$$\left.\frac{dQ_{\bar{\theta},\bar{\beta}}}{dP}\right|_{\mathcal{F}_t} \triangleq \mathcal{E}(\tilde{G}_{\theta_1,\beta_1} + \tilde{H}_{\theta_2,\beta_2})(t), \quad t \in [0,T],$$

where $\mathcal{E}(\cdot)$ denotes the stochastic exponential.

- Recall that, if M is a semimartingale, the stochastic exponential of M is the unique strong solution of

$$\begin{aligned} d\mathcal{E}(M)(t) &= \mathcal{E}(M)(t-)dM(t), \quad t \in [0,T], \\ \mathcal{E}(M)(t) &= 1, \end{aligned}$$

which is given by

$$\mathcal{E}(M)(t) = \exp\left(M(t) - \frac{1}{2}[M^c, M^c](t) - \sum_{0 \le s \le t} \Delta M(s) - \log(1 + \Delta M(s))\right).$$

The change of measure

- If M is a local martingale, then $\mathcal{E}(M)$ is also a local martingale.
- If $\mathcal{E}(M)$ is a positive local martingale, then $\mathcal{E}(M)$ is also a supermartingale and $\mathbb{E}_P[\mathcal{E}(M)(t)] \leq 1, t \in [0, T]$.
- To have a well defined change of measure we need to ensure that

$$\mathbb{E}_P[\mathcal{E}(M)(T)] = 1$$

  and

$$\mathcal{E}(M)(t) > 0, t \in [0, T].$$

- **Yor's Formula**: Let M_1 and M_2 two semimartingales starting at 0. Then,

$$\mathcal{E}(M_1 + M_2 + [M_1, M_2])(t) = \mathcal{E}(M_1)\mathcal{E}(M_2), \quad 0 \leq t \leq T.$$

The change of measure

- As $[\tilde{G}_{\theta_1,\beta_1}, \tilde{H}_{\theta_2,\beta_2}] \equiv 0$, by Yor's formula we can write

$$\mathcal{E}(\tilde{G}_{\theta_1,\beta_1} + \tilde{H}_{\theta_2,\beta_2})(t) = \mathcal{E}(\tilde{G}_{\theta_1,\beta_1})(t)\mathcal{E}(\tilde{H}_{\theta_2,\beta_2})(t), \quad t \in [0, T].$$

- As L and W are independent, we have

$$\mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1,\beta_1} + \tilde{H}_{\theta_2,\beta_2})(T)] = \mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1,\beta_1})(T)]\mathbb{E}_P[\mathcal{E}(\tilde{H}_{\theta_2,\beta_2})(T)],$$

  and the problem is reduced to show that $\mathcal{E}(\tilde{G}_{\theta_1,\beta_1})$ and $\mathcal{E}(\tilde{H}_{\theta_2,\beta_2})$ are true martingales.
- $\mathcal{E}(\tilde{G}_{\theta_1,\beta_1} + \tilde{H}_{\theta_2,\beta_2}) > 0$ if and only if $\mathcal{E}(\tilde{H}_{\theta_2,\beta_2}) > 0$.
- As $\mathcal{E}(\tilde{H}_{\theta_2,\beta_2})(t) = e^{\tilde{H}_{\theta_2,\beta_2}(t)} \prod_{0<s\leq t}(1 + \Delta\tilde{H}_{\theta_2,\beta_2}(s))e^{-\Delta\tilde{H}_{\theta_2,\beta_2}(s)}$, we have that $\mathcal{E}(\tilde{H}_{\theta_2,\beta_2}) > 0$ if and only if $\Delta\tilde{H}_{\theta_2,\beta_2} > -1$, which yields the condition

$$P\left(\frac{\alpha_Y \beta_2}{\kappa_L''(\theta_2)}(\Delta L(t))Y(t-) > -1, t \in [0, T]\right) = 1.$$

Main result

Theorem

Let $\bar{\theta} = (\theta_1, \theta_2) \in \mathbb{R} \times D_L$ and $\beta = (\beta_1, \beta_2) \in [0,1]^2$. Then $\mathcal{E}(\tilde{G}_{\theta_1,\beta_1}) = \{\mathcal{E}(\tilde{G}_{\theta_1,\beta_1})(t)\}_{t\in[0,T]}$ and $\mathcal{E}(\tilde{H}_{\theta_2,\beta_2}) = \{\mathcal{E}(\tilde{H}_{\theta_2,\beta_2})(t)\}_{t\in[0,T]}$ are martingales under P.

- For $\mathcal{E}(\tilde{G}_{\theta_1,\beta_1})$: a localised version of Novikov's criterion works. That is, there exists a partition $0 < t_1 < \cdots < t_n = T$ such that

$$\mathbb{E}_P[\exp\left(\frac{1}{2}\int_{t_k}^{t_{k+1}} G^2_{\theta_1,\beta_1}(s)ds\right)] < \infty, \quad k = 0, ..., n-1.$$

- For $\mathcal{E}(\tilde{H}_{\theta_2,\beta_2})$: Lépingle-Mémin's criterion also works but is not sharp. That is, we can assume conditions on θ_2, β_2 and $\ell(dz)$ such that the compensator of

$$\sum_{s\leq t}\{(1+\Delta\tilde{H}_{\theta_2,\beta_2}(s))\log(1+\Delta\tilde{H}_{\theta_2,\beta_2}(s)) - \Delta\tilde{H}_{\theta_2,\beta_2}(s)\}$$

has finite exponential moments of order 1.

Sketch of the proof of the main result

Let M be $\tilde{G}_{\theta_1,\beta_1}$ or $\tilde{H}_{\theta_2,\beta_2}$:

- Localise $\mathcal{E}(M)$ using a reducing sequence $\{\tau_n\}_{n\geq 1}$.
- For any $n \geq 1, \mathcal{E}(M)^{\tau_n} = \{\mathcal{E}(M)(t \wedge \tau_n)\}_{t\in[0,T]}$ is a true martingale and induces a change of measure .
- Test the uniform integrability of $\{\mathcal{E}(M)^{\tau_n}(T)\}_{n\geq 1}$ with $G(x) = x\log(x)$, i.e.

$$\sup_n \mathbb{E}_P[G(\mathcal{E}(M)^{\tau_n}(T))] < \infty.$$

- But this can be rewritten as

$$\sup_n \mathbb{E}_{Q^n}[\log(\mathcal{E}(M)^{\tau_n}(T))] < \infty.$$

- We can eliminate the ordinary exponential in $\mathcal{E}(M)^{\tau_n}(T)$.
- The problem is reduced to find a uniform bound for the second moment of X and Y under Q^n.

The dynamics under the new pricing measure

- By Girsanov's theorem for semimartingales, we can write

$$X(t) = X(0) + B^X_{Q_{\bar\theta,\bar\beta}}(t) + \sigma_X W_{Q_{\bar\theta,\bar\beta}}(t), \quad t \in [0, T],$$
$$Y(t) = Y(0) + B^Y_{Q_{\bar\theta,\bar\beta}}(t) + \int_0^t \int_0^\infty z \tilde{N}^L_{Q_{\bar\theta,\bar\beta}}(ds, dz), \quad t \in [0, T],$$

where

$$B^X_{Q_{\bar\theta,\bar\beta}}(t) = \int_0^t (\theta_1 - \alpha_X(1-\beta_1)X(s))ds, \quad t \in [0, T],$$

and

$$B^Y_{Q_{\bar\theta,\bar\beta}}(t) = \int_0^t \left(\kappa_L'(\theta_2) - \alpha_Y(1-\beta_2)Y(s)\right) ds, \quad t \in [0, T].$$

The dynamics under the new pricing measure

- The $Q_{\bar{\theta},\bar{\beta}}$-compensator measure of Y is given by

$$\nu^{Y}_{Q_{\bar{\theta},\bar{\beta}}}(dt,dz) = e^{\theta_2 z}\left(1+\frac{\alpha_Y \beta_2}{\kappa_L''(\theta_2)} zY(t-)\right)\ell(dz)dt.$$

- Using integration by parts again, we get

$$\begin{aligned} X(T) &= X(t)e^{-\alpha_X(1-\beta_1)(T-t)} + \frac{\theta_1}{\alpha_X(1-\beta_1)}(1-e^{-\alpha_X(1-\beta_1)(T-t)}) \\ &\quad + \sigma_X \int_t^T e^{-\alpha_X(1-\beta_1)(T-s)} dW_{Q_{\bar{\theta},\bar{\beta}}}(s), \\ Y(T) &= Y(t)e^{-\alpha_Y(1-\beta_2)(T-t)} + \frac{\kappa_L'(\theta_2)}{\alpha_Y(1-\beta_2)}(1-e^{-\alpha_Y(1-\beta_2)(T-t)}) \\ &\quad + \int_t^T \int_0^\infty e^{-\alpha_Y(1-\beta_2)(T-s)} z\tilde{N}^{L}_{Q_{\bar{\theta},\bar{\beta}}}(ds,dz), \end{aligned}$$

where $0 \le t \le T$.

Outline

Forward price formula

- Recall that $S(t) = \Lambda_a(t) + X(t) + Y(t), \quad t \in [0, T^*]$.

Theorem

The forward price $F_{Q_{\bar{\theta},\bar{\beta}}}(t,T)$ in the arithmetic spot model is given by

$$F_{Q_{\bar{\theta},\bar{\beta}}}(t,T) = \Lambda_a(T) + X(t)e^{-\alpha_X(1-\beta_1)(T-t)} + Y(t)e^{-\alpha_Y(1-\beta_2)(T-t)} + \frac{\theta_1}{\alpha_X(1-\beta_1)}(1 - e^{-\alpha_X(1-\beta_1)(T-t)}) + \frac{\kappa_L'(\theta_2)}{\alpha_Y(1-\beta_2)}(1 - e^{-\alpha_Y(1-\beta_2)(T-t)}).$$

- Follows easily by showing that $\int_0^t \int_0^\infty e^{-\alpha_Y(1-\beta_2)(T-s)} z \tilde{N}^L_{Q_{\bar{\theta},\bar{\beta}}}(ds, dz)$ is a $Q_{\bar{\theta},\bar{\beta}}$-martingale.
- This pricing formula allows to model the spot price with stationary factors and obtain non-deterministic forward prices for large time to maturity $\tau \triangleq T - t$.

Risk premium formula

Theorem

The risk premium for the forward price in the arithmetic spot model is given by

$$
\begin{aligned}
R^F_{a,Q_{\bar\theta,\bar\beta}}(t,\tau) &= X(t)e^{-\alpha_X\tau}(e^{\alpha_X\beta_1\tau}-1)+Y(t)e^{-\alpha_Y\tau}(e^{\alpha_Y\beta_2\tau}-1)\\
&+\frac{\theta_1}{\alpha_X(1-\beta_1)}(1-e^{-\alpha_X(1-\beta_1)\tau})\\
&+\frac{\kappa_L'(\theta_2)}{\alpha_Y(1-\beta_2)}(1-e^{-\alpha_Y(1-\beta_2)\tau})-\frac{\kappa_L'(0)}{\alpha_Y}(1-e^{-\alpha_Y\tau}).
\end{aligned}
$$

Moreover, if $\alpha_X<\alpha_Y$, then

$$
\begin{aligned}
\lim_{\tau\to\infty} R^F_{a,Q_{\bar\theta,\bar\beta}}(t,\tau) &= \frac{\theta_1}{\alpha_X(1-\beta_1)}+\frac{\kappa_L'(\theta_2)-\kappa_L'(0)}{\alpha_Y(1-\beta_2)}+\frac{\kappa_L'(0)}{\alpha_Y}\frac{\beta_2}{1-\beta_2},\\
\lim_{\tau\to 0}\frac{\partial}{\partial\tau}R^F_{a,Q_{\bar\theta,\bar\beta}}(t,\tau) &= X(t)\alpha_X\beta_1+Y(t)\alpha_Y\beta_2+\theta_1+\kappa_L'(\theta_2)-\kappa_L'(0).
\end{aligned}
$$

- Analysis of possible risk profiles in **Benth and O.-L.** (2013).

Outline

Forward price formula

- Recall that $S(t) = \Lambda_g(t) \exp(X(t) + Y(t)), \quad t \in [0, T^*]$.
- Due to the independence of X and Y, one can show that

$$\mathbb{E}_{Q_{\bar{\theta},\bar{\beta}}}[S(T)|\mathcal{F}_t] = \Lambda_g(T)\mathbb{E}_{Q_{\bar{\theta},\bar{\beta}}}[\exp(X(T))|\mathcal{F}_t]\mathbb{E}_{Q_{\bar{\theta},\bar{\beta}}}[\exp(Y(T))|\mathcal{F}_t].$$

- As $X(T)$ is Gaussian, it has finite exponential moments and it is easy to prove that

$$\begin{aligned}
&\mathbb{E}_{Q_{\bar{\theta},\bar{\beta}}}[\exp(X(T))|\mathcal{F}_t] \\
&= \exp\left(X(t) e^{-\alpha_X(1-\beta_1)(T-t)} + \frac{\theta_1}{\alpha_X(1-\beta_1)}(1 - e^{-\alpha_X(1-\beta_1)(T-t)}) \right) \\
&\quad \times \exp\left(\frac{\sigma_X^2}{4\alpha_X(1-\beta_1)}(1 - e^{-2\alpha_X(1-\beta_1)(T-t)}) \right).
\end{aligned}$$

Forward price formula

- To compute $\mathbb{E}_{Q_{\bar{\theta},\bar{\beta}}}[\exp(Y(T))|\mathcal{F}_t]$ is more difficult because L is not a Lévy process under $Q_{\bar{\theta},\bar{\beta}}$.
- The predictable semimartingale characteristics of Y,with respect to $h(x)=x$, can be written as an affine function of Y, i.e,

$$\begin{aligned} & (B^Y(t), C^Y(t), \nu^Y(dt,dz)) \\ = \ & (\int_0^t (\beta_0+\beta_1 Y(s-))\,ds, \int_0^t (\gamma_0+\gamma_1 Y(s-))\,ds, \\ & (\varphi_0(dz)+\varphi_1(dz)Y(t-))\,dt), \end{aligned}$$

where

$$(\beta_0,\gamma_0,\varphi_0(dz)) = (\kappa_L'(\theta_2), 0, \mathbf{1}_{(0,\infty)}e^{\theta_2 z}\ell(dz))$$

$$(\beta_1,\gamma_1,\varphi_1(dz)) = (-\alpha_Y(1-\beta_2), 0, \frac{\alpha_Y\beta_2}{\kappa_L''(\theta_2)}\mathbf{1}_{(0,\infty)}ze^{\theta_2 z}\ell(dz)),$$

- Hence, Y is an affine $Q_{\bar{\theta},\bar{\beta}}$-semimartingale.

Forward price formula

- Associated to the previous characteristics we have the following Lévy exponents

$$
\begin{aligned}
\Lambda_0^{\theta_2,\beta_2}(u) &= \kappa_L'(\theta_2)u + \int_0^\infty (e^{uz} - 1 - uz)e^{\theta_2 z}\ell(dz) \\
&= \int_0^\infty (e^{uz} - 1)e^{\theta_2 z}\ell(dz) = \kappa_L(u+\theta_2) - \kappa_L(\theta_2), \\
\Lambda_1^{\theta_2,\beta_2}(u) &= -\alpha_Y(1-\beta_2)u + \frac{\alpha_Y\beta_2}{\kappa_L''(\theta_2)} \int_0^\infty (e^{uz} - 1 - uz)ze^{\theta_2 z}\ell(dz) \\
&= -\alpha_Y u + \frac{\alpha_Y\beta_2}{\kappa_L''(\theta_2)} \int_0^\infty (e^{uz} - 1)ze^{\theta_2 z}\ell(dz) \\
&= -\alpha_Y u + \frac{\alpha_Y\beta_2}{\kappa_L''(\theta_2)} \left(\kappa_L'(u+\theta_2) - \kappa_L'(\theta_2)\right).
\end{aligned}
$$

Forward price formula

Theorem

Let $\beta_2 \in [0,1], \theta_2 \in D_L^g \triangleq (-\infty, (\Theta_L - 1) \wedge (\Theta_L/2))$. *Assume* $\Theta_L > 1$, *that* $\Psi^0_{\theta_2,\beta_2}, \Psi^1_{\theta_2,\beta_2} \in C^1([0,T],\mathbb{R})$ *satisfy the ODE*

$$\begin{aligned} \frac{d}{dt}\Psi^1_{\theta_2,\beta_2}(t) &= \Lambda_1^{\theta_2,\beta_2}(\Psi^1_{\theta_2,\beta_2}(t)), \qquad \Psi^1_{\theta_2,\beta_2}(0) = 1, \\ \frac{d}{dt}\Psi^0_{\theta_2,\beta_2}(t) &= \Lambda_0^{\theta_2,\beta_2}(\Psi^1_{\theta_2,\beta_2}(t)), \qquad \Psi^0_{\theta_2,\beta_2}(0) = 0, \end{aligned} \tag{1}$$

and that the integrability condition $\kappa_L''(\theta_2 + \sup_{t\in[0,T]} \Psi^1_{\theta_2,\beta_2}(t)) < \infty$, *holds. Then,*

$$\mathbb{E}_{Q_{\bar{\theta},\bar{\beta}}}[\exp(Y(T))|\mathcal{F}_t] = \exp\left(Y(t)\Psi^1_{\theta_2,\beta_2}(T-t) + \Psi^0_{\theta_2,\beta_2}(T-t)\right).$$

- The ODEs (1) are called generalised Riccati equations.
- The proof follows by applying a result by **Kallsen and Muhle-Karbe** (2010).

Forward price formula

Theorem

Under the hypothesis of the previous theorem, the forward price $F_{Q_{\bar{\theta},\bar{\beta}}}(t,T)$ in the geometric spot model is given by

$$
\begin{aligned}
F_{Q_{\bar{\theta},\bar{\beta}}}(t,T) = {} & \Lambda_g(T) \exp\left(X(t) e^{-\alpha_X (1-\beta_1)(T-t)} \right) \\
& \times \exp\left(\frac{\theta_1}{\alpha_X(1-\beta_1)} (1 - e^{-\alpha_X(1-\beta_1)(T-t)}) \right) \\
& \times \exp\left(\frac{\sigma_X^2}{4\alpha_X(1-\beta_1)} (1 - e^{-2\alpha_X(1-\beta_1)(T-t)}) \right) \\
& \times \exp(Y(t)\Psi^1_{\theta_2,\beta_2}(T-t) + \Psi^0_{\theta_2,\beta_2}(T-t)).
\end{aligned}
$$

Risk premium formula

Theorem

The risk premium for the forward price $R^F_{g,Q_{\bar\theta,\bar\beta}}(t,T)$ *is given by*

$$
\begin{aligned}
R^F_{g,Q_{\bar\theta,\bar\beta}}(t,T) = \mathbb{E}_P[S(T)|\mathcal{F}_t]\{&\exp(X(t)e^{-\alpha_X(T-t)}(e^{\alpha_X\beta_1(T-t)}-1)) \\
&\times \exp(Y(t)(\Psi^1_{\theta_2,\beta_2}(T-t)-e^{-\alpha_Y(T-t)})) \\
&\times \exp\left(\frac{\theta_1}{\alpha_X(1-\beta_1)}(1-e^{-\alpha_X(1-\beta_1)(T-t)})\right) \\
&\times \exp\left(\frac{\sigma_X^2}{4\alpha_X(1-\beta_1)}(1-e^{-2\alpha_X(1-\beta_1)(T-t)})\right) \\
&\times \exp\left(-\frac{\sigma_X^2}{4\alpha_X}(1-e^{-2\alpha_X(T-t)})\right) \\
&\times \exp\left(\Psi^0_{\theta_2,\beta_2}(T-t)-\int_0^{T-t}\int_0^\infty(e^{ze^{-\alpha_Y s}}-1)\ell(dz)ds\right) -
\end{aligned}
$$

Study of the generalised Riccati equation

Theorem

Assume that $\Theta_L > 1$. *For any* $\delta > 0$, *the system of ODEs* (1) *with* $\beta_2 \in (0,1)$ *and*

$$\theta_2 \in D_L^g(\delta) \triangleq (-\infty, (\Theta_L - 1 - \delta) \wedge (\Theta_L/2))$$

admits a unique local solution $\Psi^0_{\theta_2,\beta_2}(t)$ *and* $\Psi^1_{\theta_2,\beta_2}(t)$. *In addition, let* $u^*(\theta_2,\beta_2)$ *be the unique strictly positive solution of the following equation*

$$u = \frac{\beta_2}{\kappa_L''(\theta_2)} \left(\kappa_L'(u+\theta_2) - \kappa_L'(\theta_2)\right). \tag{2}$$

The behaviour of $\Psi^0_{\theta_2,\beta_2}(t)$ *and* $\Psi^1_{\theta_2,\beta_2}(t)$ *is characterised as follows:*

Study of the generalised Riccati equation

Theorem (*Continue*)

- *If* $u^*(\theta_2, \beta_2) > 1$, *then* $\Psi^0_{\theta_2,\beta_2}(t)$ *and* $\Psi^1_{\theta_2,\beta_2}(t)$ *are globally defined, satisfy*

$$0 < \Psi^1_{\theta_2,\beta_2}(t) \leq 1, \qquad 0 \leq \Psi^0_{\theta_2,\beta_2}(t) \leq \int_0^\infty \Lambda_0^{\theta_2,\beta_2}(\Psi^1_{\theta_2,\beta_2}(s))ds < \infty,$$

  and

$$\lim_{t\to\infty} \frac{1}{t} \log(\Psi^1_{\theta_2,\beta_2}(t)) = -\alpha_Y(1-\beta_2),$$
$$\lim_{t\to\infty} \Psi^0_{\theta_2,\beta_2}(t) = \int_0^\infty \Lambda_0^{\theta_2,\beta_2}(\Psi^1_{\theta_2,\beta_2}(s))ds < \infty.$$

- *If* $u^*(\theta_2, \beta_2) = 1$, *then* $\Psi^1_{\theta_2,\beta_2}(t) \equiv 1$ *and* $\Psi^0_{\theta_2,\beta_2}(t) = \{\kappa_L(1+\theta_2) - \kappa_L(\theta_2)\}t$.

Study of the generalised Riccati equation

Theorem (*Continue*)

- *If* $u^*(\theta_2, \beta_2) < 1$, *then* $\Psi^0_{\theta_2,\beta_2}(t)$ *and* $\Psi^1_{\theta_2,\beta_2}(t)$ *can be defined on* $[0, t_\infty)$, *where* $0 < t_\infty = \int_1^{\Theta_L - \theta_2} (\Lambda_1^{\theta_2,\beta_2}(u))^{-1} du < \infty$. *In addition,*

$$\lim_{t \uparrow t_\infty} \Psi^1_{\theta_2,\beta_2}(t) = \Theta_L - \theta_2, \quad \lim_{t \uparrow t_\infty} \Psi^0_{\theta_2,\beta_2}(t) = \int_0^{t_\infty} \Lambda_0^{\theta_2,\beta_2}(\Psi^1_{\theta_2,\beta_2}(s))ds.$$

Corollary

Under the hypothesis of the previous theorem and for $\theta_2 \in D_L^g(\delta)$ *fixed, a sufficient condition for* $u^*(\theta_2, \beta_2) > 1$ *is that*

$$\beta_2 < \frac{\kappa_L''(\theta_2)}{\kappa_L'(1+\theta_2) - \kappa_L'(\theta_2)}.$$

Examples: CPP with positive jumps of size one

- In this case $\ell(dz) = \delta_{\{1\}}, \Theta_L = \infty$. We have that $\kappa_L(\theta_2) = e^{\theta_2} - 1$ and $\kappa_L^{(n)}(\theta_2) = e^{\theta_2}, n \in \mathbb{N}$. Therefore,

$$\Lambda_0^{\theta_2,\beta_2}(u) = e^{u+\theta_2} - e^{\theta_2},$$
$$\Lambda_1^{\theta_2,\beta_2}(u) = -\alpha_Y u + \alpha_Y \beta_2(e^u - 1).$$

First, we have to solve

$$\frac{d}{dt}\Psi^1_{\theta_2,\beta_2}(t) = -\alpha_Y \Psi^1_{\theta_2,\beta_2}(t) + \alpha_Y \beta_2 (e^{\Psi^1_{\theta_2,\beta_2}(t)} - 1),$$
$$\Psi^1_{\theta_2,\beta_2}(0) = 1.$$

and then integrate $\Lambda_0^{\theta_2,\beta_2}(\Psi^1_{\theta_2,\beta_2}(s))$ from 0 to t.

Examples: CPP with positive jumps of size one

- The equation for $u^*(\theta_2, \beta_2)$ reads

$$u = \frac{\beta_2}{e^{\theta_2}} \left(e^{u+\theta_2} - e^{\theta_2} \right) = \beta_2 (e^u - 1),$$

which can only be solved numerically.

- Applying the previous Corollary, we can guarantee that $\Psi^1_{\theta_2,\beta_2}(t)$ converges to zero if

$$\beta_2 < \frac{\kappa_L''(\theta_2)}{\kappa_L'(1+\theta_2) - \kappa_L'(\theta_2)} = \frac{e^{\theta_2}}{e^{1+\theta_2} - e^{\theta_2}} = (e-1)^{-1}.$$

Examples: CPP with exponential jump sizes

- In this case $\ell(dz) = ce^{-\lambda z}\mathbf{1}_{(0,\infty)}(z)$ and $\Theta_L = \lambda$. We have that $\kappa_L(\theta_2) = \frac{c\theta_2}{\lambda(\lambda-\theta_2)}$, $\kappa_L^{(n)}(\theta_2) = \frac{cn!}{(\lambda-\theta_2)^{n+1}}$, $n \in \mathbb{N}$ and

$$\Lambda_0^{\theta_2,\beta_2}(u) = \frac{c(u+\theta_2)}{\lambda(\lambda-\theta_2-u)} - \frac{c\theta_2}{\lambda(\lambda-\theta_2)},$$

$$\Lambda_1^{\theta_2,\beta_2}(u) = -\alpha_Y u + \frac{\alpha_Y\beta_2(\lambda-\theta_2)^3}{2}\left\{\frac{1}{(\lambda-\theta_2-u)^2} - \frac{1}{(\lambda-\theta_2)^2}\right\}.$$

- Hence, we have to solve

$$\frac{d}{dt}\Psi^1_{\theta_2,\beta_2}(t) = -\alpha_Y\Psi^1_{\theta_2,\beta_2}(t) + \frac{\alpha_Y\beta_2(\lambda-\theta_2)^3}{2}\left\{\frac{1}{(\lambda-\theta_2-\Psi^1_{\theta_2,\beta_2}(t))^2} - \frac{1}{(\lambda-\theta_2)^2}\right\},$$

$$\Psi^1_{\theta_2,\beta_2}(0) = 1,$$

and then integrate $\Lambda_0^{\theta_2,\beta_2}(\Psi^1_{\theta_2,\beta_2}(s))$ from 0 to t.

Examples: CPP with exponential jump sizes

- The equation for $u^*(\theta_2, \beta_2)$ reads

$$u = \beta_2 \frac{(\lambda - \theta_2)^3}{2} \left(\frac{1}{(\lambda - \theta_2 - u)^2} - \frac{1}{(\lambda - \theta_2)^2} \right),$$

which has roots $u_0 = 0$ and

$$u_\pm = \frac{\lambda - \theta_2}{4} \left(4 - \beta_2 \pm \sqrt{\beta_2^2 + 8\beta_2} \right).$$

- We are just interested in the root $u_- \in (0, \lambda - \theta_2)$, note that $u_+ > \lambda - \theta_2$. The inequality $\lambda - \theta_2 > u_- > 1$ yields

$$0 < \beta_2 < 2 \frac{(\lambda - \theta_2 - 1)^2}{(\lambda - \theta_2)(2(\lambda - \theta_2) - 1)}.$$

Analysis of the risk premium

Lemma

If $\alpha_X < \alpha_Y$, we have that the sign of the risk premium $R^F_{g,Q}(t,\tau)$ will be the same as the sign of

$$\Sigma(t,\tau) \triangleq X(t)e^{-\alpha_X\tau}(e^{\alpha_X\beta_1\tau}-1) + Y(t)(\Psi^1_{\theta_2,\beta_2}(\tau) - \Psi^1_{0,0}(\tau)) + \frac{\theta_1}{\alpha_X(1-\beta_1)}(1-e^{-\alpha_X(1-\beta_1)\tau}) + \frac{\sigma_X^2}{4\alpha_X}\Lambda(2\alpha_X\tau, 1-\beta_2) + \Psi^0_{\theta_2,\beta_2}(\tau) - \Psi^0_{0,0}(\tau),$$

where

$$\Lambda(x,y) = \frac{1-e^{-xy}}{y} - (1-e^{-x}).$$

Analysis of the risk premium

Lemma (*Continue*)

In addition,

$$\lim_{\tau\to\infty}\Sigma(t,\tau) = \frac{\theta_1}{\alpha_X(1-\beta_1)} + \frac{\sigma_X^2}{4\alpha_X}\frac{\beta_1}{1-\beta_1} + \int_0^\infty \kappa_L(\Psi^1_{\theta_2,\beta_2}(t)+\theta_2) - \kappa_L(\theta_2) - \kappa_L(e^{-\alpha_Y t})dt,$$

$$\lim_{\tau\to 0}\frac{\partial}{\partial\tau}\Sigma(t,\tau) = X(t)\alpha_X\beta_1 + Y(t)\alpha_Y\beta_2\frac{\kappa_L'(1+\theta_2)-\kappa_L'(\theta_2)}{\kappa_L''(\theta_2)} + \theta_1 + \kappa_L(1+\theta_2) - \kappa_L(\theta_2) - \kappa_L(1).$$

- Analysis of possible risk profiles in **Benth and O.-L.** (2013).

Conclusions and future research

Conclusions

- A new change of measure is introduced.
- It allows to modify simultaneously the level and the speed of mean reversion in a OU process driven by a Brownian motion and/or a Lévy subordinator.
- We have found relatively explicit formulae for forward prices in the associated arithmetic and geometric models.
- We have discussed the possible risk premium profiles in those models.

Future research

- Estimation procedures for the parameters of the OU processes.
- Calibration procedures for the parameters of the measure change.
- Extend the pricing measure to other commodity models:
  - CARMA processes.
  - BNS stochastic volatility models.

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