Density Analysis of BSDEs

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Joint work in progress with Dylan Possamaï and Anthony Réveillac

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Financial market model:

- $W := (W_t)_{t\in[0,T]}$ a Brownian motion defined on the probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$
- Risk-free asset $S^0 := (S^0_t)_{t\in[0,T]}$,

$$dS^0_t = S^0_t r\, dt$$

- Asset $S := (S_t)_{t\in[0,T]}$,

$$dS_t = S_t\Big(\theta_t\, dt + dW_t\Big),$$

where θ is predictable and bounded.

Investing strategy ($r = 0$): $(x, (\Pi_t)_t)$ such that the associated wealth process denoted $(X_t^{x,\Pi})_t$ and defined for all $t \in [0, T]$ by:

$$X_t^{x,\Pi} := x + \int_0^t \Pi_u \frac{dS_u}{S_u} = x + \int_0^t \Pi_u (dW_u + \theta_u du).$$

Motivation: pricing and hedging problems in finance

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- Utility function $U(x) := -e^{-\alpha x}$
- Utility maximisation problem:

$$V(x) := \sup_{\Pi \in \mathcal{A}} \mathbb{E}[U(X_T^{x,\Pi} - F)],$$

where F is a $\mathcal{F}_T$ measurable variable (the liability of the investor).

Motivation: pricing and hedging problems in finance

Hu, Imkeller and Müller have showed that it can be reduced to solve a BSDE (Backward Stochastic Differential Equation) of the form:

$$Y_t = F + \int_t^T h(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad Y_T = F$$

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The value is given by $V(x) = -e^{-\alpha(x - Y_0)}$.

Optimal strategies are characterized by Z_t.

Problem

If we are in the Markovian case, we consider the Forward BSDE:

$$\begin{cases} X_t &= X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \\ Y_t &= g(X_T) + \int_t^T h(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T] \end{cases}$$

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Idea: Get the existence of densities for the Y process and for the Z process with estimates of these densities.

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- The Malliavin derivative DF of F is the $\mathcal{H}$-valued random variable defined as:

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- We denote by $\mathbb{D}^{1,2}$ the closure of $\mathcal{C}$ with respect to the Sobolev norm $\|\cdot\|_{1,2}$ defined as:

$$\|F\|_{1,2} := \mathbb{E}[|F|^2] + \mathbb{E}\left[\int_0^T |D_t F|^2 dt\right].$$

Theorem (Bouleau-Hirsch)

Assume that $\|DF\|_{L^2([0,T])} > 0$ *a.s., then* F *has a probability distribution which is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}$*, denoted by* ρ_F*.*

Malliavin calculus and densities estimates (F centered)

Assume that $DF = \Phi_F(W)$ where $\Phi_F : \mathbb{R}^{\mathcal{H}} \to \mathcal{H}$. We set:

$$g_F(x) = \int_0^{+\infty} e^{-u}\mathbb{E}\left[\mathbb{E}^*[\langle \Phi_F(W), \widetilde{\Phi_F^u}(W)\rangle_{L^2([0,T])}]|F = x\right] du$$

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- Where $\widetilde{\Phi_F^u}(W) := \Phi_F(e^{-u}W + \sqrt{1-e^{-2u}}W^*)$
- With W^* an independent copy of W defined on a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$
- Where $\mathbb{E}^*$ is the expectation under $\mathbb{P}^*$.

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Theorem (Nourdin-Viens)

F has a density ρ_F with respect to the Lebesgue measure if and only if the random variable $g_F(F)$ is positive a.s.. In this case, the support of ρ_F is a closed interval of $\mathbb{R}$ and for all $x \in supp(\rho_F)$:

$$\rho_F(x) = \frac{\mathbb{E}(|F|)}{2g_F(x)} \exp\left(-\int_0^x \frac{udu}{g_F(u)}\right).$$

We make the classical assumption:

L) $h : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ is Lipschitz in (x, y, z) with Lipschitz constants respectively k_x, k_y, k_z, *i.e.* for all $(x, x', y, y', z, z') \in \mathbb{R}^6$:

$$|h(x, y, z) - h(x', y', z')| \leq k_x|x - x'| + k_y|y - y'| + k_z|z - z'|.$$

Theorem (Antonelli-Kohatsu Higa (2005))

Assume that L) holds (plus some conditions on the coefficients b, σ, g and h). We set $K := k_b + k_y + k_\sigma k_z$. Let $t \in (0, T]$. If for some $A \subset \mathbb{R}$ such that $\mathbb{P}(X_T \in A | X_t) > 0$:

$$\begin{cases} \underline{g} e^{-sgn(\underline{g})KT} + \underline{h}(t) \int_t^T e^{-sgn(\underline{h}(s))Ks} ds & \geq 0 \\ \underline{g}^A e^{-sgn(\underline{g}^A)KT} + \underline{h}(t) \int_t^T e^{-sgn(\underline{h}(s))Ks} ds & > 0 \end{cases} \quad (1)$$

or

$$\begin{cases} \overline{g} e^{-sgn(\overline{g})KT} + \overline{h}(t) \int_t^T e^{-sgn(\overline{h}(s))Ks} ds & \leq 0 \\ \overline{g}^A e^{-sgn(\overline{g}^A)KT} + \overline{h}(t) \int_t^T e^{-sgn(\overline{h}(s))Ks} ds & < 0, \end{cases} \quad (2)$$

is met, then Y_t has a probability distribution that is absolutely continuous with respect to the Lebesgue measure.

- These conditions are sufficient but not necessary. As shown in the following example.

Densities existence for BSDEs: previous results

- These conditions are sufficient but not necessary. As shown in the following example.
- Example (M., Possamaï, Réveillac).

  Let $T = 1$, $g(x) = x$, $X = W$, $h(s, x, y, z) = (s - 2)x$.

  (2) is not satisfied for any $t \in [0, T]$.

  (1) is not satisfied for any $t \in [0, \frac{3-\sqrt{5}}{2})$.

  However for all $t \in [0, 1]$:

$$\begin{aligned} Y_t &= \mathbb{E}\left[W_1 + \int_t^1 (s-2) W_s ds \Big| \mathcal{F}_t \right] \\ &= W_t(1 + \int_t^1 (s-2) ds) = W_t(-\frac{1}{2} + 2t - \frac{t^2}{2}), \end{aligned}$$

  admits a density with respect to the Lebesgue measure except when $t = 2 - \sqrt{3}$.

Densities existence for BSDEs: previous results

Antonelli and Kohatsu-Higa have proved an other theorem with upper order conditions on h when it does not depend on z. Let:

$$\tilde{g}(x) := g'(x) + (T-t)h_x(T, x, g(x)),$$

$$\tilde{h}(s,x,y,z) := -\Big(h_{xt} - hh_{xy} + \frac{1}{2}(h_{xxx} + 2zh_{xxy} + z^2 h_{xxy}) \\ + h_y h_x + \sigma_x h_{xx} + z\sigma_x h_{xy})\Big)(s,x,y).$$

Theorem (Antonelli-Kohatsu Higa)

Assume that h does not depend on the variable z and suppose that L) holds (plus some conditions on the coefficients). Let $t \in (0, T]$. If for some $A \in \mathbb{R}$ such that $\mathbb{P}(X_T \in A | X_t) > 0$:

$$\begin{cases} \underline{\tilde{g}} e^{-sgn(\underline{\tilde{g}})KT} + \underline{\tilde{h}}(t) \int_t^T e^{-sgn(\underline{\tilde{h}}(s))Ks}(T-s)ds & \geq 0 \\ \underline{\tilde{g}}^A e^{-sgn(\underline{\tilde{g}}^A)KT} + \underline{\tilde{h}}(t) \int_t^T e^{-sgn(\underline{\tilde{h}}(s))Ks}(T-s)ds & > 0 \end{cases}$$

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We study now the existence of a density for Y and Gaussian estimates of this density in the general case.

$$\begin{cases} H1 : \text{For all } \theta \leq T, g \in \mathcal{C}_b^1(\mathbb{R}),\ 0 < c \leq g'(X_T)D_\theta X_T \leq C, a.s. \\ H2 : 0 \leq h_x \leq C \\ H3 : 0 \leq \sigma \leq C \text{ and } |[b,\sigma]| \leq M\sigma \end{cases}$$

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Theorem (Aboura-Bourguin (2012))

Under the above assumptions H1),H2) and H3), Y_t has a density for $t \in (0,T)$ denoted by ρ_{Y_t} satisfying:

$$\frac{\mathbb{E}[|Y_t - \mathbb{E}[Y_t]|]}{2ct} \exp\left(-\frac{(y-\mathbb{E}[Y_t])^2}{2Ct}\right) \leq \rho_{Y_t}(y)$$

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Densities existence: our contribution for Y

We study the quadratic case under the following assumption:

Q) $h : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ such that for all $(t,x,y,z) \in ([0,T] \times \mathbb{R})$: $|h(t,x,y,z)| \leq K(1+|y|+|z|^2)$ for some $K > 0$.

Theorem (M., Possamaï, Réveillac)

Assume that Q) holds with some conditions on the coefficients (but not on the sign of DX_T). Fix $t \in (0,T]$. If for some $A \subset \mathbb{R}$ such that $\mathbb{P}(X_T \in A | X_t) > 0$, $g' \geq 0$, $g'_A > 0$ and $\underline{h}(t) \geq 0$ (resp. $g' \leq 0, g'_A < 0$ and $\overline{h}(t) \leq 0$), then Y_t has a probability distribution which is absolutely continuous with Lebesgue measure.

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Notice that in this theorem we do not need a sign for DX_T.

$\hookrightarrow$ In the proof we just need to control the norm of DX_T.

Densities existence: our contribution for Y

Example Consider the BSDE:

$$Y_t = W_T + \int_t^T \frac{1}{2}|Z_s|^2 ds - \int_t^T Z_s dW_s.$$

Then:

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Indeed, by the uniqueness of the solution to this BSDE: $Y_t = W_t + \frac{1}{2}(T - t)$, $Z_t = 1$ and Y_t admits a density for all $t \in (0, T]$.

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Density existence for the Z process ?

Densities existence: previous results for Z

$\hookrightarrow$ Aboura and Bourguin proved that Z_t admits a density under convexity and growth conditions for the terminal condition g and for the generator h when $h(x, y, z) = \tilde{f}(x, y) + \alpha z$ where α is constant.

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- $\hookrightarrow$ They have Gaussian estimates of this density when $h \in \mathcal{C}_b^2(\mathbb{R})$ and $g \in \mathcal{C}_b^2(\mathbb{R})$.
- $\hookrightarrow$ Using the fact that Z_t can be represented by the Clark-Ocone formula and after, taking the Malliavin derivative of Z_t.

Densities existence: our contribution for Z

We consider the following FBSDE:

$$\begin{cases} X_t & = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \\ Y_t & = g(X_T) + \int_t^T (\tilde{f}(s, X_s, Y_s) + h(Z_s))ds - \int_t^T Z_s dW_s. \end{cases}$$

Theorem (M., Possamaï, Réveillac)

Assume that Q) holds with some conditions which ensure that $DX_T > 0$ and $D^2 X_T \geq 0$ and assume that $\tilde{f}_x, \tilde{f}_{xx}, \tilde{f}_{xy}, \tilde{f}_{yy} \geq 0$. Then, if there exists $A \subset \mathbb{R}$ such that $\mathbb{P}(X_T \in A | X_t) > 0$, $g' \geq 0$, $g'' \geq 0$, $g''_A > 0$ and $h'' \geq 0$ then, for all $t \in (0, T]$, Z_t has a density with respect to the Lebesgue measure.

Densities existence: our contribution for Z

Assume now that there exists a function $f \in \mathcal{C}^2(\mathbb{R})$ such that for all $t \in [0, T]$: $X_t = f(t, W_t)$.

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Under this assumption, for all $0 \leq r, s \leq t \leq T$: $D_r Y_t = D_s Y_t$ and $D_r Z_t = D_s Z_t$, $\mathbb{P}$-a.s..

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To simplify assume that $\tilde{f} \equiv 0$ (the generator of the BSDE depends only on z through h).

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To simplify assume that $\tilde{f} \equiv 0$ (the generator of the BSDE depends only on z through h).

Theorem (M., Possamaï, Réveillac)

Assume that Q) and conditions on coefficients hold. Assume that $h'' \geq 0$ and $(g \circ f)'' \geq 0$. Then, if there exists $A \subset \mathbb{R}$ such that $\mathbb{P}(X_T \in A | X_t) > 0$ and $(g \circ f)''_A > 0$, then for all $t \in (0, T]$ Z_t has a density with respect to the Lebesgue measure.

Densities estimates: linear Feynman-Kac's formula

$$\begin{cases} \partial_t v(t,x) + b(t,x)\cdot Dv(t,x) + \frac{1}{2} Tr.[\sigma\sigma^T(t,x)D^2 v(t,x)] = 0 \\ v(T,\cdot) = g(\cdot). \end{cases}$$

$$\text{"}\Leftrightarrow\text{"}$$

$$\begin{cases} dX_s^{t,x} = b(s,X_s^{t,x})ds + \sigma(s,X_s^{t,x})dW_s \\ X_t^{t,x} = x. \end{cases}$$

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$$v(t,x) = \mathbb{E}[g(X_T^{t,x})] = P_{t,T} g(x), \quad (v \in \mathcal{C}^{1,2})$$

Densities estimates: semi-linear Feyman-Kac's formula

$$\begin{cases} \partial_t v(t,x) + b(t,x)\cdot Dv(t,x) + \frac{1}{2}Tr.[\sigma\sigma^T(t,x)D^2v(t,x)] = h(t,\cdot,v,\sigma^T\cdot Dv) \\ v(T,\cdot) = g(\cdot). \end{cases}$$

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$$v(t,x) = Y_t^{t,x}, \quad (v \in \mathcal{C}^{1,2})$$

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$$\begin{cases} \partial_t v(t,x) + b(t,x)\cdot Dv(t,x) + \frac{1}{2} Tr.[\sigma\sigma^T(t,x) D^2 v(t,x)] = h(t,\cdot,v,\sigma^T\cdot Dv) \\ v(T,\cdot) = g(\cdot). \end{cases}$$

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$$\begin{cases} dX_s^{t,x} = b(s,X_s^{t,x})ds + \sigma(s,X_s^{t,x})dW_s; \quad X_t^{t,x} = x. \\ dY_s^{t,x} = h(t,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})ds - Z_s^{t,x}dW_s; \quad Y_T^{t,x} = g(X_T^{t,x}). \end{cases}$$

$$v(t,x) = Y_t^{t,x}, \quad (v \in \mathcal{C}^{1,2})$$

Rk.: $h \equiv 0 \Longrightarrow Y_s^{t,x} = \mathbb{E}[g(X_T^{t,x})|\mathcal{F}_s]$.

Densities estimates

Let $u(t, W_t) := v(t, X_t)$ where $X_t =: f(t, W_t)$. Then, $u(t, W_t) := Y_t$ and $u'(t, W_t) = v'(t, X_t)f'(t, W_t) := Z_t$.

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Theorem (M., Possamaï, Réveillac)

Assume Q) and P) hold. Suppose that there exists $\overline{\delta} > 0$ such that $(g \circ f)'' \geq \overline{\delta} > 0$ and $h'' \geq 0$. Then, there exist $C > 0$, $\delta > 0$ and $\gamma \in (0,1)$ such that for all $t \in (0, T]$ the probability distribution of Z_t has a law which admits a density ρ_{Z_t} such that for all $z \in \mathbb{R}$:

$$\frac{\mathbb{E}[|Z_t - \mathbb{E}[Z_t]|]}{2tC(1+|z|^{2\gamma})} e^{\frac{-z^2}{2t\delta^2}} \leq \rho_{Z_t}(z) \leq \frac{\mathbb{E}[|Z_t - \mathbb{E}[Z_t]|]}{2t\delta^2} e^{-\frac{1}{2t\delta^2}\int_0^{z-\mathbb{E}[Z_t]} \frac{xdx}{1+|x|^{2\gamma}}}$$

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A same result holds for Y_t (we just use the first derivative of u).