

# ASYMPTOTIC FORMULAS FOR THE IMPLIED VOLATILITY AT EXTREME STRIKES

A minicourse at Barcelona University, June 2014.

Archil Gulisashvili, Ohio University

The course consists of two lectures (1.5 hours each). It will be explained in the lectures how the implied volatility (the smile) behaves at large strikes. We will discuss and prove several known results describing the smile asymptotics. These results include Lee's moment formulas, the tail-wing formula of Benaim and Friz, and the asymptotic formulas with error estimates due to the author. The presentation of the material in the course follows that in Chapters 9 and 10 of the book "A. Gulisashvili, *Analytically Tractable Stochastic Stock Price Models*, Springer Finance, 2012".

## LECTURE 1

*What will be explained in the lecture:*

Call Pricing Functions, the Black-Scholes Model, Implied Volatility,

---

A Model-Free Asymptotic Formula.

---

The implied volatility was first introduced under the name "the implied standard deviation" in the paper "H. A. Latané and R. J. Rendleman, Standard deviations of stock price ratios implied in option prices, *Journal of Finance* 31 (1976), pp. 369-381". Latané and Rendleman studied standard deviations of asset returns, which are implied in actual call option prices when investors price options according to the Black-Scholes model. For a general model of call option prices, the implied volatility can be obtained by inverting the Black-Scholes call pricing function with respect to the volatility variable and composing the resulting inverse function with the original call pricing function.

## 1. General Call Pricing Functions.

---

We model the random behavior of the asset price by an adapted positive stochastic process  $X$  defined on a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}^*)$ . It is assumed that the following conditions are satisfied:

- The interest rate  $r$  is a nonnegative constant.
- The process  $X$  starts at  $x_0 > 0$ .
- The process  $X$  is integrable. This means that  $\mathbb{E}^* [X_t] < \infty$  for every  $t \geq 0$ .
- $\mathbb{P}^*$  is a risk-neutral measure. More precisely, the discounted stock price process  $\{e^{-rt}X_t\}_{t \geq 0}$  is an  $(\{\mathcal{F}_t\}, \mathbb{P}^*)$ -martingale.

In the model described above, the asset price distributions are modeled by the marginal distributions of the process  $X$  with respect to the probability measure  $\mathbb{P}^*$ . Note that the integrability condition for  $X$  implies the existence of asset price moments only for the orders between zero and one, while the martingality condition for the discounted asset price process leads to fair pricing formulas for European call and put options.

Let  $X$  be an asset price process under a risk-neutral measure  $\mathbb{P}^*$ . For every real number  $u$  set  $u^+ = \max\{u, 0\}$ . An European style call option on the underlying asset, with strike price  $K$  and maturity  $T$ , is a special contract, which gives its holder the right, but not the obligation, to buy one unit of the asset from the seller of the option, for the price  $K$  on the date  $T$ . The price that the buyer of the option pays for the contract is called the option premium. Note that call options can be exercised only on the expiration date. Buying options is less risky than buying units of underlying asset, because the holder of the option has no obligation to exercise it if things go wrong.

In a risk-neutral environment, a natural way to price an European style option is to choose the expected value of the discounted payoff of the option at maturity to be the option premium. For a call option, this payoff is given by  $(X_T - K)^+$ . Combining the premiums for all maturities and strikes, we obtain the so-called pricing function associated with the option.

**Definition 1.** *The European call option pricing function  $C$  in a stochastic asset price model is defined as follows:*

$$C(T, K) = e^{-rT} \mathbb{E}^* \left[ (X_T - K)^+ \right], \quad T \geq 0, \quad K \geq 0.$$

## 2. The Black-Scholes Call Pricing Function.

---

For every strike  $K > 0$  and maturity  $T$ ,

$$C_{BS}(T, K) = \frac{x_0}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-\frac{y^2}{2}} dy - \frac{Ke^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-\frac{y^2}{2}} dy,$$

where

$$d_1 = \frac{\log x_0 - \log K + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}$$

and

$$d_2 = \frac{\log x_0 - \log K + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}.$$

## 3. Implied Volatility.

---

**Definition 2.** Let  $C$  be a call pricing function. For  $(T, K) \in (0, \infty)^2$ , the implied volatility  $I(T, K)$  associated with  $C$  is the value of the volatility  $\sigma$  in the Black-Scholes model for which  $C(T, K) = C_{BS}(T, K, \sigma)$ . The implied volatility  $I(T, K)$  is defined only if such a number  $\sigma$  exists and is unique.

In the next definition, we introduce a special class of call pricing functions.

**Definition 3.** The class  $PF_\infty$  consists of all call pricing functions  $C$ , for which one of the following equivalent conditions holds:

- (1)  $C(T, K) > 0$  for all  $T > 0$  and  $K > 0$  with  $x_0 e^{rT} \leq K$ .
- (2) For every  $T > 0$  and all  $a > 0$  the random variable  $X_T$  is such that  $\mathbb{P}^*[X_T < a] < 1$ .

**Remark 4.** Suppose the maturity  $T > 0$  is fixed, and consider the pricing function  $C$  and the implied volatility  $I$  as functions of the strike price  $K$ . If  $C \in PF_\infty$ , then the implied volatility  $I(K)$  is defined for large values of  $K$ . This allows to study the asymptotic behavior of the implied volatility as  $K \rightarrow \infty$ .

#### 4. A Model-Free Asymptotic Formula for the Implied Volatility.

---

The following assertion was obtained in “A. Gulisashvili, Asymptotic formulas with error estimates for call pricing functions and the implied volatility at extreme strikes, *SIAM Journal on Financial Mathematics* 1 (2010), 609-641”.

**Theorem 5.** For any call pricing function  $C \in PF_\infty$ ,

$$I(K) = \frac{\sqrt{2}}{\sqrt{T}} \left[ \sqrt{\log K + \log \frac{1}{C(K)}} - \sqrt{\log \frac{1}{C(K)}} \right] \\ + O \left( \left( \log \frac{1}{C(K)} \right)^{-\frac{1}{2}} \log \log \frac{1}{C(K)} \right)$$

as  $K \rightarrow \infty$ .

We call the formula in the previous theorem a model-free asymptotic formula for the implied volatility with an error estimate.

## LECTURE 2

*What will be explained in the lecture:*

Lee’s Moment Formula for the Implied Volatility at Large Strikes,

---

Tail-Wing Formulas Due to Benaim and Friz,

---

Special Models.

---

### 1. Lee’s Moment Formula.

---

The moment formulas obtained by R. Lee are arguably the first model-free formulas for the implied volatility at extreme strikes. In this lecture, we will show how to derive Lee’s moment formula for large strikes from the model-free formula for the implied volatility with an error estimate discussed in the first lecture. Lee’s formula is a certain relation between the implied volatility and the order of the first exploding moment of the stock price.

**Definition 6** (Moments of random variables). Let  $X$  be a nonnegative random variable on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . The moment  $m_p(X)$  of order  $p \in \mathbb{R}$  of the random variable  $X$  is defined as follows:

$$m_p(X) = \mathbb{E}^* [X^p].$$

The next assertion was established in “R. Lee, The moment formula for implied volatility at extreme strikes, *Mathematical Finance* 14 (2004), 469-480”.

**Theorem 7.** Let  $C$  be a call pricing function, and let  $I$  be the implied volatility associated with  $C$ . Fix  $T > 0$ , and define the number  $\tilde{p}$  by

$$\tilde{p} = \sup \{p \geq 0 : m_{1+p}(X_T) < \infty\}.$$

Then the following equality holds:

$$\limsup_{K \rightarrow \infty} \frac{TI(K)^2}{\log K} = \psi(\tilde{p})$$

where the function  $\psi$  is given by

$$\psi(u) = 2 - 4 \left( \sqrt{u^2 + u} - u \right), \quad u \geq 0.$$

Lee’s formula identifies the leading term in the asymptotic expansion of the implied volatility as  $K \rightarrow \infty$ . It shows that for models with moment explosions, the implied volatility  $K \mapsto I(K)$  behaves at infinity like a constant multiple of the function  $K \mapsto \sqrt{\log K}$ .

## 2. Tail-Wing Formulas.

---

Tail-wing formulas characterize the asymptotics of the implied volatility at extreme strikes (the wing asymptotics) in terms of the tail behavior of the stock price density. We will next formulate and discuss the tail-wing formulas established in “S. Benaim and P. Friz, Regular variation and smile asymptotics, *Mathematical Finance* 19 (2009), 1-12”.

We will denote by  $\bar{F}$  the complementary cumulative distribution function of the stock price  $X_T$ , and by  $D$  the distribution density of  $X_T$  (if this density exists).

**Definition 8** (Regular variation). A positive function  $f$  on  $[a, \infty)$ , where  $a > 0$ , belongs to the class  $R_\alpha$  with  $\alpha \in \mathbb{R}$  if for every  $\lambda > 0$ ,

$$\lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha.$$

Functions from the class  $R_\alpha$  are called regularly varying with index  $\alpha$ .

Functions of regular variation play an important role in the work of Benaim and Friz.

**Theorem 9** (Benaim-Friz). *Let  $C$  be a call pricing function, and suppose the stock price  $X_T$  satisfies the condition*

$$m_{1+\varepsilon}(X_T) < \infty \quad \text{for some } \varepsilon > 0.$$

Then the following are true:

- (1) If  $C(K) = \exp\{-\eta(\log K)\}$  with  $\eta \in R_\alpha$ ,  $\alpha > 0$ , then

$$I(K) \sim \frac{\sqrt{\log K}}{\sqrt{T}} \sqrt{\psi\left(-\frac{\log C(K)}{\log K}\right)} \quad \text{as } K \rightarrow \infty.$$

- (2) If  $\bar{F}(y) = \exp\{-\rho(\log y)\}$  with  $\rho \in R_\alpha$ ,  $\alpha > 0$ , then

$$I(K) \sim \frac{\sqrt{\log K}}{\sqrt{T}} \sqrt{\psi\left(-\frac{\log[K\bar{F}(K)]}{\log K}\right)} \quad \text{as } K \rightarrow \infty.$$

- (3) If the distribution  $\mu_T$  of the stock price  $X_T$  admits a density  $D$  and if

$$D(x) = \frac{1}{x} \exp\{-h(\log x)\}$$

as  $x \rightarrow \infty$ , where  $h \in R_\alpha$ ,  $\alpha > 0$ , then

$$I(K) \sim \frac{\sqrt{\log K}}{\sqrt{T}} \sqrt{\psi\left(-\frac{\log[K^2D(K)]}{\log K}\right)} \quad \text{as } K \rightarrow \infty.$$

The tail-wing formulas can be derived from the model-free formula for the implied volatility with an error estimate. We will also explain how to obtain a little stronger formulas than those established by Benaim and Friz using similar ideas.

### 3. Special Models.

---

The model-free formulas discussed in the lectures will be applied to popular option pricing models with stochastic volatility. It will be explained how the implied volatility behaves in the case of the Stein-Stein model and the Heston model.