# Euler-Poisson schemes for Lévy processes

**A. Ferreiro-Castilla**

School of Mathematical Sciences, QMUL

# Motivation

- Lévy process. A (one dimensional) process with stationary and independent increments which has paths which are right continuous with left limits. (incl. Brownian motion with drift, compound Poisson processes, stable processes amongst many others.)
- A popular (and often criticised) model in mathematical finance for the evolution of a risky asset is

$$S_t := e^{X_t},\ t \geq 0$$

  where $\{X_t : t \geq 0\}$ is a Lévy process. (other applications: queuing theory, genetics, population models, random fluctuations with jump, also used in insurance risk models!)
- Of interest in several fields are quantities involving the first passage time (American options), the overshoot (debt at ruin in actuarial models), the undershoot (nature of the ruin event), the supremum (Barrier options), etc ... (Computing these quantities accurately is notoriously hard for general Lévy processes.)

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  (Computing these quantities accurately is notoriously hard for general Lévy processes.)
- The first approach to compute a path quantity for a stochastic process is to produce a skeleton approximating the path of the stochastic process at an equally spaced grid points (Euler scheme), missing the possible excursions of the process. (Even for a BM, where the Euler scheme is exact computing the supremum in this way leads to a significant bias)[1]

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# Original WH scheme: Kuznetsov et al. (2011)[2]

(Produces a path skeleton for Lévy processes which captures the value of the process and its supremum)

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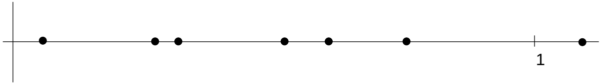
- Consider a Poisson process with arrival rate *n*. Denote by $\tau_1, \tau_2, \cdots$ the arrival times.

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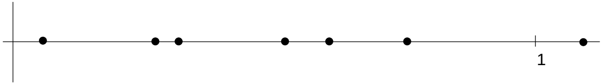


- Note that $\tau_n$ is the sum of $n$ i.i.d exponential random variables, each with mean $1/n$. We could therefore write

$$\tau_n = \sum_{i=1}^{n} \frac{1}{n}\mathcal{E}_i \overset{d}{=} \mathbf{g}(n, n),$$

where $\mathcal{E}_i$ are i.i.d. exponential random variables with unit mean. Hence by SLLN

$$\tau_n \to 1 \qquad \text{almost surely.}$$

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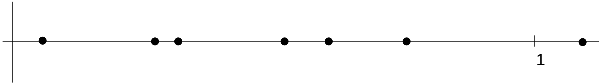


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$$\tau_n \to 1 \qquad \text{almost surely.}$$

- Hence for a suitably large $n$, we have in distribution

$$(X_{\tau_n}, \overline{X}_{\tau_n}) \simeq (X_1, \overline{X}_1).$$

Since 1 is not a jump time $(X_{\tau_n}, \overline{X}_{\tau_n}) \to (X_1, \overline{X}_1)$ a.s.

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## Wiener-Hopf factorization

- A formulation of the Wiener-Hopf factorization states that

$$X_{\mathcal{E}(q)} \stackrel{d}{=} S^{(q)} + I^{(q)}$$

where $S^{(q)}$ is independent of $I^{(q)}$ and they are respectively equal in distribution to $\overline{X}_{\mathcal{E}(q)}$ and $\underline{X}_{\mathcal{E}(q)}$; $\mathcal{E}(q)$ is an exponential distribution with mean $q^{-1}$ and independent of $X$.

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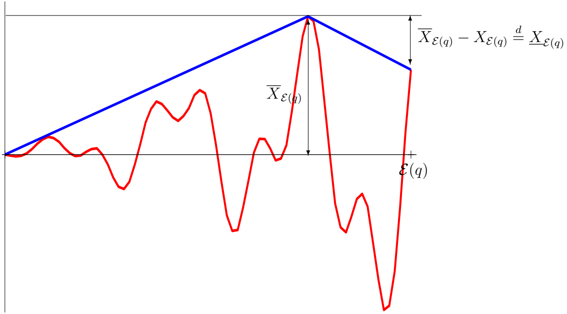

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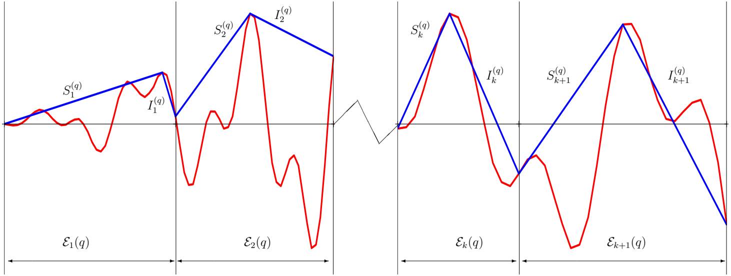

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- For many years only a few number of Lévy process with explicit Wiener-Hopf factors were known. Some of these included stable processes, subclasses of spectrally one sided processes and a handful more.

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- Very recent results have incorporated a new family to this class: Meromorphic Lévy processes.
  - Let $X$ be a Lévy process, then it is characterized by

$$\log \mathbb{E}[e^{\mathrm{i}\theta X_t}] = -t\psi(\theta) = -t\left(iA\theta + \frac{1}{2}\Sigma^2\theta^2 + \int_{\mathbb{R}}\left(1 - e^{i\theta x} + i\theta x \mathbf{1}_{\{|x|<1\}}\right)\nu(\mathrm{d}x)\right),$$

  where $(A, \Sigma, \nu)$ are called the Lévy triplet and $\psi$ the Lévy exponent

$$\nu(\{0\}) = 0\,, \qquad \int_{\mathbb{R}}(1 \wedge x^2)\nu(\mathrm{d}x) < \infty\,.$$

  - A Lévy process $X$ is said to belong to the meromorphic class (M-class) if its Lévy measure $\nu$ decomposes as $\nu^+(x) = \nu((x, \infty))$, $\nu^-((-\infty, -x))$ for $x > 0$ and $\nu^+, \nu^-$ are discrete completely monotone functions.

$$f(x) = \int_{[0,\infty)} e^{-xz}\mu(\mathrm{d}z), \quad x > 0\,.$$

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- The previous scheme has proved to be extremely flexible.
  - Many popular models in finance can be approximated by members of the M-class keeping the desired stylized features (VG, Meixner, CGMY, etc ...)
  - The original scheme can be improved and modified easily (multilevel Monte Carlo schemes, discrete schemes for Lévy driven SDEs, compute other quantities than the supremum, etc ... )

(Ask me later if you want more detail)

# Heuristics behind Euler-Poisson schemes (I)

The Wiener-Hopf scheme should be a good scheme for path dependent quantities

[3] Doney, R. A. (2004) Stochastic bounds for Lévy processes. *Ann. Probab.* **32**(2), 1545–1552.

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- [3]Let $\{X_{\tau_k}\}_{k\geq 0}$, then the random variables

$$M_k := \sup_{\tau_k \leq t < \tau_{k+1}} X_t \quad \text{and} \quad m_k := \inf_{\tau_k \leq t < \tau_{k+1}} X_t$$

can be written as

$$M_k = S_0^{(n)} + Y_k^{(+)} \quad \text{and} \quad m_k = I_0^{(n)} + Y_k^{(-)} ,$$

where $\{Y_k^{(+)}\}_{k\geq 0}$ and $\{Y_k^{(-)}\}_{k\geq 0}$ are random walks with the same distribution as $\{X_{\tau_k}\}_{k\geq 0}$ and independent of $S_0^{(n)}$ and $I_0^{(n)}$ respectively.

Stochastic bound/sausage.

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# Heuristics behind Euler-Poisson schemes (II)

- [4] [Feynman-Kac] Under *some conditions* if $u(t,x)$ is a classical solution of the PIDE

$$\frac{\partial}{\partial t}u(t,x) = \mathcal{A}_X u(t,x) \quad \text{and} \quad u(0,x) = f(x)\ , \tag{1}$$

then $u(1-t,x) = \mathbb{E}[f(X_1)|X_t = x] := \mathbb{E}_x[f(X_{1-t})]$. The converse also holds with *stronger assumptions*.

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- Consider the Laplace-Carlson transform, $\mathcal{L}$, of $u(t,x)$, that is
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- Then

$$\frac{\mathcal{L}[u](x) - f(x)}{T/n} = \mathcal{A}_Y \mathcal{L}[u](x) \,.$$

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- Then
$$\frac{\mathcal{L}[u](x) - f(x)}{T/n} = \mathcal{A}_Y\mathcal{L}[u](x)\ .$$
- Consider the discretization of (1) given by
$$\frac{u_i(x) - u_{i-1}(x)}{T/n} = \mathcal{A}_X u_i(x)\ ,$$
for $i = 1, \ldots, n$ with $u_0(x) = f(x)$. Then, for all $i = 1, \ldots, n$,
$$u_i(x) = \mathbb{E}_x[f(X_{\tau_i})]\ .$$

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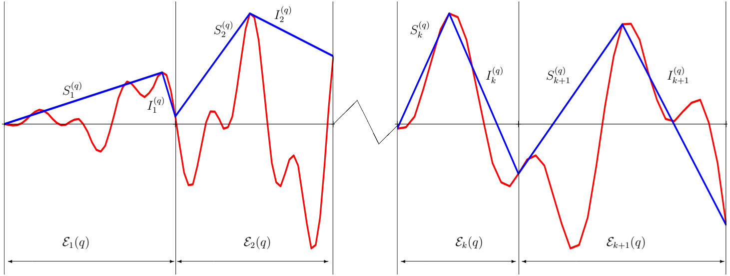
# Some applications (I): Other fluctuation quantities Actuarial science

Consider the following random variables:

$$V_r^{(q)} := V_{r-1}^{(q)} + (S_j^{(q)} + I_j^{(q)}) \qquad \text{and} \qquad J_r^{(q)} := J_{r-1}^{(q)} \bigvee (V_{r-1}^{(q)} + S_r^{(q)})$$



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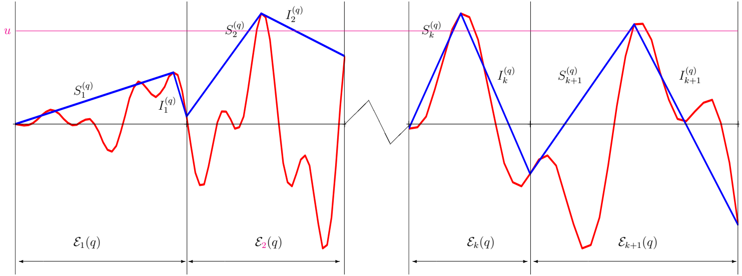

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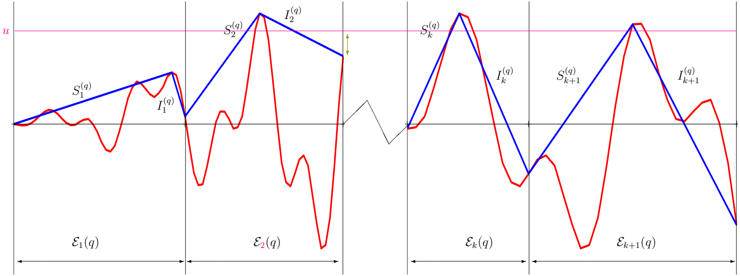

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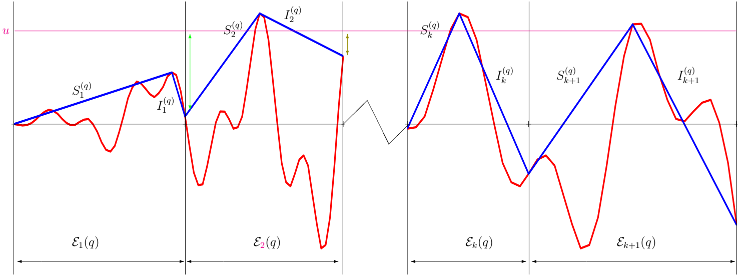

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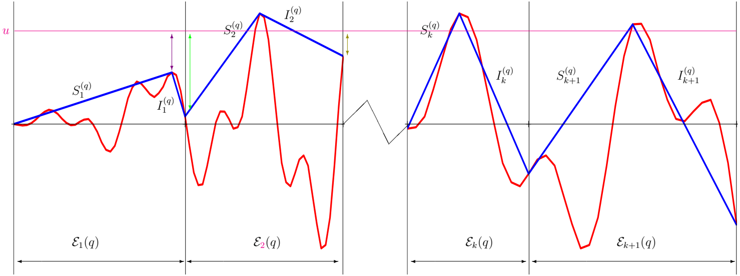
- Last maximum before the crossing $u - J_{\kappa_u^{(n)}-1}^{(n/t)}$



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Under the previous notation we have the following result:

## Theorem

Let $X$ be any Lévy process. Fix some $t > 0$ and $u > 0$. Set for all $n \in \mathbb{N}$

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(where as usual we understand $\inf \emptyset = \infty$). Then we have as $n \to \infty$

$$\left(\frac{t}{n}(\kappa_u^{(n)} \wedge n), V^{(n/t)}_{\kappa_u^{(n)} \wedge n} - u, u - V^{(n/t)}_{(\kappa_u^{(n)}-1)\wedge n}, u - J^{(n/t)}_{(\kappa_u^{(n)}-1)\wedge n}\right) \xrightarrow{d} \left(\tau_u \wedge t, X_{\tau_u \wedge t} - u, u - X_{(\tau_u \wedge t)-}, u - \overline{X}_{(\tau_u \wedge t)-}\right) .$$

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## Theorem

Using the same notation, we have

$$\mathbb{E}\left[\left(\frac{t}{n}(\kappa_u^{(n)} \wedge n) - \tau \wedge t\right)^2\right] \leq \frac{2t^2}{n}$$

# Some applications (II): Numerical schemes for Lévy driven SDEs

Let $Y := \{Y_t\}_{t\in[0,T]}$ be the solution of the stochastic differential equation

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The Euler-Poisson scheme is then given by the discrete Markov chain $\widetilde{Y} := \{\widetilde{Y}_{\tau_i}\}_{i\geq 0}$ defined recursively by

$$\widetilde{Y}_{\tau_i} := \widetilde{Y}_{\tau_{i-1}} + a(\widetilde{Y}_{\tau_{i-1}})\Delta X_{\mathcal{E}_i(n/T)} \qquad \text{for} \quad i \geq 1 \quad \text{and} \quad \widetilde{Y}_0 := y_0 ,$$

where $\Delta X_{\mathcal{E}_i} := X_{\mathcal{E}_i(n/T)} - X_{\mathcal{E}_{i-1}(n/T)} \overset{d}{=} X_{\mathcal{E}(n/T)}$ and $\tau_i := \sum_{j=0}^{i} \mathcal{E}_j(n/T)$ .

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$$Z_t = y_0 + \int_0^t a(Z_{s-}, \overline{X}_{s-})\mathrm{d}X_s \qquad \text{or} \qquad Z_t = y_0 + \int_0^t a(Z_{s-}, \underline{X}_{s-})\mathrm{d}X_s \, .$$

Chemical reactions, stochastic population models, ...

# Some applications (II): ctd.

## Theorem

Let $a := \mathbb{R}^{d_Y} \to \mathbb{R}^{d_Y} \otimes \mathbb{R}^{d_X}$ be a measurable function such that

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for $x, x' \in \mathbb{R}^{d_Y}$ and $k' \in \mathbb{R}^+$. And the driving process $X$ (Lévy–Khinchine) satisfies

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- Hitting: what happens in $[T, \tau_n]$
  - Centered moments of a Gamma distribution: $\tau_n \stackrel{d}{=} \mathbf{g}(n, n/t)$

## Some applications (III): Fast computation of derivatives Pricing barrier options

E.g. pricing a Barrier option by a Monte Carlo method:

$$\mathbb{E}[f(X_t)\mathbb{I}_{\{\overline{X}_t > b\}}] =: \mathbb{E}[F(X_t, \overline{X}_t)] \approx \mathbb{E}[F(X_{\tau_n}, \overline{X}_{\tau_n})] := \mathbb{E}[F^n]$$

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- Multilevel Monte Carlo estimator: $(M_L = n)$

$$\begin{aligned} \widehat{F}^{n_0,L,\{M_\ell\}}_{ML} &:= \frac{1}{M_0}\sum_{i=1}^{M_0} F^{n_0,(i)} + \sum_{\ell=1}^{L}\frac{1}{M_\ell}\sum_{i=1}^{M_\ell}(F^{n_\ell,(i)} - F^{n_{\ell-1},(i)}). \\ e(\widehat{F}^{n_0,L,\{M_\ell\}}_{ML})^2 &= \frac{1}{M_0}\mathbb{V}(F^{n_0}) + \sum_{\ell=1}^{L}\frac{1}{M_\ell}\mathbb{V}(F^{n_\ell} - F^{n_{\ell-1}}) + \left(\mathbb{E}[F^n - F(X_1, \overline{X}_1)]\right)^2. \end{aligned}$$

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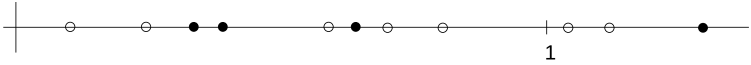
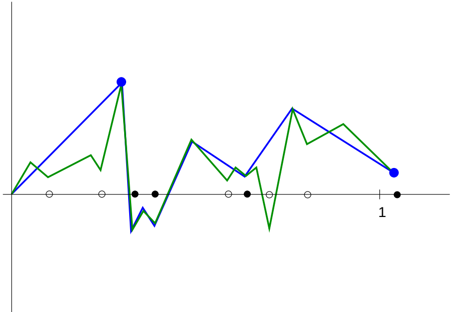
- In the WHMC method how do we introduce ”levels”?
- It is **crucial** to have a Poisson process for the time randomisations on **all** levels! How do we sample on two consecutive levels?
- Suppose the ”level $\ell$” grid is based on a Poisson process of rate $n_\ell$. Then by tossing a coin (thinning) and rejecting arrivals with probability $1/2$ we end up with a Poisson process of rate $n_{\ell-1}$: our new coarser ”level $\ell - 1$” Poisson grid.

(**Not** a new idea! Also used by **[Glasserman, Merener, 2003], [Giles, Xia, 2012], . . .**)



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Assume $\exists \alpha, \beta > 0$ with $\alpha \geq \frac{1}{2}\min\{\beta, 1\}$ such that

(i) $|\mathbb{E}[F^{n_\ell} - F(X_t, \overline{X}_t)]| \lesssim n_\ell^{-\alpha}$

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(iii) $\mathbb{E}[\mathcal{C}_{n_\ell}] \lesssim n_\ell$.

Then, $\forall \nu \in \mathbb{N}$ $\exists L$ and $\{M_\ell\}_{\ell=0}^{L}$ s.t. $\mathbb{E}\left[\mathcal{C}\left(\widehat{F}_{\mathrm{ML}}^{n_0, L, \{M_\ell\}}\right)\right] \lesssim \nu$ and $L^2$ error

$$\text{R.M.S.E.} = e\left(\widehat{F}_{\mathrm{ML}}^{n_0, L, \{M_\ell\}}\right) \lesssim \begin{cases} \nu^{-\frac{1}{2}} & \text{if } \beta > 1, \\ \nu^{-\frac{1}{2}} \log^2 \nu & \text{if } \beta = 1, \\ \nu^{-\frac{1}{2+(1-\beta)/\alpha}} & \text{if } \beta < 1. \end{cases}$$

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- An important feature of multilevel schemes is that they give a regime of optimal performance.
- The $M$-class satisfies Assumption (iii).
- We can prove that $\alpha = 1/4$ & $\beta = 1/2 \ \Rightarrow\ \mathcal{O}(\nu^{-\frac{1}{4}})$.

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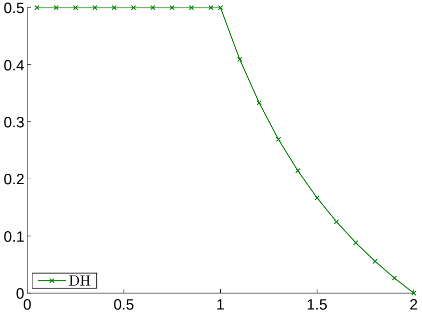

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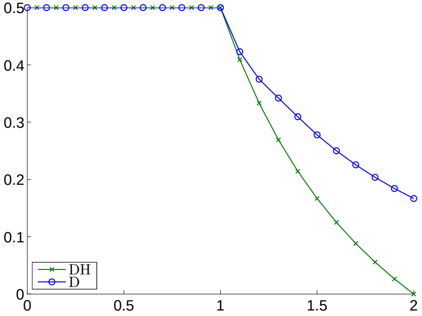

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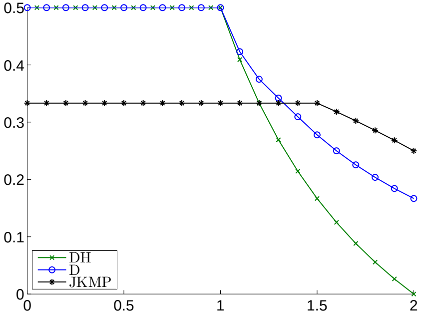

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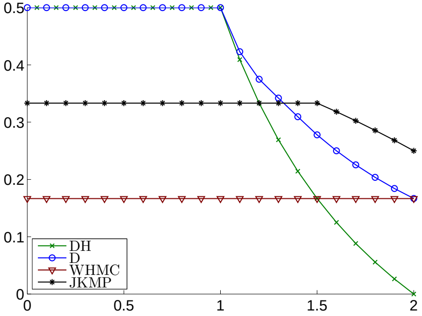

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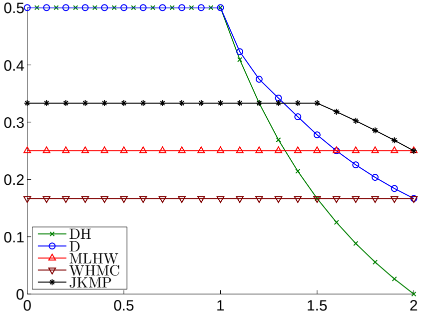

# Talk based in the papers:


- F-C, A., Kyprianou, A.E., Scheichl, R. and Suryanarayana, G. (2014) Multilevel Monte Carlo simulation for Lévy processes based on the Wiener-Hopf factorization. *Stoch. Proc. Appl.*, 124 **(2)**, 985–1010.
- F-C, A., Kyprianou, A.E. and Scheichl, R. (2013) Euler-Poisson scheme for Lévy driven SDEs. *Preprint*
- F-C, A., and van Schaik, K. (2014) Applying the Wiener-Hopf Monte Carlo simulation technique for Lévy processes to path functionals. *J. Appl. Probab.* (To appear)


Thanks!

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Equivalent definition:

(i) The characteristic exponent $\Psi(z)$ is a meromorphic function which has poles at points $\{-\mathrm{i}\rho_n, \mathrm{i}\hat{\rho}_n\}_{n\geq 1}$, where $\rho_n$ and $\hat{\rho}_n$ are positive real numbers.

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$$... - \rho_2 < -\zeta_2 < -\rho_1 < -\zeta_1 < 0 < \hat{\zeta}_1 < \hat{\rho}_1 < \hat{\zeta}_2 < \hat{\rho}_2 < ...$$

(iv) The Wiener-Hopf factors are expressed as convergent infinite products,

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- For $x \geq 0$

$$\mathbb{P}(\overline{X}_{\mathcal{E}_q} \in \mathrm{d}x) = \mathrm{a}_0(\rho,\zeta)\delta_0(\mathrm{d}x) + \sum_{n=1}^{\infty} \mathrm{a}_n(\rho,\zeta)\zeta_n \mathrm{e}^{-\zeta_n x}\mathrm{d}x$$

- Here

$$\mathrm{a}_0(\rho,\zeta) = \lim_{n\to+\infty} \prod_{k=1}^{n} \frac{\zeta_k}{\rho_k}, \quad \mathrm{a}_n(\rho,\zeta) = \left(1 - \frac{\zeta_n}{\rho_n}\right) \prod_{\substack{k\geq 1\\ k\neq n}} \frac{1 - \frac{\zeta_n}{\rho_k}}{1 - \frac{\zeta_n}{\zeta_k}}$$

# Feasibility ctd.

Kuznetsov's $\beta$-family

- The corresponding Lévy measure $\nu$ has density

$$\pi(x) = c_1 \frac{e^{-\alpha_1 \beta_1 x}}{(1 - e^{-\beta_1 x})^{\lambda_1}} \mathbf{1}_{\{x>0\}} + c_2 \frac{e^{\alpha_2 \beta_2 x}}{(1 - e^{\beta_2 x})^{\lambda_2}} \mathbf{1}_{\{x<0\}}.$$

  The $\beta$-class of Lévy processes includes another recently introduced family of Lévy processes known as Lamperti-stable processes.