On the closed-form approximation of short-time random strike options

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"Two-day Workshop on Finance and Stochastics", 26-27 November, 2013, Barcelona

The problem: Random strike options with payoff of the form:

$$(S_T - K_T)_+$$

In particular: two-assets and three-assets spread options

$$K_t = S_t' + K$$

$$K_t = S_t^1 + S_t^2 + K$$

Motivation examples:

$$K_t = S_t' + K$$

Price at another time plus the storage cost (**calendar spread option**)

Price at another end location plus transportation cost (**geographical spread option**)

$$K_t = S_t^1 + S_t^2 + K$$

Fuel price

Emission price

Production cost

State of the art

Two-assets spread options, K=0

(Margrabe formula, exact)

$$a^2 := \sigma^2 - 2\rho\sigma\sigma' + (\sigma')^2 .$$

Two-assets spread options, general case

(Kirk's formula, approximation)

$$a^2 := \sigma^2 - 2\rho\sigma\sigma' \frac{S'_t}{S'_t + K} + (\sigma')^2 \frac{(S'_t)^2}{(S'_t + K)^2} .$$

(here r=0 for simplicity)

Three-assets spread options
(Alòs, Eydeland and Laurence, approximation)

$$
\begin{aligned}
a^2 \quad &= \quad \sigma^2 - 2\rho_{1,3}\sigma\sigma_1 \frac{S_t^1}{S_t^1 + S_t^2 + K} - 2\tilde{\rho}_{2,3}\sigma\sigma_2 \frac{S_t^2}{S_t^1 + S_t^2 + K} \\
&\quad + \frac{\left(S_t^1\sigma_1\right)^2}{\left(S_t^1 + S_t^2 + K\right)^2} + \frac{2\rho_{1,2}\left(S_t^1\sigma_1\right)\left(S_t^2\sigma_2\right)}{\left(S_t^1 + S_t^2 + K\right)^2} + \frac{\left(S_t^2\sigma_2\right)^2}{\left(S_t^1 + S_t^2 + K\right)^2}
\end{aligned}
$$

(here r=0 for simplicity)

Some problems

The above formulas are constant volatility approximations, that are extremely simple and very accurate… except for very high positive correlations

T1. Moderately high positive correlations *Source: authors*

Correlations are $(\rho_{12}, \rho_{1,3}, \rho_{23}) = (0.95, 0.80, 0.70)$. Volatilities are $\{\sigma_1, \sigma_2, \sigma_3\} = \{0.5, 0.45, 0.2\}$ and times to expiration are in the first column. $\{F_0^1, F_0^2, F_0^3\} = \{60, 60, 2\}$, $K = 2.5$. Spread is close to, but out-of-the-money

	Exact	Decomp	Error (%)	Laplace	Error (%)
$T = 0.5$	1.142	1.1429	–0.0090	1.1407	–0.0814
$T = 1$	2.16	2.1632	0.1477	2.16339	–0.1569
$T = 2$	3.707	3.7086	0.0449	3.714	–0.0448

(from Alòs, Eydeland and Laurence (2011))

T3. High positive correlations *Source: authors*

High correlations are $\{\rho_{12}, \rho_{1,3}, \rho_{23}\} = \{0.99, 0.96, 0.94\}$. Volatilities are $(\sigma_1, \sigma_2, \sigma_3) = (0.5, 0.45, 2)$ and times to expiration are in the first column. $\{F_0^1, F_0^2, F_0^3\} = \{60, 60, 0.2\}$, $K = 2.5$. Spread is slightly out-of-the-money

	Exact	Decomp	Error (%)	Laplace	Error (%)
$T = 0.5$	0.324	0.3391	4.6709	0.32394	0.01852
$T = 1$	0.8	0.83246	4.05728354	0.8046	–0.5812
$T = 2$	1.615	1.673	3.548	1.629	-0.0867

(from Alòs, Eydeland and Laurence (2011))

Why? And…How can we improve these formulas in a simple way?

The procedure

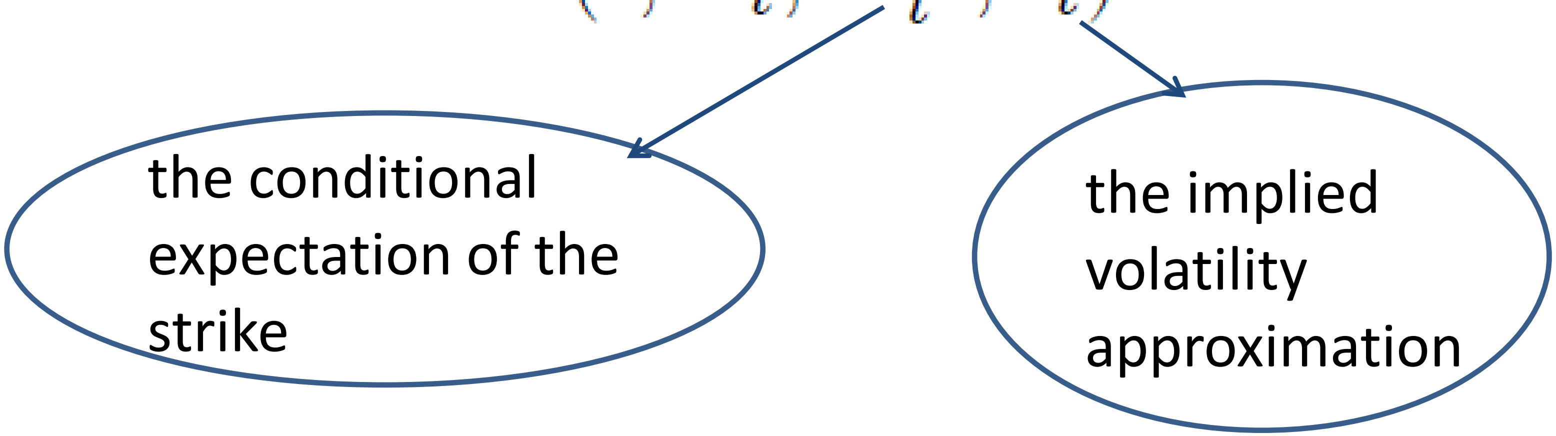
a) We obtain a first approximation formula (where the volatility is constant with respect to the stock price)

a) We study the dependence of the implied volatility with respect to the stock price (the at-the-money slope of the implied volatility)

a) We use the results in b) to correct the results in a) and to obtain a second-order approximation formula

A first-order approximation formula

$$BS(t, X_t, M_t^T, v_t)$$



$v_t := \left(\frac{Y_t}{T-t}\right)^{\frac{1}{2}}$, with $Y_t := \int_t^T a_s^2 ds$, where $a_s^2 ds := \sigma_s^2 ds - 2\frac{d\langle M^T, X\rangle_s}{M_s^T} + \frac{d\langle M^T, M^T\rangle_s}{(M_s^T)^2}$.

$$BS(T, X_T, M_T^T, v_T) = (S_T - K_T)_+$$

The decomposition formula

$$
\begin{aligned}
V_t &= E\left(BS(t, X_t, M_t^T, v_t) \middle| \mathcal{F}_t \right) \\
&+ \frac{1}{2} E\left(\sum_{i=1}^{d} \left\{ \rho_{i,d+1} \int_t^T e^{-r(s-t)} \left(\partial_{xxx}^3 - \partial_{xx}^2 \right) BS(s, X_s, M_s^T, v_s) \sigma_s \Lambda_s^{W^i} ds \right. \right. \\
&\left. \left. + \int_t^T e^{-r(s-t)} \partial_K \left(\partial_{xx}^2 - \partial_x \right) BS(s, X_s, M_s^T, v_s) \Lambda_s^{W^i} m^i(T, s) ds \right\} \middle| \mathcal{F}_t \right)
\end{aligned}
$$

$$
\textit{where } \Lambda_s^{W^i} := \left[D_s^{W^i} \int_s^T a^2(r) dr \right], \; i = 1, ..., d.
$$

Malliavin derivatives

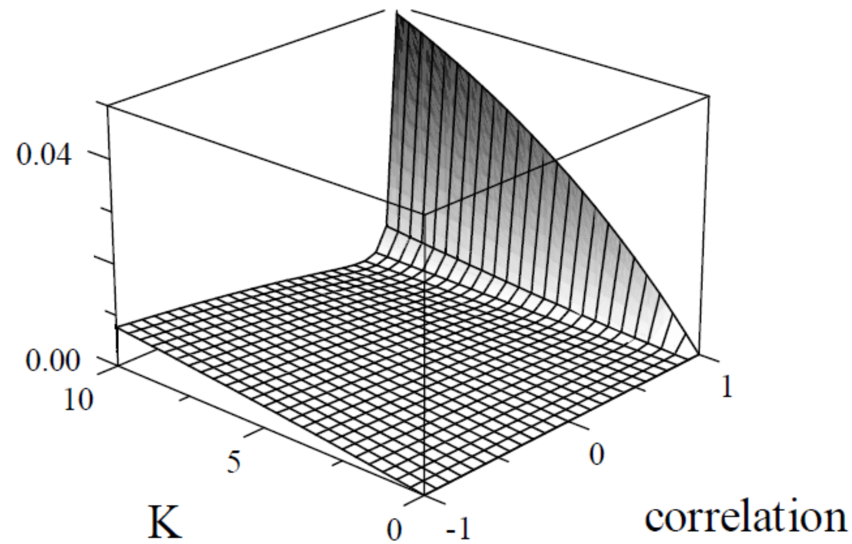
The implied volatility slope

From the decomposition formula we can prove that

$$\lim_{T\to t}\frac{\partial I_t}{\partial X_t}(x_t^*)=\frac{1}{2}\left(\frac{\sum_{i=1}^{d} m^i(t,t)D_t^{W^i+}a_t}{K_t}-\sigma_t\sum_{i=1}^{d}\rho_{i,d+1}D_t^{W^i+}a_t\right)\frac{1}{\tilde{a}_t^2}.$$

Two-assets spread options

(r=0 for simplicity)



$$
\begin{aligned}
&\lim_{T\to t} \frac{\partial I_t}{\partial X_t}(x_t^*) \\
= \quad & \frac{1}{2}\left(\frac{m^1(t,t)}{K_t} - \rho\sigma\right)\frac{D_t^+ a_t}{\tilde{a}_t^2} \\
= \quad & \frac{1}{2}\left(\sigma'\left(\frac{S_t'}{S_t'+K}\right) - \rho\sigma\right)^2 \frac{1}{\left(\sqrt{a_t^2}\right)^3}\left(\sigma'\right)^2 \frac{S_t' K}{\left(S_t'+K\right)^2}.
\end{aligned}
$$

The effect of the correlation

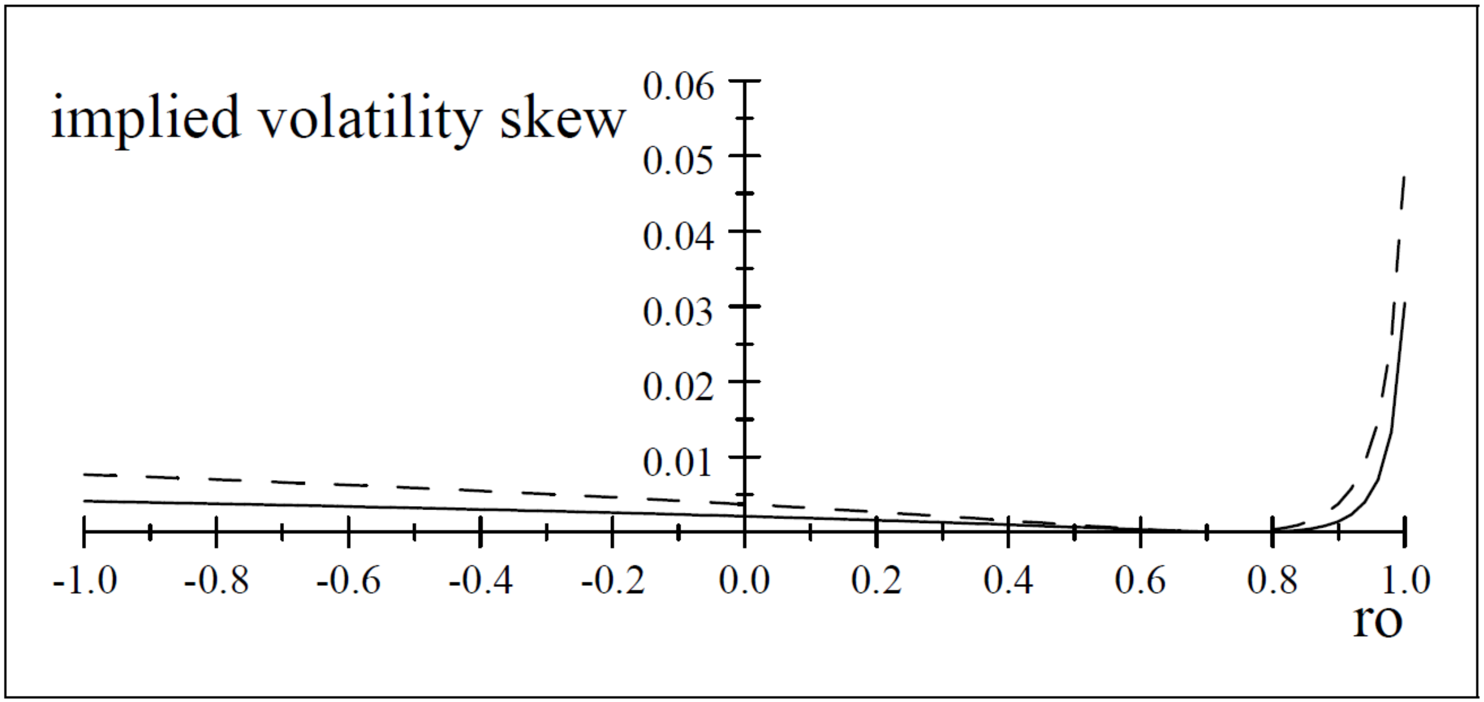


Figure 2: $\lim_{T\to t} \frac{\partial I_t}{\partial X_t}(x_t^*)$ as a function of ρ for $K = 5$ (solid) and $K = 10$ (dash). Here $\sigma = 0.5, \sigma' = 0.4$.

An improvement of Kirk's formula

$$\hat{I}_t(X_t) := \sqrt{a_t^2} + \frac{1}{2}\left(\sigma'\left(\frac{S'_t}{S'_t + K}\right) - \rho\sigma\right)^2 \frac{1}{\left(\sqrt{a_t^2}\right)^3} (\sigma')^2 \frac{S'_t K}{(S'_t + K)^2} (X_t - x_t^*) \,.$$

And now we can consider the modified Kirk approximation given by

$$BS(t, X_t, M_t^T, \hat{I}_t(X_t)).$$

K/ρ		0.60	0.98	0.99	0.999
	Monte-Carlo	9,4564	2,1890	1,8386	1,5011
	Kirk	9,4260	2,2159	1,8775	1,5420
5	error (Kirk)	*−0,321%*	*1,230%*	*2,117%*	*2,725%*
	Modified Kirk	9,4255	2,2067	1,8309	1,4829
	error (Modified Kirk)	*−0,327%*	*0,809%*	*0,804%*	*0,414%*
	Monte-Carlo	7,6404	1,2714	1,0207	0,7934
	Kirk	7,6070	1,3326	1,1015	0,8848
10	error Kirk	*−0,437%*	*4,814%*	*7,913%*	*11,516%*
	Modified Kirk	7,6060	1,2888	1,0367	0,8210
	error Modified Kirk	*−0,451%*	*1,368%*	*1,660%*	*1,400%*

The three-assets case

Example 18 *Take* $T = 0.5, (\rho_{1,2}, \rho_{1,3}, \tilde{\rho}_{2,3}) = (0.99, 0.96, 0.94)$, $(\sigma_1, \sigma_2, \sigma) = (0.5, 0.45, 0.2)$, $\left(S_0^1, S_0^2, K\right) = (50, 2, 1)$. *In the following table we compare the errors given by the extended Kirk's approximation prices obtained in Alòs, Eydeland and Laurence (2011) (AEL) with the modified Alòs, Eydeland and Laurence approximation (MAEL) given by*

$$BS(t, X_t, M_t^T, \hat{I}_t(X_t)).$$

S_0	Monte Carlo	AEL	Error(AEL)	MAEL	Error(MAEL)
48	0.09256	0.00988	6.749 1%	0.009 14	−1.234 2%
50	0.34575	0.355 34	2.773 7%	0.345 97	0.0636%
52	0.93606	0.944 11	0.8600%	0.939 68	0.386 7%

Many thanks!