Functional Limit theorems for the quadratic variation of a continuous time random walk and for certain stochastic integrals

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Outline

1. Introduction

2. FCLT for the quadratic variation of Compound Renewal Processes

3. FCLT for the stochastic integrals driven by a time-changed symmetric α-stable Lévy process

Scaling Limits

Consider a sequence of i.i.d. centered random variables ξ_i. Define the centered random walk:

$$S_n := \sum_{i=1}^{n} \xi_i.$$

(a) How does S_n behave when n is large?

(b) What is the limit after rescaling?

Lévy-Lindeberg Central Limit Theorem (CLT)

Given a sequence of random variables $(\xi_i)_{i\in\mathbb{N}}$ i.i.d. with mean μ and finite, positive variance σ^2, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Then

$$\frac{S_n - n\mu}{\sqrt{n}} \overset{\mathcal{L}}{\Rightarrow} Y, \qquad \textit{with} \qquad Y \sim N(0, \sigma^2).$$

Donsker's Theorem

The classical CLT was generalized to a FCLT by Donsker (1951).

Donsker's Theorem (1951)

Given a sequence of random variables $(\xi_i)_{i\in\mathbb{N}}$ i.i.d. with mean 0 and finite, positive variance σ^2, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.The random functions defined by

$$X_n(t,\omega) := \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor}(\omega) + (nt - \lfloor nt \rfloor) \frac{1}{\sigma\sqrt{n}} \xi_{\lfloor nt \rfloor + 1}(\omega)$$

satisfy that

$$(X_n(t),\ t \in [0, T]) \overset{\mathcal{L}}{\Rightarrow} (B(t),\ t \in [0, T])$$

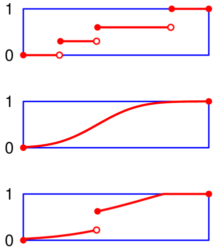
where B is a standard Brownian motion.

The Skorokhod space

The Skorokhod space, denoted by $\mathbb{D} = D([0, T], \mathbb{R})$ (with $T > 0$), is the space of real functions $x : [0, T] \to \mathbb{R}$ that are right-continuous with left limits:

1. For $t \in [0, T)$, $x(t+) = \lim_{s \downarrow t} x(s)$ exists and $x(t+) = x(t)$.
2. For $t \in (0, T]$, $x(t-) = \lim_{s \uparrow t} x(s)$ exists.

Functions satisfying these properties are called càdlàg functions.



Skorokhod topologies

The Skorokhod space provides a natural and convenient formalism for describing the trajectories of stochastic processes with jumps: Poisson process, Lévy processes, martingales and semimartingales, empirical distribution functions, discretizations of stochastic processes, etc.

It can be assigned a topology that, intuitively allows us to *wiggle space and time a bit* (whereas the traditional topology of uniform convergence only allows us to *wiggle space a bit*).

Skorokhod (1965) proposed four metric separable topologies on $\mathbb{D}$, denoted by J_1, J_2, M_1 and M_2.

A. Skorokhod.
Limit Theorems for Stochastic Processes.
Theor. Probability Appl. **1**, *261–290*, 1956.

The Skorokhod J_1-topology

For $T > 0$, let

$$\Lambda := \{\lambda : [0,T] \to [0,T],\ \textit{strictly increasing and continuous}\}.$$

If $\lambda \in \Lambda$, then $\lambda(0) = 0$ and $\lambda(T) = T$.
For $x, y \in \mathbb{D}$, the Skorokhod J_1-metric is

$$d_{J_1}(x,y) := \inf_{\lambda \in \Lambda}\{\sup_{t\in[0,T]} |\lambda(t) - t|, \sup_{t\in[0,T]} |x(t) - y(\lambda(t))|\} \tag{1}$$

Convergence in J_1-topology

The sequence $x_n(t) \in \mathbb{D}$ converges to $x_0(t) \in \mathbb{D}$ in the J_1-topology if there exists a sequence of increasing homeomorphisms $\lambda_n : [0,T] \to [0,T]$ such that

$$\sup_{t\in[0,T]} |\lambda_n(t) - t| \to 0, \sup_{t\in[0,T]} |x_n(\lambda_n(t)) - x_0(t)| \to 0, \tag{2}$$

as $n \to \infty$.

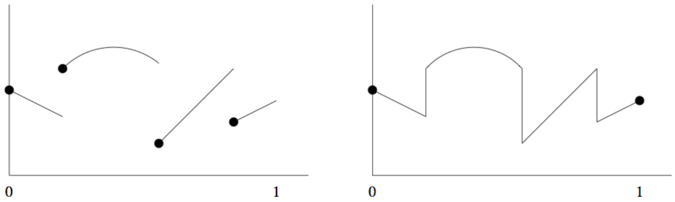
The Skorokhod M_1-topology

We use the M_1-topology in order to be able to establish stochastic process limits with unmatched jumps in the limit process.

We define the M_1-metric using the completed graph of the functions. For $x \in \mathbb{D}$, the completed graph of x is

$$\Gamma_x^{(a)} = \{(t, z) \in [0, T] \times \mathbb{R} : z = ax(t-) + (1-a)x(t) \textit{ for some } a \in [0, 1]\},$$

where $x(t-)$ is the left limit of x at t and $x(0-) := x(0)$.



A function in $D([0, 1], \mathbb{R})$ and its completed graph

The Skorokhod M_1-topology

We define the M_1 metric using the uniform metric defined on parametric representations of the completed graphs of the functions.

A parametric representation of $\Gamma_x^{(a)}$ is a continuous nondecreasing function

$$(r, u) : [0, 1] \to \Gamma_x^{(a)},$$

with r being the time component and u being the spatial component.

Denote $\Pi(x)$ the set of parametric representations of $\Gamma_x^{(a)}$ in $\mathbb{D}$.

For $x_1,\ x_2 \in \mathbb{D}$, the Skorokhod M_1-metric on $\mathbb{D}$ is

$$d_{M_1}(x_1, x_2) := \inf_{\substack{(r_i,u_i)\in\Pi(x_i)\\ i=1,2}} \{\|r_1 - r_2\|_{[0,T]} \vee \|u_1 - u_2\|_{[0,T]}\}. \tag{3}$$

Convergence in M_1-topology

Convergence in M_1-topology

The sequence $x_n(t) \in \mathbb{D}$ converges to $x_0(t) \in \mathbb{D}$ in the M_1-topology if

$$\lim_{n\to+\infty} d_{M_1}(x_n(t), x_0(t)) = 0. \tag{4}$$

In other words, we have the convergence in M_1-topology if there exist parametric representations $(y(s), t(s))$ of the graph $\Gamma_{x_0(t)}$ and $(y_n(s), t_n(s))$ of the graph $\Gamma_{x_n(t)}$ such that

$$\lim_{n\to\infty} \|(y_n, t_n) - (y, t)\|_{[0,T]} = 0. \tag{5}$$

Characterization for the M_1-convergence (Silvestrov(2004))

If the following two conditions are satisfied:

(i) Let A be a dense subset in $[0, +\infty)$ which contains 0.

$$\{X_n(t)\}_{t \in A} \overset{\mathcal{L}}{\Rightarrow} \{X(t)\}_{t \in A} \text{ as } n \to +\infty.$$

(ii) Condition on M_1-compactness:

$$\lim_{\delta \to 0} \limsup_{n \to +\infty} w(X_n, \delta) = 0, \tag{6}$$

where $w(X_n, \delta) := \sup_{t \in A} w(X_n, t, \delta)$, and

$$w(X_n, t, \delta) := \sup_{0 \vee (t-\delta) \leqslant t_1 < t_2 < t_3 \leqslant (t+\delta) \wedge T} \{\| X_n(t_2) - [X_n(t_1), X_n(t_3)] \|\}.$$

Then,

$$\{X_n(t)\}_{t \geqslant 0} \underset{n \to +\infty}{\overset{M_1 - top}{\Rightarrow}} \{X(t)\}_{t \geqslant 0}.$$

Some remarks

For $x, y \in \mathbb{R}$ denote the standard segment as

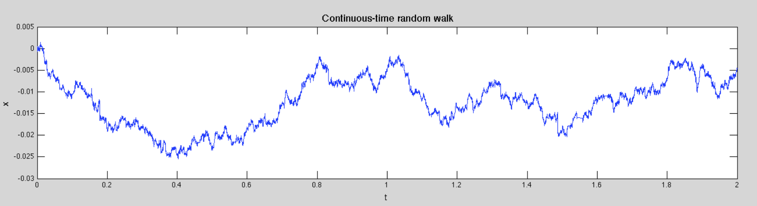
$$[x, y] := \{ax + (1 - a)y,\ a \in [0, 1]\}.$$

The modulus of M_1-compactness plays the same role for càdlàg functions as the modulus of continuity for continuous functions.

D.S Silvestrov.
Limit Theorems for Randomly Stopped Stochastic Processes.
Probability and its Applications. Springer, New York, 2004.

Continuous-Time Random Walks (CTRW)

A continuous time random walk (CTRW) is a pure jump process given by a sum of i.i.d. random jumps $(Y_i)_{i\in\mathbb{N}}$ separated by i.i.d. random waiting times (positive random variables) $(J_i)_{i\in\mathbb{N}}$.



Compound Poisson Process

Let $X_n = \sum_{i=1}^{n} Y_i$ denote the position of a diffusing particle after n jumps and $T_n = \sum_{i=1}^{n} J_i$ be the epoch of the n-th jump.

The corresponding counting process $N(t)$ is defined by

$$N(t) \stackrel{\text{def}}{=} \max\{n : \; T_n \leqslant t\}. \tag{7}$$

Then the position of a particle at time $t > 0$ can be expressed as the sum of the jumps up to time t

$$X(t) = X_{N(t)} \stackrel{\text{def}}{=} \sum_{i=1}^{N(t)} Y_i. \tag{8}$$

It is called compound Poisson process. It is a Markov and Lévy process.

The functional limit is an α-stable Lévy process .

α-stable Lévy processes

A continuous-time process $L = \{L_t\}_{t \geqslant 0}$ with values in $\mathbb{R}$ is called a Lévy process if its sample paths are càdlàg at every time point t, and it has stationary, independent increments, that is:

(a) For all $0 = t_0 < t_1 < \cdots < t_k$, the increments $L_{t_i} - L_{t_{i-1}}$ are independent.

(b) For all $0 \leqslant s \leqslant t$ the random variables $L_t - L_s$ and $L_{t-s} - L_0$ have the same distribution.

An α-stable process is a real-valued Lévy process $L_\alpha = \{L_\alpha(t)\}_{t \geqslant 0}$ with initial value $L_\alpha(0)$ that satisfies the self-similarity property

$$\frac{1}{t^{1/\alpha}} L_\alpha(t) \overset{\mathcal{L}}{=} L_\alpha(1), \quad \forall t > 0.$$

If $\alpha = 2$ then the α-stable Lévy process is the Wiener process.

Compound Fractional Poisson Process

Consider a **CTRW** whose i.i.d. jumps $(Y_i)_{i\in\mathbb{N}}$ have **symmetric** α**-stable distribution** with $\alpha \in (1,2]$, and whose i.i.d waiting times $(J_i)_{i\in\mathbb{N}}$ satisfy

$$\mathbb{P}(J_i > t) = E_\beta(-t^\beta), \tag{9}$$

for $\beta \in (0,1]$, where

$$E_\beta(z) = \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(1+\beta j)},$$

denotes the **Mittag-Leffler function**.

If $\beta = 1$, the waiting times are exponentially distributed with parameter $\lambda = 1$ and the counting process is the Poisson process.

Compound Fractional Poisson Process

The counting process associated is called the fractional Poisson process

$$N_\beta(t) = \max\{n : \; T_n \leqslant t\}.$$

If we subordinate a CTRW to the fractional Poisson process, we obtain the compound fractional Poisson process, which is not Markov

$$X_{N_\beta(t)} = \sum_{i=1}^{N_\beta(t)} Y_i. \tag{10}$$

The functional limit of the compound fractional Poisson process is an α-stable Lévy process subordinated to the fractional Poisson process.

These processes are possible models for tick-by-tick financial data.

β-stable subordinator and its functional inverse

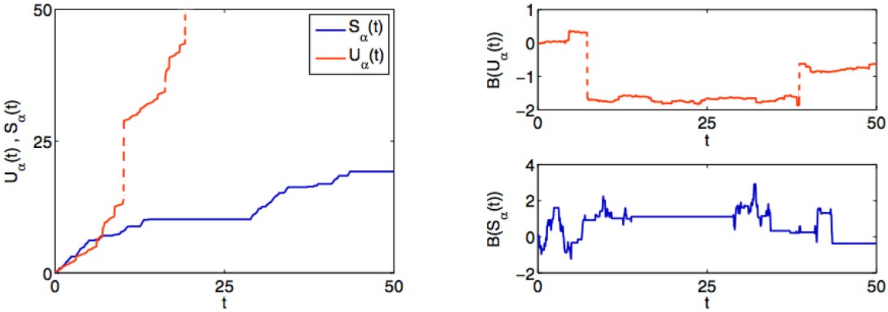
A β-stable subordinator $\{D_\beta\}_{t\geqslant 0}$ is a real-valued β-stable Lévy process with nondecreasing sample paths.

The functional inverse of $\{D_\beta\}_{t\geqslant 0}$ can be defined as

$$D_\beta^{-1}(t) := \inf\{x \geqslant 0 : D_\beta(x) > t\}.$$

It has almost surely continuous non-decreasing sample paths and without stationary and independent increments.

Magdziard & Weron, 2006



About scaling limits

- **Scaling limit of a CTRW**: the limit process resulting from appropriate scaling in time and space according to a functional central limit theorem (FCLT).
- The limit behavior of the CTRW depends on the distribution of the jumps and the waiting times.
- If the **waiting times have finite mean**, the CTRW behaves like a random walk in the limit. So, by Donskers Theorem, if the waiting times have finite mean and the jumps have finite variance then the scaled CTRW converges in distribution to a Brownian motion.
- If the **waiting times have finite mean** and the **jumps** are in the **DOA of an α-stable random variable, with** $\alpha \in (0, 2)$, then the appropriately scaled CTRW converges in distribution to an α-stable Lévy motion.

About scaling limits

- If the **waiting times have an infinite mean**, the CTRW limit behavior is more complex. Meerschaert and Scheffler proved a FCLT which identifies the limit process as a composition of an α-stable Lévy motion $L_\alpha(t)$ and the inverse of a β-stable subordinator, $D_\beta^{-1}(t)$, where $\alpha \in (0, 2]$ and $\beta \in (0, 1)$

M. Meerschaert, H. P. Scheffler.
Limit Theorems for continuous time random walks.
Available at http://www.mathematik.uni-dortmund.de/lsiv/scheffler/ctrw1.pdf., 2001.

M. Meerschaert, H. P. Scheffler.
Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice.
Wiley Series in Probability and Statistics., 2001.

Convergence to the inverse β-stable subordinator

For $t \geqslant 0$, we define

$$T_t := \sum_{i=1}^{\lfloor t \rfloor} J_i.$$

We have

$$\{c^{-1/\beta} T_{ct}\}_{t \geqslant 0} \stackrel{\mathcal{L}}{\Rightarrow} \{D_\beta(t)\}_{t \geqslant 0}, \qquad \text{as} \quad c \to +\infty.$$

For any integer $n \geqslant 0$ and any $t \geqslant 0$: $\{T_n \leqslant t\} = \{N_\beta(t) \leqslant n\}$.

Theorem (Meerschaert & Scheffler (2001))

$$\{c^{-1/\beta} N_\beta(ct)\}_{t \geqslant 0} \stackrel{\mathcal{L}}{\Rightarrow} \{D_\beta^{-1}(t)\}_{t \geqslant 0}, \quad \text{as} \quad c \to +\infty.$$

Theorem (Meerschaert & Scheffler (2001))

$$\{c^{-1/\beta} N_\beta(ct)\}_{t \geqslant 0} \stackrel{J_1-top}{\Rightarrow} \{D_\beta^{-1}(t)\}_{t \geqslant 0}, \quad \text{as} \quad c \to +\infty.$$

Convergence to the symmetric α-stable Lévy process

Assume the jumps Y_i belong to the strict generalized domain of attraction of some stable law with $\alpha \in (0,2)$, then $\exists a_n > 0$ such that

$$a_n \sum_{i=1}^{n} Y_i \overset{\mathcal{L}}{\Rightarrow} \widetilde{L}_\alpha, \quad \text{as} \quad c \to +\infty.$$

Theorem (Meerschaert & Scheffler (2001))

$$\left\{ c^{-1/\alpha} \sum_{i=1}^{[ct]} Y_i \right\}_{t \geqslant 0} \overset{\mathcal{L}}{\Rightarrow} \{L_\alpha(t)\}_{t \geqslant 0}, \quad \text{when} \quad c \to +\infty.$$

Corollary (Meerschaert & Scheffler (2004))

$$\left\{ c^{-1/\alpha} \sum_{i=1}^{[ct]} Y_i \right\}_{t \geqslant 0} \overset{J_1-top}{\Rightarrow} \{L_\alpha(t)\}_{t \geqslant 0}, \quad \text{when} \quad c \to +\infty.$$

Functional Central Limit Theorem

Theorem (Meerschaert & Scheffler (2004))

Under the distributional assumptions considered above for the waiting times J_i and the jumps Y_i, we have

$$\left\{ c^{-\beta/\alpha} \sum_{i=1}^{N_\beta(t)} Y_i \right\}_{t \geqslant 0} \overset{M_1-top}{\Rightarrow} \{L_\alpha(D_\beta^{-1}(t))\}_{t\geqslant 0}, \quad \text{when} \quad c \to +\infty, \tag{11}$$

in the Skorokhod space $D([0,+\infty),\mathbb{R})$ endowed with the M_1-topology.

M. Meerschaert, H. P. Scheffler.
Limit theorems for continuous-time random walks with infinite mean waiting times.
J. Appl. Probab., **41** (3), 623–638, 2004.

Idea of the proof

Apply

$$\{c^{-1/\beta} N_\beta(ct)\}_{t \geqslant 0} \stackrel{J_1-top}{\Rightarrow} \{D_\beta^{-1}(t)\}_{t \geqslant 0}, \quad \text{as} \quad c \to +\infty.$$

and

$$\left\{ c^{-1/\alpha} \sum_{i=1}^{[ct]} Y_i \right\}_{t \geqslant 0} \stackrel{J_1-top}{\Rightarrow} \{L_\alpha(t)\}_{t \geqslant 0}, \quad \text{when} \quad c \to +\infty.$$

$$\left\{ \left(c^{-1/\alpha} \sum_{i=1}^{[ct]} Y_i, c^{-1/\beta} N_\beta(ct) \right) \right\}_{t \geqslant 0} \underset{c \to +\infty}{\stackrel{J_1-top}{\Rightarrow}} \{(L_\alpha(t), D_\beta^{-1}(t))\}_{t \geqslant 0}.$$

Idea of the proof

The proof uses a continuous mapping approach.

Continuous Mapping Theorem (Whitt 2002)

Suppose that $(x_n, y_n) \to (x, y)$ in $D([0,a], \mathbb{R}^k) \times D^1_{\uparrow}$ (where $D^1_{\uparrow}$ is the subset of functions nondecreasing and with $x^i(0) \geqslant 0$). If y is continuous and strictly increasing at t whenever $y(t) \in Disc(x)$ and x is monotone on $[y(t-), y(t)]$ and $y(t-), y(t) \notin Disc(x)$ whenever $t \in Disc(y)$, then $x_n \circ y_n \to x \circ y$ in $D([0,a], \mathbb{R}^k)$, where the topology throughout is M_1 or M_2.

The convergence result only holds in weaker M_1-topology since the composition map is continuous in M_1-topology but not in J_1 at (L_α, D_β^{-1}).

W. Whitt,
Stochastic-Process Limits: An Introduction to Stochastic-Process Limits and Their Application to Queues.
Springer, New York (2002).

FCLT for the quadratic variation of Compound Renewal Processes

Quadratic Variation

Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables (also independent of the J_is) then the compound process $X(t)$ defined by

$$X(t) = \sum_{i=1}^{N_\beta(t)} Y_i \tag{12}$$

The quadratic variation of X is

$$[X](t) = [X,X](t) = \sum_{i=1}^{N_\beta(t)} [X(T_i) - X(T_{i-1})]^2 = \sum_{i=1}^{N_\beta(t)} Y_i^2. \tag{13}$$

FCLT for the Quadratic Variation

Theorem (Scalas & V. (2012))

Under the distributional assumptions considered above for the waiting times J_i and the jumps Y_i, we have

$$\left\{ \left(\frac{1}{n^{2/\alpha}} \sum_{i=1}^{[nt]} Y_i^2, \frac{1}{n^{1/\beta}} T_{nt} \right) \right\}_{t \geqslant 0} \overset{J_1-top}{\underset{n \to +\infty}{\Rightarrow}} \{(L^+_{\alpha/2}(t), D_\beta(t))\}_{t \geqslant 0}, \qquad (14)$$

in the Skorokhod space $D([0,+\infty), \mathbb{R}_+ \times \mathbb{R}_+)$ endowed with the J_1-topology. Moreover, we have also

$$\sum_{i=1}^{N_\beta(nt)} \frac{Y_i^2}{n^{2\beta/\alpha}} \overset{M_1-top}{\Rightarrow} L^+_{\alpha/2}(D_\beta^{-1}(t)), \quad as \quad n \to +\infty,$$

in the Skorokhod space $D([0,+\infty), \mathbb{R}_+)$ with the M_1-topology, where $L^+_{\alpha/2}(t)$ denotes an $\alpha/2$-stable positive Lévy process.

E. Scalas, N. Viles,
On the Convergence of Quadratic variation for Compound Fractional Poisson Processes.
Fractional Calculus and Applied Analysis, **15**, 314–331 (2012).

FCLT for the stochastic integrals driven by a time-changed symmetric α-stable Lévy process

Damped harmonic oscillator subject to a random force

The equation of motion is informally given by

$$\ddot{x}(t) + \gamma\dot{x}(t) + kx(t) = \xi(t), \tag{15}$$

where $x(t)$ is the position of the oscillating particle with unit mass at time t, $\gamma > 0$ is the damping coefficient, $k > 0$ is the spring constant and $\xi(t)$ represents white Lévy noise (formal derivative symmetric $L_\alpha(t)$).

I. M. Sokolov,
Harmonic oscillator under Lévy noise: Unexpected properties in the phase space.
Phys. Rev. E. Stat. Nonlin Soft Matter Phys **83**, 041118 (2011).

The formal solution is

$$x(t) = F(t) + \int_{-\infty}^{t} G(t - t')\xi(t')dt', \tag{16}$$

where $G(t)$ is the Green function for the homogeneous equation.
The solution for the velocity component can be written as

$$v(t) = F_v(t) + \int_{-\infty}^{t} G_v(t - t')\xi(t')dt', \tag{17}$$

where $F_v(t) = \frac{d}{dt}F(t)$ and $G_v(t) = \frac{d}{dt}G(t)$.

Consider the compound renewal process given by

$$X(t) = \sum_{i=1}^{N_\beta(t)} Y_i = \sum_{i \geqslant 1} Y_i \mathbf{1}_{\{T_i \leqslant t\}}$$

The corresponding white noise can be formally written as

$$\Xi(t) = dX(t)/dt = \sum_{i=1}^{N_\beta(t)} Y_i \delta(t - T_i) = \sum_{i \geqslant 1} Y_i \delta(t - T_i) \mathbf{1}_{\{T_i \leqslant t\}}.$$

Our goal

To study the convergence of the integral of a deterministic continuous and bounded function with respect to a properly rescaled CTRW.

We aim to prove that under a proper scaling and distributional assumptions:

$$\left\{ \sum_{i=1}^{N_\beta(nt)} G\left(t - \frac{T_i}{n}\right) \frac{Y_i}{n^{\beta/\alpha}} \right\}_{t\geqslant 0} \overset{M_1-top}{\Rightarrow} \left\{ \int_0^t G(t-s) dL_\alpha(D_\beta^{-1}(s)) \right\}_{t\geqslant 0},$$

and

$$\left\{ \sum_{i=1}^{N_\beta(nt)} G_v\left(t - \frac{T_i}{n}\right) \frac{Y_i}{n^{\beta/\alpha}} \right\}_{t\geqslant 0} \overset{M_1-top}{\Rightarrow} \left\{ \int_0^t G_v(t-s) dL_\alpha(D_\beta^{-1}(s)) \right\}_{t\geqslant 0},$$

when $n \to +\infty$, in the Skorokhod space $D([0,+\infty),\mathbb{R})$ endowed with the M_1-topology.

Rescaled CTRW

Let h and r be two positive scaling factors such that

$$\lim_{h,r\to 0} \frac{h^\alpha}{r^\beta} = 1, \tag{18}$$

with $\alpha \in (1,2]$ and $\beta \in (0,1]$.
We rescale the duration J and the jump by positive scaling factors r and h:

$$J_r := rJ, \qquad Y_h := hY.$$

The rescaled CTRW denoted:

$$X_{r,h}(t) = \sum_{i=1}^{N_\beta(t/r)} hY_i,$$

where $N_\beta = \{N_\beta(t)\}_{t\geqslant 0}$ is the fractional Poisson process.

Distributional assumptions

- Jumps $\{Y_i\}_{i\in\mathbb{N}}$: i.i.d. symmetric α-stable random variables such that Y_1 belongs to DOA of an α-stable random variable with $\alpha \in (1, 2]$.
- Watiting times $\{J_i\}_{i\in\mathbb{N}}$: i.i.d. random variables such that J_1 belongs to DOA of some β-stable random variables with $\beta \in (0, 1)$.

FCLT for stochastic integrals driven by a time-changed symmetric α-stable Lévy process

Theorem (Scalas & V.)

Let $f \in \mathcal{C}_b(\mathbb{R})$. Under the distributional assumptions and the scaling,

$$\left\{ \sum_{i=1}^{N_\beta(nt)} f\left(\frac{T_i}{n}\right) \frac{Y_i}{n^{\beta/\alpha}} \right\}_{t \geqslant 0} \overset{M_1-top}{\underset{n\to+\infty}{\Rightarrow}} \left\{ \int_0^t f(s) dL_\alpha(D_\beta^{-1}(s)) \right\}_{t \geqslant 0},$$

in $D([0,+\infty),\mathbb{R})$ with M_1-topology.

Sketch of the proof

- ✓ Check M_1-compactness condition for the integral process

$$\left\{ I_n(t) := \sum_{k=1}^{N_\beta(nt)} f\left(\frac{T_k}{n}\right) \frac{Y_k}{n^{\beta/\alpha}} \right\}_{t\geqslant 0} .$$

- ✓ Prove the convergence in law of the family of processes $\{I_n(t)\}_{t\geqslant 0}$ when $n \to +\infty$.
  - ✓ $\{X^{(n)}(t)\}_{t\geqslant 0}$ is uniformly tight or a *good sequence* .
  - ✓ Apply the Continuous Mapping Theorem (CMT) taking as a continuous mapping the composition function.
- ✓ Apply Characterization of the M_1-convergence.

M_1-compactness condition

Lemma (Scalas & V.)

Let $f \in \mathcal{C}_b(\mathbb{R})$*. Let* $\{Y_i\}_{i\in\mathbb{N}}$ *be i.i.d. symmetric* α*-stable random variables. Assume that* Y_1 *belongs DOA of* S_α*, with* $\alpha \in (1,2]$*. Let* $\{J_i\}_{i\in\mathbb{N}}$ *be i.i.d. such that* J_1 *belongs to the strict DOA of* S_β *with* $\beta \in (0,1)$*. Consider*

$$I_n(t) := \sum_{k=1}^{N_\beta(nt)} f\left(\frac{T_k}{n}\right) \frac{Y_k}{n^{\beta/\alpha}}. \tag{19}$$

If

$$\lim_{\delta\to 0} \limsup_{n\to+\infty} w_s(X_n, \delta) = 0,$$

where $X_n(t) := \sum_{k=1}^{N_\beta(nt)} \frac{Y_k}{n^{\beta/\alpha}}$*. Then,*

$$\lim_{\delta\to 0} \limsup_{n\to+\infty} w_s(I_n, \delta) = 0. \tag{20}$$

Now, to see the convergence in the M_1-topology it only remains to prove

$$\sum_{k=1}^{N_\beta(nt)} f\left(\frac{T_k}{n}\right)\frac{Y_k}{n^{\beta/\alpha}} \overset{\mathcal{L}}{\Rightarrow} \int_0^t f(s)dL_\alpha(D_\beta^{-1}(s)), \qquad n \to +\infty.$$

A fundamental question is to know under what conditions the convergence in law of (H^n, X^n) to (H, X) implies that X is a semimartingale and that $\int_0^t H^n(s-)dX_s^n$ converges in law to $\int_0^t H(s-)dX_s$.

Good sequence

Let $(X^n)_{n\in\mathbb{N}}$ be an $\mathbb{R}^k$-valued process defined on $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ s.t. it is a $\mathcal{F}_t^n$-semimartingale. Assume that $(X^n)_{n\in\mathbb{N}} \overset{\mathcal{L}}{\Rightarrow} X$ in the Skorokhod topology.

The sequence $(X^n)_{n\in\mathbb{N}}$ is said to be good if for any sequence $(H^n)_{n\in\mathbb{N}}$ of $\mathbb{M}^{km}$-valued, càdlàg processes, H^n $\mathcal{F}_t^n$-adapted, such that

$$(H^n, X^n) \overset{\mathcal{L}}{\Rightarrow} (H, X)$$

in the Skorokhod topology on $D_{\mathbb{M}^{km}\times\mathbb{R}^m}([0,\infty))$, $\exists$ a filtration $\mathcal{F}_t$ such that H is $\mathcal{F}_t$-adapted, X is a $\mathcal{F}_t$-semimartingale, and

$$\int_0^t H^n(s-)dX_s^n \overset{\mathcal{L}}{\Rightarrow} \int_0^t H(s-)dX_s,$$

when $n \to \infty$.

$(X^{(n)})_{n\in\mathbb{N}}$ uniformly tight

Lemma

If $(X^n)_{n\in\mathbb{N}}$ is a sequence of local martingales and the following condition

$$\sup_n \mathbb{E}^n \left[\sup_{s\leqslant t} |\Delta X^{(n)}(s)|\right] < +\infty,$$

holds for each $t < +\infty$, where

$$\Delta X^{(n)}(s) := X^{(n)}(s) - X^{(n)}(s-) \tag{21}$$

denotes the increment of $X^{(n)}$ in s, then the sequence is uniformly tight.

$(X^{(n)})_{n\in\mathbb{N}}$ uniformly tight

Lemma (Scalas & V.)

Assume that $(Y_i)_{i\in\mathbb{N}}$ be i.i.d. symmetric α-stable random variables, with $\alpha \in (1,2]$. Let

$$X^{(n)}(t) := \sum_{i=1}^{\lfloor n^{\beta} t \rfloor} \frac{Y_i}{n^{\beta/\alpha}} \tag{22}$$

be defined on the probability space $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$. Then $X^n(t)$ is a $\mathcal{F}^n_t$-martingale (with respect the natural filtration of $X^{(n)}$) and

$$\sup_n \mathbb{E}^n \left[\sup_{s\leqslant t} \left| \Delta X^{(n)}(s) \right| \right] < +\infty,$$

for each $t < +\infty$.

Convergence in law

Proposition (Scalas & V.)

Let $f \in \mathcal{C}_b(\mathbb{R})$. Under the distributional assumptions and the scaling considered above we have that

$$\sum_{i=1}^{\lfloor n^\beta t \rfloor} f\left(\frac{T_i}{n}\right) \frac{Y_i}{n^{\beta/\alpha}} \overset{\mathcal{L}}{\Rightarrow} \int_0^t f(D_\beta(s)) dL_\alpha(s), \qquad n \to +\infty.$$

Proposition (Scalas & V.)

Let $f \in \mathcal{C}_b(\mathbb{R})$. Under the distributional assumptions and scaling,

$$\left\{ \sum_{i=1}^{N_\beta(nt)} f\left(\frac{T_i}{n}\right) Y_i \right\}_{t \geqslant 0} \overset{\mathcal{L}}{\Rightarrow} \left\{ \int_0^{D_\beta^{-1}(t)} f(D_\beta(s)) dL_\alpha(s) \right\}_{t \geqslant 0}$$

as $n \to +\infty$, where $\int_0^{D_\beta^{-1}(t)} f(D_\beta(s)) dL_\alpha(s) \overset{a.s.}{=} \int_0^t f(s) dL_\alpha(D_\beta^{-1}(s))$.

Applications

Corollary (Scalas & V.)

$$\left\{ \sum_{i=1}^{N_\beta(nt)} G\left(t - \frac{T_i}{n}\right) \frac{Y_i}{n^{\beta/\alpha}} \right\}_{t\geqslant 0} \overset{M_1-top}{\Rightarrow} \left\{ \int_0^t G(t-s) dL_\alpha(D_\beta^{-1}(s)) \right\}_{t\geqslant 0},$$

and

$$\left\{ \sum_{i=1}^{N_\beta(nt)} G_v\left(t - \frac{T_i}{n}\right) \frac{Y_i}{n^{\beta/\alpha}} \right\}_{t\geqslant 0} \underset{n\to+\infty}{\overset{M_1-top}{\Rightarrow}} \left\{ \int_0^t G_v(t-s) dL_\alpha(D_\beta^{-1}(s)) \right\}_{t\geqslant 0},$$

in $D([0,+\infty),\mathbb{R})$ *with* M_1*-topology.*

Summary:

- We have studied the convergence of a class of stochastic integrals with respect to the Compound Fractional Poisson Process.
- Under proper scaling hypotheses, these integrals converge to the integrals w.r.t a symmetric α-stable process subordinated to the inverse β-stable subordinator.

Future work:

- It is possible to approximate some of the integrals discussed in Kobayashi (2010) by means of simple Monte Carlo simulations. This will be the subject of a forthcoming applied paper.
- To extend this result to the integration of stochastic processes instead of deterministic functions.

K. Kobayashi.
Stochastic Calculus for a Time-changed Semimartingale and the Associated Stochastic Differential Equations.
Journal of Theoretical Probability **24**, 789–820 (2010).

Thank you for your attention

Future work:

The functional convergence of quadratic variation leads to the following conjecture on the integrals defined as:

$$I_a(t) = \sum_{i=1}^{N_t} [(1-a)G(X(T_{i-1})) + aG(X(T_i))](X(T_i) - X(T_{i-1}))$$
$$= I_{1/2}(t) + \left(a - \frac{1}{2}\right)[X, G(X)](t),$$

where $G(x)$ is a sufficiently smooth ordinary function and $a \in [0,1]$ and

$$[X, G(X)](t) = \sum_{i=1}^{N_t} [X(T_i) - X(T_{i-1})][G(X(T_i)) - G(X(T_{i-1}))].$$

It might be possible to prove that, under proper scaling, the integral converges in some sense to a stochastic integral driven by the semimartingale measure $L_\alpha(D_\beta^{-1}(t))$.