Stochastic integration in UMD Banach spaces

Jan van Neerven
Technical University Delft, The Netherlands
`http://aw.twi.tudelft.nl/~neerven`

In these lectures we shall present an introduction of the theory of stochastic integration in UMD Banach spaces and some of its applications.

The Hilbert space approach to stochastic partial differential equations (SPDEs) was pioneered in the 1980s by Da Prato and Zabczyk. Under suitable Lipschitz conditions, mild solutions of semilinear SPDEs in Hilbert spaces can be obtained by solving a fixed point equation involving suitable Hilbert space-valued stochastic integrals.

In order to extend this theory to SPDEs in more general classes of Banach spaces, such as L^p, one has to overcome the fundamental obstruction that the construction of the Itô stochastic integral depends essentially on an L^2-isometry. Like many other L^2-isometries in Analysis (another example is the Fourier-Plancherel isometry in Fourier Analysis), a direct vector-valued extension is possible only for Hilbert spaces.

In these lectures, based on recent work in collaboration with Mark Veraar and Lutz Weis, we show how this problem can be solved in a very natural way in the setting of UMD Banach spaces. This class of Banach spaces was introduced and studied by Burkholder in the 1980s and has established itself in many respects as the 'correct' class for handling vector-valued martingale inequalities and singular integrals.

We shall introduce the class of UMD Banach spaces, discuss the relevant examples, and construct the Itô stochastic integral of a process taking values in a UMD space. The main idea is to identify square integrable functions (respectively, processes) with values in a Hilbert space H as (random) Hilbert–Schmidt operators acting from L^2 into H, and then to use Gaussian techniques to generalise the notion of a (random) Hilbert-Schmidt operator to (UMD) Banach spaces. We then prove two-sided estimates which extend the Itô isometry and the Burkholder–Davis–Gundy inequalities. Possible applications include vector-valued Malliavin calculus and the maximal L^p-regularity problem for parabolic SPDEs; the choice will depend on the interests of the audience.

We shall start from scratch, assuming only some standard background material from Functional Analysis, Measure Theory, and Probability Theory. The course will be based upon my Internet Seminar lecture notes (available at `ocw.tudelft.nl/courses/mathematics/` and a forthcoming book with Hytönen, Veraar, and Weis.