Integration theory for infinite dimensional volatility modulated Volterra processes

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This is a joint work with Fred Espen Benth
which was done on a *estancia breve* at the University of Oslo.

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Our aim: derive a stochastic calculus with respect to these objects.
Today's question is: Given a random field Y, what is

$$Z(t,x) = \int_0^t \int_{\mathbb{R}^d} Y(t,s;x,y)X(\mathrm{d}s,\mathrm{d}y)?$$

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where

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Furthermore, we are given stochastic process $(Y(t))_{t\in[0,T]}$ from H_1 to H_2. Then our integral looks like

$$Z(t) = \int_0^t Y(s)\mathrm{d}X(s).$$

A few examples

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fBM Choose $\sigma = Q^{1/2}$ where Q is a nonnegative, selfadjoint, trace-class operator and

$$g(t,s) = c_H(t-s)^{H-1/2} + c_H\left(\frac{1}{2} - H\right)\int_s^t (u-s)^{H-3/2}\left(1-(s/u)^{1/2-H}\right)\mathrm{d}u.$$

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OU process For $g(t,s) = \exp(-(t-s)A)$ we rediscover OU processes

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S(P)DE X is the mild solution of $\mathrm{d}X_t = AX_t + \sigma(X(t))\mathrm{d}B(t)$, i.e.

$$X(t) = \int_0^t g(t-s)\sigma(X(s))\mathrm{d}B(s).$$

Idea behind this integral I

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For the first term a basic rule of Malliavin calculus yields

$$Y(t)X(t) = \int_0^t Y(t)g(t,s)\sigma(s)\delta B(s) + \int_0^t D_s Y(t)g(t,s)\sigma(s)\mathrm{d}s.$$

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Similarly for the second term (and a stochastic Fubini)

$$\begin{aligned}
&\int_0^t \frac{\mathrm{d}Y}{\mathrm{d}s}(s)X(s)\mathrm{d}s \\
&\quad = \int_0^t \left(\int_u^t \frac{\mathrm{d}Y}{\mathrm{d}s}(s)g(s,u)\mathrm{d}s \right)\sigma(u)\delta B(u) \\
&\qquad + \int_0^t \left(\int_u^t D_u\left(\frac{\mathrm{d}Y}{\mathrm{d}s}(s)\right)g(s,u)\mathrm{d}s \right)\sigma(u)\mathrm{d}u.
\end{aligned}$$

Idea behind this integral II

So, putting this together and deterministic IbP yield

$$\int_0^t \left(Y(t)g(t,s) - \int_s^t \frac{\mathrm{d}Y}{\mathrm{d}u}(u)g(u,s)\mathrm{d}u \right)\sigma(s)\delta B(s)$$
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From this we read off the following kernel

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and can define the stochastic integral as

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In some cases we need to or can rewrite the kernel $\mathcal{K}_g$.

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&= Y(s)g(t,s) + \int_s^t \big(Y(u) - Y(s)\big)\frac{\partial g}{\partial u}(u,s)\mathrm{d}u \\
&= Y(s)g(t-s) + \int_0^{t-s} \big(Y(u+s) - Y(s)\big)g(\mathrm{d}u)
\end{aligned}$$

Formal Definition

Definition

Fix some $t \in [0, T]$. We say that a stochastic process $(Y(s))_{s\in[0,t]}$ belongs to the domain of the stochastic integral with respect to X if

1. the process $(Y(u) - Y(s))_{u\in(s,t]}$ is integrable with respect to $g(\mathrm{d}u, s)$ almost surely,

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3. $\mathcal{K}_g(Y)(t,s)$ is Malliavin differentiable for all $s \in [0,t]$ and the Hilbert-valued stochastic process $s \mapsto \mathrm{tr}_{\mathcal{H}} D_s\big(\mathcal{K}_g(Y)(t,s)\big)\sigma(s)$ is Bochner integrable on $[0,t]$ almost surely.

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We denote this by $Y \in \mathcal{I}^X(0, t)$.

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Basic calculus rules I

Linearity If $Y, Z \in \mathcal{I}^X(0,t)$ and $a, b \in \mathbb{R}$ then

$$\int_0^t (aY(s) + bZ(s))\mathrm{d}X(s) = a\int_0^t Y(s)\mathrm{d}X(s) + b\int_0^t Z(s)\mathrm{d}X(s).$$

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Indicator integrands Choosing an indicator function in time gives us the increment of the integrator, i.e.

$$\int_0^t 1_{[u,v]}(s)\mathrm{d}X(s) = \int_0^t 1_{[u,v]}(s)\,\mathrm{id}\,\mathrm{d}X(s) = X(v) - X(u).$$

Basic calculus rules II

Bounded linear operators Let Z be a random linear operator which is almost surely bounded (no special measurabilty conditions) such that $s \mapsto ZY(s) \in \mathcal{I}^X(0,t)$. Then

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Projection I Let $(e_k)_{k\in\mathbb{N}}$ be a CONS of H_1. Then

$$\mathcal{K}_g(Y)(t,s)(e_k) = \mathcal{K}_{g(e_k)}(Y)(t,s),$$

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Projection II Let $(e_k)_{k\in\mathbb{N}}$ be a CONS of H_1 and let $X^k := \langle X, e_k\rangle$. Then

$$\int_0^t Y(s)\mathrm{d}X^k(s) = \int_0^t \mathcal{K}_{\langle g,e_k\rangle}(Y)(t,s)\delta B(s) + \mathrm{tr}_{\mathcal{H}}\int_0^t D_s\mathcal{K}_{\langle g,e_k\rangle}(Y)(t,s)\mathrm{d}s.$$

Basic calculus rules III

Shift of domain For $0 \leq u < v \leq t$ and $Y \in \mathcal{I}^X(0,u) \cap \mathcal{I}^X(0,v)$

$$\int_0^t Y(s) 1_{[u,v]}(s) \mathrm{d}X(s) = \int_0^v Y(s) \mathrm{d}X(s) - \int_0^u Y(s) \mathrm{d}X(s).$$

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Simple processes Let Y be a simple process, i.e. $Y = \sum_{j=1}^{n-1} Z_j 1_{(t_j, t_{j+1}]}$. Then $Y \in \mathcal{I}^X(0,t)$ and

$$\int_0^t Y(s) \mathrm{d}X(s) = \sum_{j=1}^{n-1} Z_j (X_{t_{j+1}} - X_{t_j}).$$

Semimartingale condition

Proposition

Let $t > 0$ and assume that $g(s,s)$ is well-defined for all $0 \leq s \leq t$. Furthermore, assume that there is a bi-measurable function $\phi : [0,T] \to L(H_1, H_1)$ such that $g(t,s) = g(s,s) + \int_s^t \phi(v,s)\mathrm{d}v$, for all $0 \leq s \leq t$, where this integral is defined in the sense of Bochner and

$$\int_0^t \|g(s,s)\|^2_{L(H_1,H_1)}\mathrm{d}s < \infty \quad \textit{and} \quad \int_0^t \int_0^u \|\phi(u,s)\|^2_{L(H_1,H_1)}\mathrm{d}u\mathrm{d}s < \infty.$$

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Let $t > 0$ *and assume that* $g(s,s)$ *is well-defined for all* $0 \le s \le t$. *Furthermore, assume that there is a bi-measurable function* $\phi : [0,T] \to L(H_1, H_1)$ *such that* $g(t,s) = g(s,s) + \int_s^t \phi(v,s)\mathrm{d}v$, *for all* $0 \le s \le t$, *where this integral is defined in the sense of Bochner and*

$$\int_0^t \|g(s,s)\|^2_{L(H_1,H_1)} \mathrm{d}s < \infty \quad \text{and} \quad \int_0^t \int_0^u \|\phi(u,s)\|^2_{L(H_1,H_1)} \mathrm{d}u\mathrm{d}s < \infty.$$

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$$X(t) = \int_0^t g(s,s)\sigma(s)\mathrm{d}B(s) + \int_0^t \int_0^s \phi(s,u)\sigma(u)\mathrm{d}B(u)\mathrm{d}s.$$

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Furthermore, $\int_0^t Y\mathrm{d}X(s) = Y \cdot X$.

Deterministic integrands and OU processes

Fix $t > 0$ and let $s \mapsto h(t, s)$ be a deterministic function, such that $u \mapsto h(t, u) - h(t, s)$ is $g(\mathrm{d}u, s)$-integrable on $[s, t]$. Then,

$$\int_0^t h(t, s)\mathrm{d}X(s) = \int_0^t \mathcal{K}_g(h)(t, s)\sigma(s)\mathrm{d}B(s).$$

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where e^{-tA} is the C_0-semigroup generated by $-A$, where we assume that $u \mapsto e^{-(u-s)A}$ to be $g(\mathrm{d}u, s)$-integrable.

Volterra processes as integrands

Now we turn to the problem what happens if the integrand is of the form (assume $\sigma \equiv 1$)

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With this one can write down a formula for

$$\int_0^t X^*(s)\mathrm{d}X(s),$$

and the Itô formula will give a connection with

$$\frac{1}{2}\langle X(t), X(t)\rangle_{H_1} = \int_0^t \int_0^s \langle g(t,v)\mathrm{d}B(v), g(t,s)\mathrm{d}B(s)\rangle_{H_1} + \int_0^t \|g(t,s)\|^2\mathrm{d}s.$$

An Itô formula

Theorem

Let $F : H_2 \to H_3$ be twice Fréchet differentiable. Furthermore assume that g satisfies the semimartingle condition. Assume that Y and σ are twice Malliavin differentiable, $Y(s)g(s,s)\sigma(s) \in \mathbb{L}^{2,p}(\mathcal{H}, H_2)$ for some $p > 4$ and

$$\int_0^s Y(s)\frac{\partial g}{\partial s}(s,u)\sigma(u)\delta B(u) + \mathrm{tr}_{\mathcal{H}}\, D_s(Y(s))g(s,s)\sigma(s) + \mathrm{tr}_{\mathcal{H}} \int_0^s D_u(Y(s))\frac{\partial g}{\partial s}(s,u)\sigma(u)\mathrm{d}u \in \mathbb{L}^{1,4}(H_2).$$

Then $F'(Z)Y \in \mathcal{I}^X(0,t)$ for all $t \in [0,T]$ and

$$F(Z_t) = F(0) + \int_0^t F'(Z(s))Y(s)\mathrm{d}X(s) - \frac{1}{2}\mathrm{tr}_{\mathcal{H}} \int_0^t F''(Z_s)\big(Y(s)g(s,s)\sigma(s)\big)\big(Y(s)g(s,s)\sigma(s)\big)\mathrm{d}s$$

Corollaries of the Itô formula

Corollary

Under the conditions of the last theorem $F'(X) \in \mathcal{I}^X(0,t)$ *for all* $t \in [0,T]$ *and*

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Corollary

Suppose the same conditions and $H_2 = \mathbb{R}$ *and* $F(x) = x^2$. *Then*

$$\frac{1}{2}(Z(t))^2 = \int_0^t Z(s)\mathrm{d}Z(s) + \operatorname{tr}_{\mathcal{H}} \int_0^t \big(D_s Z(s)\big) Y(s) g(s,s)\sigma(s)\mathrm{d}s$$
$$- \frac{1}{2}\int_0^t \|Y(s)g(s,s)\sigma(s)\|^2_{L_2(\mathcal{H},H_1)}\mathrm{d}s$$

Contents

How do SPDE enter?

Consider SDEs in infinite dimensions

$$\mathrm{d}X(t) = AX(t) + \sigma(t)\mathrm{d}B(t) + b(t)\mathrm{d}t$$

and its corresponding mild solutions

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Then g is a C_0-semigoup of linear operators generated by A and its (singular) fundamental solution Λ, i.e.

$$g(t,s)f = \int_D \Lambda(t,s,\cdot,y)f(y)\mathrm{d}y,$$

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if the Hilbert space is $L^2(D)$. Then we want to now about the conditions on h under which the operator $\mathcal{K}_g(h)(t,s)$ is well-defined for a large class of σ (or the other way round).

An example using the wave kernel

For $d = 3$ the fundamental solution of the wave equation is given by $g(t,s) = c\sigma^3_{t-s}/(t-s)$. The question is: what is $\mathcal{K}_g(h)$?

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Now σ^d_v grows as v^{d-1}, so this operator makes sense if $h \in \mathcal{C}^1((0,t-s])$ and h' has a singularity at zero of less than v^{-2}. This also works for larger d, then h has to be smoother but the highest derivative can be more singular at zero.

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Further Research

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any more ideas?

Thank you very much for your attention!