Anticipating Linear Stochastic Differential Equations Driven by Lévy Processes

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Contents

1. Introduction
2. Preliminaries
3. Canonical Lévy Space
4. Malliavin Calculus
5. Girsanov Transformations
6. Anticipative Linear SDE on Canonical Wiener Space
7. Existence and Uniqueness of the Solution

Contents

1 Introduction

2 Preliminaries

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Equation

In this talk, we consider

$$
\begin{aligned}
X_t &= X_0 + \int_0^t b_s X_s \, ds + \int_0^t a_s X_s \, \delta W_s \\
&\quad + \int_0^t \int_{\{|y|>1\}} v_s(y) X_{s-} \, dN(s,y) \\
&\quad + \int_0^t \int_{\{0<|y|\le 1\}} v_s(y) X_{s-} \, d\tilde{N}(s,y), \qquad 0 \le t \le T.
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W is the canonical Wiener process, N is the canonical Poisson random measure with parameter ν and $d\tilde{N}(t,y) := dN(t,y) - dt\,\nu(dy)$.

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\end{aligned}
$$

The stochastic integral with respect to W (resp. N and $\tilde{N}$) is in the Skorohod sense (resp. are pathwise defined).

Equation

$$\begin{aligned} X_t^\varepsilon &= X_0 + \int_0^t b_s X_s^\varepsilon \, ds + \int_0^t a_s X_s^\varepsilon \, \delta W_s \\ &\quad + \int_0^t \int_{\{|y|>1\}} v_s(y) X_{s-}^\varepsilon \, dN(s,y) \\ &\quad + \int_0^t \int_{\{\varepsilon<|y|\leq 1\}} v_s(y) X_{s-}^\varepsilon \, d\tilde{N}(s,y), \qquad 0 \leq t \leq T. \end{aligned}$$

Contents

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2 Preliminaries

3 Canonical Lévy Space

4 Malliavin Calculus

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$$Y_t = \gamma t + \sigma W_t + \int_{(0,t]\times\{|x|>1\}} x dN(s,x) + \lim_{\varepsilon \downarrow 0} \int_{(0,t]\times\{\varepsilon<|x|\leq 1\}} x d\widetilde{N}(s,x).$$

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Here, $W = \{W_t : t \in [0, T]\}$ is a standard Brownian motion.

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Here,

$$\tilde{N} = N - dtd\nu$$

and

$$N(B) = \#\{t : (t, \Delta Y_t) \in B\}, \quad B \in \mathcal{B}([0,T] \times \mathbb{R}_0)$$

with

$$\mathbb{R}_0 = \mathbb{R} - \{0\}.$$

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i) For $B \in \mathcal{B}([0,T]\times\mathbb{R}_0)$, $N(B)$ is Poisson distributed with parameter $\lambda \otimes \nu(B)$.

ii) If $B_1, \ldots, B_n$ are pairwisedisjoints, then $N(B_1), \ldots, N(B_n)$ are independent random variables.

Example

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where $N = \{N_t : t \geq 0\}$ is a Poisson process with parameter τ and $\{Y_i : i \geq 1\}$ are i.i.d. random variables, with distribution Q, which are also independent of N. Then Z is a Lévy process with triplet $(0, 0, \tau Q)$

Contents

1. Introduction
2. Preliminaries
3. Canonical Lévy Space
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Canonical Lévy space for a compound Poisson process

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$(\Omega_T, \mathcal{F}_T, P_T)$ is the Canonical Lévy space.

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$$Y_t(\omega) = \begin{cases} \sum_{j=1}^n x_j 1_{[0,t]}(t_j), & \text{if} \quad \omega = ((t_1, x_1), \cdots, (t_n, x_n)), \\ 0, & \text{if} \quad \omega = \alpha. \end{cases}$$

Canonical Lévy space for a pure jump Lévy process

Let $\{Y_t^{(k)} : t \in [0, T]\}$ be the canonical compound Poisson process on $(\Omega^{(k)}, \mathcal{F}^{(k)}, P^{(k)})$ with intensity $\tau_k = \nu(S_k)$ and probability measure $Q_k = \frac{\nu(\cdot \cap S_k)}{\nu(S_k)}$.

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$$S_1 = \{x \in \mathbb{R} : \varepsilon_1 < |x|\} \quad \text{and} \quad S_k = \{x \in \mathbb{R} : \varepsilon_k < |x| \leq \varepsilon_{k-1}\}.$$

and $\varepsilon_n \downarrow 0$

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$$(\Omega_N, \mathcal{F}_N, \mathcal{P}_N) = \bigotimes_{k\geq 1} (\Omega^{(k)}, \mathcal{F}^{(k)}, P^{(k)}),$$

$$J_t(\omega) = Y_t^{(1)}(\omega^1) + \lim_{n\to\infty} \sum_{k=2}^{n} \left(Y_t^{(k)}(\omega^k) - t \int_{S_k} x d\nu(x) \right)$$

with $\omega = (\omega^k)_{k\geq 1} \in \Omega_J$.

Canonical Lévy space

$$(\Omega, \mathcal{F}, P) = (\Omega_W \otimes \Omega_N, \mathcal{F}_W \otimes \mathcal{F}_N, P_W \otimes P_N),$$

where $(\Omega_W, \mathcal{F}_W, P_W)$ is the canonical Wiener space.

Canonical Lévy space

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where $(\Omega_W, \mathcal{F}_W, P_W)$ is the canonical Wiener space.

$$Y_t(\omega) = \gamma t + \sigma \omega'(t) + J_t(\omega''), \quad \omega = (\omega', \omega''),$$

is a Lévy process with triplet (γ, σ^2, ν).

Contents

1 Introduction

2 Preliminaries

3 Canonical Lévy Space

4 Malliavin Calculus

5 Girsanov Transformations

6 Anticipative Linear SDE on Canonical Wiener Space

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Derivative operator

$\mathcal{S}^W(L^2(\Omega_N))$ is the set of all smooth $L^2(\Omega_N)$-random variables of the form

$$F = \sum_{i=1}^{n} f_i\Big(\int_0^T h_{1,i}(s)dW_s, \ldots, \int_0^T h_{n_i,i}(s)dW_s\Big)G_i.$$

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Here $h_{j,i} \in L^2([0,T])$, $G_i \in L^2(\Omega_N)$ and $f_i \in \mathcal{C}_b^\infty(\mathbb{R}^{n_i})$.

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The derivative of the random variable F with respect to W is the $L^2(\Omega_N \times [0,T])$-valued random variable

$$D^W F = \sum_{i=1}^{n}\sum_{j=1}^{n_i}(\partial_j f_i)\left(\int_0^T h_{1,i}(s)dW_s, \ldots, \int_0^T h_{n_i,i}(s)dW_s\right)h_{j,i}G_i.$$

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Hence

$$D^W : \mathbb{D}^W_{1,2}(L^2(\Omega_N)) \subset L^2(\Omega) \to L^2(\Omega \times [0,T]).$$

Divergence operator with respect to W

Skorohod integral with respect to W, denoted by δ^W, is the adjoint of the derivative operator

$$D^W : \mathbb{D}^W_{1,\infty}(L^\infty(\Omega_N)) \subset L^\infty(\Omega) \to L^\infty(\Omega \times [0, T]) .$$

Divergence operator with respect to W

u is in $Dom\ \delta^W$ if and only if $u \in L^1(\Omega \times [0, T])$ and there exists a random variable $\delta^W(u) \in L^1(\Omega)$ satisfying the duality relation

$$\mathbb{E}\left[\int_0^T u_t D_t^W F dt\right] = \mathbb{E}\left[\delta^W(u) F\right] \quad \text{for every} \quad F \in \mathbb{D}^W_{1,\infty}(L^\infty(\Omega_N)).$$

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For $u \in Dom\ \delta^W$, we will make use of the notation

$$\delta^W(u) = \int_0^T u_t \delta W_t$$

and for $u1\!\!1_{[0,t]} \in Dom\ \delta^W$, we will write

$$\delta^W(u1\!\!1_{[0,t]}) = \int_0^t u_s \delta W_s.$$

Properties of D and δ

Lemma

Let $F \in \mathbb{D}^W_{1,2}(L^2(\Omega_N))$ *and* $u \in Dom\ \delta^W$*. Then, for almost all* $\omega'' \in \Omega_N$, $F(\cdot,\omega'') \in \mathbb{D}^W_{1,2}$, $u(\cdot,\omega'') \in Dom\ \delta^W \cap L^1(\Omega_W \times [0,T])$,

$$D^W F(\cdot,\omega'') = (D^W F)(\cdot,\omega'')$$

and

$$\delta^W(u(\cdot,\omega'')) = \delta^W(u)(\cdot,\omega'').$$

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Lemma

Let $F \in \mathbb{D}^W_{1,2}(L^2(\Omega_N))$ and $u \in Dom\ \delta^W$. Then, for almost all $\omega'' \in \Omega_N$, $F(\cdot, \omega'') \in \mathbb{D}^W_{1,2}$, $u(\cdot, \omega'') \in Dom\ \delta^W \cap L^1(\Omega_W \times [0, T])$,

$$D^W F(\cdot, \omega'') = (D^W F)(\cdot, \omega'')$$

and

$$\delta^W(u(\cdot, \omega'')) = \delta^W(u)(\cdot, \omega'').$$

Proof : $H \in \mathcal{S}^W$ and $G \in L^\infty(\Omega_N)$. Then,

$$\mathbb{E}\left[G \int_0^T u_t D^W_t H dt\right] = \mathbb{E}\left[G\delta^W(u)H)\right].$$

Properties of D and δ

Lemma

Let $F \in \mathbb{D}^W_{1,2}(L^2(\Omega_N))$ *and* $u \in Dom\ \delta^W$. *Then, for almost all* $\omega'' \in \Omega_N$, $F(\cdot,\omega'') \in \mathbb{D}^W_{1,2}$, $u(\cdot,\omega'') \in Dom\ \delta^W \cap L^1(\Omega_W \times [0,T])$,

$$D^W F(\cdot,\omega'') = (D^W F)(\cdot,\omega'')$$

and

$$\delta^W(u(\cdot,\omega'')) = \delta^W(u)(\cdot,\omega'').$$

Proof : Consequently, for a.a. $\omega'' \in \Omega_N$,

$$\mathbb{E}_W\left[\int_0^T u_t(\cdot,\omega'') D^W_t H dt\right] = \mathbb{E}_W\left[\delta^W(u)(\cdot,\omega'')H(\cdot,\omega'')\right].$$

□

Hypothesis

$\mathbb{D}^W_{1,\infty}(L^2(\Omega_N))$ represents the elements F in $\mathbb{D}^W_{1,2}(L^2(\Omega_N))$ such that

$$||F||_{1,\infty} := ||F||_\infty + ||\,|D^W F|_2\,||_\infty < \infty.$$

Contents

1 Introduction

2 Preliminaries

3 Canonical Lévy Space

4 Malliavin Calculus

5 Girsanov Transformations

6 Anticipative Linear SDE on Canonical Wiener Space

7 Existence and Uniqueness of the Solution

Girsanov trannsformations

For $a \in L^2([0,T];\mathbb{D}^W_{1,\infty}(L^2(\Omega_N)))$, we consider $\{T_t : \Omega \to \Omega_W : 0 \le t \le T\}$ and $\{A_{s,t} : \Omega \to \Omega_W : 0 \le s \le t \le T\}$, which are the solutions of the equations

$$(T_t\,\omega)_\cdot = \omega'_\cdot + \int_0^{t\wedge\cdot} a_s(T_s\omega, \omega'')\, ds.$$

and

$$(A_{s,t}\,\omega)_\cdot = \omega'_\cdot - \int_{s\wedge\cdot}^{t\wedge\cdot} a_r(A_{r,t}\omega, \omega'')\, dr,$$

respectively.

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and

$$(A_{s,t}\,\omega)_{\cdot} = \omega'_{\cdot} - \int_{s\wedge\cdot}^{t\wedge\cdot} a_r(A_{r,t}\omega,\omega'')\,dr,$$

Proposition

Let $a \in L^2([0,T];\mathbb{D}^W_{1,\infty}(L^2(\Omega_N)))$. Then, there exist two unique families of absolutely continuous transformations $\{T_t : 0 \le t \le T\}$ and $\{A_{s,t} : 0 \le s \le t \le T\}$ that satisfy above equations. Moreover, for each $s,t \in [0,T]$, $s < t$, $A_{s,t} = T_s A_t$, with $A_t = A_{0,t}$, T_t is invertible with inverse A_t and $a_{\cdot}(T_{\cdot}(*,\omega''),\omega'') \in L^2([0,T];\mathbb{D}^W_{1,\infty})$, for a.a. $\omega'' \in \Omega_N$.

Auxiliary result

For an absolutely continuous functions ω' of Ω_W, with square-integrable derivatives,

$$|\omega'|_{CM} := \left(\int_0^T \dot{\omega}'(t)^2 dt \right)^{\frac{1}{2}}.$$

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Lemma

Let $a \in L^2([0, T]; \mathbb{D}^W_{1,\infty}(L^2(\Omega_N)))$. Then, for any $u \leq s \leq t$, we have

$$|A_{u,t}\omega - A_{u,s}\omega|^2_{CM} \leq 2 \left(\int_s^t ||a_r||^2_\infty dr \right) \exp \left\{ 2 \int_0^T || \, |D^W a_r|^2_2 \, ||_\infty dr \right\}.$$

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Proof Let $u \le s \le t$. Then,

$$\begin{aligned} & |A_{u,s}\omega - A_{u,t}\omega|^2_{CM} \\ = & \left| \int_{u\wedge\cdot}^{s\wedge\cdot} a_r(A_{r,s}\omega, \omega'')dr - \int_{u\wedge\cdot}^{t\wedge\cdot} a_r(A_{r,t}\omega, \omega'')dr \right|^2_{CM} \end{aligned}$$

Auxiliary result

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Let $a \in L^2([0,T]; \mathbb{D}^W_{1,\infty}(L^2(\Omega_N)))$. Then, for any $u \le s \le t$, we have

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Proof Let $u \le s \le t$. Then,

$$\begin{aligned}
&|A_{u,s}\omega - A_{u,t}\omega|^2_{CM} \\
&= \int_0^T |1\!\!1_{(u,s]}(r)a_r(A_{r,s}\omega,\omega'') - 1\!\!1_{(u,t]}(r)a_r(A_{r,t}\omega,\omega'')|^2 dr \\
&\le 2\int_s^t |a_r(A_{r,t}\omega,\omega'')|^2 dr + 2\int_u^s |a_r(A_{r,s}\omega,\omega'') - a_r(A_{r,t}\omega,\omega'')|^2 dr \\
&\le 2\int_s^t ||a_r||^2_\infty dr + 2\int_u^s || \, |D^W a_r|^2_2 \, ||_\infty |A_{r,s}\omega - A_{r,t}\omega|^2_{CM} dr.
\end{aligned}$$

Buckdahn result

Let

$$(T_t\,\omega)_\cdot = \omega'_\cdot + \int_0^{t\wedge\cdot} a_s(T_s\omega, \omega'')\; ds.$$

and

$$(A_{s,t}\,\omega)_\cdot = \omega'_\cdot - \int_{s\wedge\cdot}^{t\wedge\cdot} a_r(A_{r,t}\omega, \omega'')\; dr,$$

then

$$\mathbb{E}\left[F(A_{s,t}\omega, \omega'')L_{s,t}(\omega)\right] = \mathbb{E}\left[F\right]$$

and

$$\mathbb{E}\left[F(A_{s,t}\omega, \omega'')\right] = \mathbb{E}\left[F\mathcal{L}_{s,t}\right],$$

Buckdahn result

$$\mathbb{E}\left[F(A_{s,t}\omega,\omega'')L_{s,t}(\omega)\right]=\mathbb{E}\left[F\right]$$

and

$$\mathbb{E}\left[F(A_{s,t}\omega,\omega'')\right]=\mathbb{E}\left[F\mathcal{L}_{s,t}\right],$$

where

$$\begin{aligned}
L_{s,t}(\omega) &= \exp\left\{\int_s^t a_r(A_{r,t}\omega,\omega'')\delta W_r-\frac{1}{2}\int_s^t a_r^2(A_{r,t}\omega,\omega'')dr\right.\\
&\left.\quad-\int_s^t\int_r^t(D_u^W a_r)(A_{r,t}\omega,\omega'')D_r^W[a_u(A_{u,t}\omega,\omega'')]dudr\right\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_{s,t}(\omega) &= \exp\left\{-\int_s^t a_r(T_tA_r\omega,\omega'')\delta W_r-\frac{1}{2}\int_s^t a_r^2(T_tA_r\omega,\omega'')dr\right.\\
&\left.\quad-\int_s^t\int_s^r(D_u^W a_r)(T_tA_r\omega,\omega'')D_r^W[a_u(T_tA_u\omega,\omega'')]dudr\right\}
\end{aligned}$$

Contents

1 Introduction

2 Preliminaries

3 Canonical Lévy Space

4 Malliavin Calculus

5 Girsanov Transformations

6 Anticipative Linear SDE on Canonical Wiener Space

7 Existence and Uniqueness of the Solution

SDE on canonical Wiener space

On the canonical Wiener space, consider the linear stochastic differential equation

$$Z_t = Z_0 + \int_0^t h_s Z_s \, ds + \int_0^t a_s Z_s \, \delta W_s, \qquad t \in [0, T].$$

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Theorem

Assume $a \in L^2([0,T], \mathbb{D}^W_{1,\infty})$, $h \in L^1([0,T], L^\infty(\Omega))$ *and* $Z_0 \in L^\infty(\Omega)$. *Then, the process* $Z = \{Z_t : t \in [0,T]\}$ *defined by*

$$Z_t := Z_0(A_{0,t}) \exp\left\{ \int_0^t h_s(A_{s,t}) \, ds \right\} L_{0,t} \qquad (1)$$

belongs to $L^1(\Omega \times [0,T])$ *and is a solution of above equation. Conversely, if* $Y \in L^1(\Omega \times [0,T])$ *is a global solution of above equation and, if, moreover,* $a, h \in L^\infty(\Omega \times [0,T])$ *and* $D^W a \in L^\infty(\Omega \times [0,T]^2)$, *then* Y *is of the form (1).*

SDE on canonical Wiener space

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$$Z_t = Z_0 + \int_0^t h_s Z_s \, ds + \int_0^t a_s Z_s \, \delta W_s, \qquad t \in [0, T],$$

Theorem

Assume $Z_0 \in \mathbb{D}^W_{1,\infty}$, $h \in L^1([0, T], \mathbb{D}^W_{1,\infty})$ and that, for some $p > 2$,

$$a \in L^{2p}([0, T], \mathbb{D}^W_{1,\infty}) \cap L^2([0, T], \mathbb{D}^W_{2,\infty}).$$

Then,

$$Z_t := Z_0(A_{0,t}) \exp\left\{ \int_0^t h_s(A_{s,t}) \, ds \right\} L_{0,t}$$

has continuous trajectories a.s.

SDE on canonical Wiener space

Theorem

$$Z_t := Z_0(A_{0,t}) \exp\left\{ \int_0^t h_s(A_{s,t})\ ds \right\}\ L_{0,t}$$

has continuous trajectories a.s.

Idea of the proof :

$$\left| \int_0^t g_r(A_{r,t}) dr - \int_0^s g_r(A_{r,s}) dr \right|$$
$$\leq \int_s^t ||g_r||_\infty dr + \int_0^s ||\,|Dg_r|_2\,||_\infty |A_{r,t} - A_{r,s}|_{CM} dr$$
$$\leq \int_s^t ||g_r||_\infty dr + \left(\int_0^T ||\,|Dg_r|_2\,||_\infty dr \right) \sup_{r\leq s} |A_{r,t} - A_{r,s}|_{CM}.$$

Contents

1 Introduction

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3 Canonical Lévy Space

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Hypotheses

(H1) Assume that $X_0 \in \mathbb{D}^W_{1,\infty}(L^\infty(\Omega_N))$, $b, v_\cdot(y) \in L^1([0,T], \mathbb{D}^W_{1,\infty}(L^\infty(\Omega_N))$, for all $y \in \mathbb{R}_0$. Moreover, there exists $p > 2$ such that

$$a \in L^2([0,T], \mathbb{D}^W_{2,\infty}(L^\infty(\Omega_N)) \cap L^{2p}([0,T], \mathbb{D}^W_{1,\infty}(L^\infty(\Omega_N)).$$

(H2) There exist a positive function $g \in L^2(\mathbb{R}_0, \nu) \cap L^1(\mathbb{R}_0, \nu)$ such that

$$|v_s(y,\omega)| \leq g(y), \qquad \text{uniformly on } \omega \text{ and } s,$$

and

$$\lim_{|y|\to 0} g(y) = 0.$$

(H3) The function g satisfies $\int_{\mathbb{R}_0} (e^{2g(y)} - 1)\nu(dy) < \infty$.

Main result

Theorem

The process

$$
\begin{aligned}
X_t &= X_0(A_{0,t})L_{0,t} \prod_{s\le t, y\in\mathbb{R}_0} \Big[1+v_s(y,A_{s,t})\Delta N(s,y)\Big] \\
&\quad \times \exp\left\{ \int_0^t b_s(A_{s,t})\, ds - \int_0^t \int_{\mathbb{R}_0} v_s(y,A_{s,t})\, \nu(dy)\, ds \right\}
\end{aligned}
$$

is the unique solution in $L^1(\Omega\times[0,T])$ *of*

$$
X_t = X_0 + \int_0^t b_s X_s ds + \int_0^t a_s X_s \delta W_s + \int_0^t \int_{\mathbb{R}_0} v_s(y) X_{s-}\, d\tilde{N}(s,y),
$$

$t\in[0,T]$.

Proof : Step 1

Theorem

Let $\varepsilon > 0$. *Then, the process*

$$\begin{aligned} X_t^{\varepsilon} &= X_0(A_{0,t}) L_{0,t} \prod_{i=1}^{N_t^{\varepsilon}} \left[1 + v_{T_i^{\varepsilon}}(y_i^{\varepsilon}, A_{T_i^{\varepsilon},t}) \right] \\ &\quad \times \exp\left\{ \int_0^t b_s(A_{s,t})\, ds - \int_0^t \int_{\{|y|>\varepsilon\}} v_s(y, A_{s,t})\, \nu(dy)\, ds \right\} \end{aligned}$$

is the unique solution in $L^1(\Omega \times [0,T])$ *of*

$$X_t^{\varepsilon} = X_0 + \int_0^t b_s X_s^{\varepsilon} ds + \int_0^t a_s X_s^{\varepsilon} \delta W_s + \int_0^t \int_{\{|y|>\varepsilon\}} v_s(y) X_{s-}^{\varepsilon}\, d\tilde{N}(s,y),$$

$t \in [0, T]$.

Proof of Step 1 : Existence

Let

$$\Phi_s^{\varepsilon}(\omega) = \prod_{r \leq s, \varepsilon < |y|} \Big[1 + v_r(y, T_r) \Delta N(r, y) \Big].$$

Proof of Step 1 : Existence

Let

$$\Phi_s^{\varepsilon}(\omega) = \prod_{r \le s, \varepsilon < |y|} \Big[1 + v_r(y, T_r)\Delta N(r, y)\Big].$$

Due to Girsanov's theorem and that $A_{r,t} = T_r \circ A_t$, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t a_s X_s^{\varepsilon} D_s^W G ds\right] \\
&= \mathbb{E}\left[\int_0^t a_s X_0(A_{0,s}) \exp\left\{\int_0^s b_r^{\varepsilon}(A_{r,s}) dr\right\} L_{0,s} \Phi_s^{\varepsilon}(A_s) D_s^W G ds\right] \\
&= \mathbb{E}\left[\int_0^t a_s(T_s) X_0 \exp\left\{\int_0^s b_r^{\varepsilon}(T_r) dr\right\} \Phi_s^{\varepsilon} \, (D_s^W G)(T_s) ds\right].
\end{aligned}$$

Proof of Step 1 : Existence

Due to Girsanov's theorem and that $A_{r,t} = T_r \circ A_t$, we have

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\begin{aligned}
&\mathbb{E}\left[\int_0^t a_s X_s^\varepsilon D_s^W G ds\right] \\
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&= \mathbb{E}\left[\int_0^t a_s(T_s) X_0 \exp\left\{\int_0^s b_r^\varepsilon(T_r)dr\right\}\Phi_s^\varepsilon\, (D_s^W G)(T_s) ds\right].
\end{aligned}
$$

Since $\frac{d}{ds}G(T_s) = a_s(T_s)(D_s^W G)(T_s)$, we get

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t a_s X_s^\varepsilon D_s^W G ds\right] \\
&= \mathbb{E}\left[\int_0^t \left(\frac{d}{ds}G(T_s)\right) X_0 \exp\left\{\int_0^s b_r^\varepsilon(T_r)dr\right\}\Phi_s^\varepsilon ds\right].
\end{aligned}
$$

Proof of Step 1 : Existence

$$\mathbb{E}\left[\int_0^t a_s X_s^\varepsilon D_s^W G ds\right] = \mathbb{E}\left[\int_0^t \left(\frac{d}{ds}G(T_s)\right) X_0 \exp\left\{\int_0^s b_r^\varepsilon(T_r)dr\right\}\Phi_s^\varepsilon ds\right].$$

Using $\int_0^t \int_{\{|y|>\varepsilon\}} dN(s,y) < \infty$ a.s., we have

$$\mathbb{E}\left[\int_0^t a_s X_s^\varepsilon D_s^W G ds\right] = \sum_{i=1}^{\infty} \mathbb{E}\left[\int_{\tau_{i-1}^\varepsilon \wedge t}^{\tau_i^\varepsilon \wedge t} \left(\frac{d}{ds}G(T_s)\right) X_0 \exp\left\{\int_0^s b_r^\varepsilon(T_r)dr\right\}\Phi_{\tau_{i-1}^\varepsilon \wedge t}^\varepsilon ds\right].$$

Proof of Step 1 : Existence

Finally, integration by parts and the fact that

$$\Phi^{\varepsilon}_{\tau^{\varepsilon}_i} = \Phi^{\varepsilon}_{\tau^{\varepsilon}_{i-1}}(1 + v_{\tau^{\varepsilon}_i}(y^{\varepsilon}_i, T_{\tau^{\varepsilon}_i}))$$

imply

$$\begin{aligned} \mathbb{E}\left[\int_0^t a_s X^{\varepsilon}_s D^W_s G ds\right] &= \mathbb{E}\Bigg[G\Big(X^{\varepsilon}_t - X_0 - \int_0^t b_s X^{\varepsilon}_s ds \\ &\quad - \int_0^t \int_{\{|y|>\varepsilon\}} v_s(y) X^{\varepsilon}_{s-} \, d\tilde{N}(s,y)\Big)\Bigg]. \end{aligned}$$

Proof : Step 1

Theorem

Let $\varepsilon > 0$. *Then, the process*

$$X_t^{\varepsilon} = X_0(A_{0,t})L_{0,t}\prod_{i=1}^{N_t^{\varepsilon}}\left[1 + v_{\tau_i^{\varepsilon}}(y_i^{\varepsilon}, A_{\tau_i^{\varepsilon},t})\right]$$
$$\times \exp\left\{\int_0^t b_s(A_{s,t})\, ds - \int_0^t \int_{\{|y|>\varepsilon\}} v_s(y, A_{s,t})\, \nu(dy)\, ds\right\}$$

is the unique solution in $L^1(\Omega \times [0, T])$ *of*

$$X_t^{\varepsilon} = X_0 + \int_0^t b_s X_s^{\varepsilon} ds + \int_0^t a_s X_s^{\varepsilon} \delta W_s + \int_0^t \int_{\{|y|>\varepsilon\}} v_s(y) X_{s-}^{\varepsilon}\, d\tilde{N}(s, y),$$

$t \in [0, T]$.

Proof of Step 1 : Uniqueness

Let Y^{ε} be a solution to

$$X_t^{\varepsilon} = X_0 + \int_0^t b_s X_s^{\varepsilon} ds + \int_0^t a_s X_s^{\varepsilon} \delta W_s + \int_0^t \int_{\{|y|>\varepsilon\}} v_s(y) X_{s-}^{\varepsilon} \, d\tilde{N}(s,y).$$

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Then, for $t \in [\tau_1^{\varepsilon}, \tau_2^{\varepsilon})$,

$$\mathbb{E}\left[Y_t^{\varepsilon} G(A_{\tau_1^{\varepsilon},t})\right] = \mathbb{E}\left[X_{\tau_1^{\varepsilon}}^{\varepsilon} G\right] + \mathbb{E}\left[\int_{\tau_1^{\varepsilon}}^t b_s^{\varepsilon} Y_s^{\varepsilon} G(A_{\tau_1^{\varepsilon},s}) ds\right].$$

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Then, for $t \in [\tau_1^{\varepsilon}, \tau_2^{\varepsilon})$,

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Thus,

$$\mathbb{E}\left[Y_t^{\varepsilon}(T_{\tau_1^{\varepsilon},t})\mathcal{L}_{\tau_1^{\varepsilon},t} G\right] = \mathbb{E}\left[X_{\tau_1^{\varepsilon}}^{\varepsilon} G\right] + \mathbb{E}\left[\int_{\tau_1^{\varepsilon}}^t b_s^{\varepsilon}(T_{\tau_1^{\varepsilon},s}) Y_s^{\varepsilon}(T_{\tau_1^{\varepsilon},s})\mathcal{L}_{\tau_1^{\varepsilon},s} G ds\right].$$

Proof of Step 1 : Uniqueness

For $t \in [\tau_1^\varepsilon, \tau_2^\varepsilon)$,

$$\mathbb{E}\left[Y_t^\varepsilon G(A_{\tau_1^\varepsilon,t})\right] = \mathbb{E}\left[X_{\tau_1^\varepsilon}^\varepsilon G\right] + \mathbb{E}\left[\int_{\tau_1^\varepsilon}^t b_s^\varepsilon Y_s^\varepsilon G(A_{\tau_1^\varepsilon,s})ds\right].$$

Thus,

$$\mathbb{E}\left[Y_t^\varepsilon(T_{\tau_1^\varepsilon,t})\mathcal{L}_{\tau_1^\varepsilon,t}G\right] = \mathbb{E}\left[X_{\tau_1^\varepsilon}^\varepsilon G\right] + \mathbb{E}\left[\int_{\tau_1^\varepsilon}^t b_s^\varepsilon(T_{\tau_1^\varepsilon,s})Y_s^\varepsilon(T_{\tau_1^\varepsilon,s})\mathcal{L}_{\tau_1^\varepsilon,s}Gds\right].$$

$$Y_t^\varepsilon(T_{\tau_1^\varepsilon,t})\mathcal{L}_{\tau_1^\varepsilon,t} = X_{\tau_1^\varepsilon}^\varepsilon + \int_{\tau_1^\varepsilon}^t b_s^\varepsilon(T_{\tau_1^\varepsilon,s})Y_s^\varepsilon(T_{\tau_1^\varepsilon,s})\mathcal{L}_{\tau_1^\varepsilon,s}ds.$$

Main result

Theorem

The process

$$X_t = X_0(A_{0,t})L_{0,t} \prod_{s\le t, y\in\mathbb{R}_0} \left[1 + v_s(y, A_{s,t})\Delta N(s,y)\right]$$
$$\times \exp\left\{ \int_0^t b_s(A_{s,t})\, ds - \int_0^t \int_{\mathbb{R}_0} v_s(y, A_{s,t})\, \nu(dy)\, ds \right\}$$

is the unique solution in $L^1(\Omega \times [0,T])$ *of*

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$t \in [0, T]$.

Step 2

We have

$$X^{\varepsilon} \to X \quad \text{in } L^1(\Omega \times [0, T]).$$

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$$\begin{aligned} \mathbb{E}\left[\int_0^t a_s X_s^{\varepsilon} D_s^W G ds\right] &= \mathbb{E}\bigg[G\Big(X_t^{\varepsilon} - X_0 - \int_0^t b_s X_s^{\varepsilon} ds \\ &\quad - \int_0^t \int_{\{|y|>\varepsilon\}} v_s(y) X_{s-}^{\varepsilon} \, d\tilde{N}(s, y)\Big)\bigg]. \end{aligned}$$

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Then,

$$\begin{aligned}\mathbb{E}\left[\int_0^t a_s X_s D_s^W G ds\right] &= \mathbb{E}\Bigg[G\Big(X_t - X_0 - \int_0^t b_s X_s ds \\ &\quad - \int_0^t \int_{\{|y|>0\}} v_s(y) X_{s-} d\tilde{N}(s,y)\Big)\Bigg].\end{aligned}$$