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Monique Jeanblanc,
Université d'Évry-Val-D'Essonne

Random times with a given Azéma supermartingale

Joint work with S. Song

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Begin at the beginning, and go on till you come to the end. Then, ...

Lewis Carroll, Alice's Adventures in Wonderland

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Problem

Motivation: In credit risk, in mathematical finance, one works with random times which represent the default times. Many studies are based on the intensity process: starting with a reference filtration $\mathbb{F}$, the intensity process of τ is the $\mathbb{F}$ **predictable increasing process** Λ (the dual predictable projection of $\mathbb{1}_{\tau \cdot t}$) such that

$$\mathbb{1}_{\tau \cdot t} - \Lambda_{t \wedge \tau}$$

is a $\mathbb{G}$-martingale, where $\mathcal{G}_t = \cap_{\epsilon>0}\mathcal{F}_{t+\epsilon} \vee \sigma(\tau \wedge (t+\epsilon))$.

Then, the problem is : given Λ, construct a random time τ which admits Λ as intensity.

The classical construction is: extend the probability space $(\Omega, \mathbb{F}, \mathbb{P})$ so that there exists a random variable Θ, with exponential law, **independent of $\mathcal{F}_1$** and define

$$\tau := \inf\{t : \Lambda_t \geq \Theta\}$$

Then,

$$\mathbb{P}(\tau > \theta | \mathcal{F}_t) = \mathbb{P}(\Lambda_\theta < \Theta | \mathcal{F}_t) = e^{i\,\Lambda_\theta}, \quad \theta \leq t$$

and, in particular

$$\mathbb{P}(\tau > t | \mathcal{F}_t) = e^{i\,\Lambda_t}$$

Moreover $\mathbb{1}_{\tau \cdot t} - \Lambda_{t \wedge \tau}$ is a $\mathbb{G}$ martingale

Moreover, under this construction, one can show that any $\mathbb{F}$ martingale is a $\mathbb{G}$ martingale: this is the so-called immersion hypothesis.

Our goal is to provide other constructions. One starts with noting that, in general,

$$Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$$

is a supermartingale (called the Azéma supermartingale) with **multiplicative decomposition $Z_t = N_t D_t$**, where N is a local martingale and D a decreasing predictable process. In this talk, **we assume that Z does not vanishes and D is continuous so that $D_t = e^{i\,\Lambda_t}$** for some continuous increasing process Λ. In that case, **the continuous process Λ is the intensity of τ.**

Problem ($\star$): let $(\Omega, \mathbb{F}, \mathbb{P})$ be a filtered probability space, Λ an increasing predictable process, N a non-negative local martingale such that, for $t > 0$,

$$0 < Z_t := N_t e^{\,i\,\Lambda_t} \leq 1$$

Construct, on the **canonical** extended space $(\Omega \times [0, \infty])$, the **canonical map** τ (such that $\tau(\omega, t) = t$) and a probability $\mathbb{Q}$ such that

1. **restriction condition** $\mathbb{Q}|_{F_\infty} = \mathbb{P}|_{F_\infty}$
2. **projection condition** $\mathbb{Q}[\tau > t|\mathcal{F}_t] = N_t e^{\,i\,\Lambda_t}$

We shall note $\mathbb{P}(X) := \mathbb{E}_{\mathbb{P}}(X)$. **We assume that** $Z_0 = 1$.

Related work

Nikeghbali, A. and Yor, M. (2006) Doob's maximal identity, multiplicative decompositions and enlargements of filtrations, *Illinois Journal of Mathematics*, 50, 791-814.

In that paper, given a supermartingale of the form $Z_t = \frac{N_t}{\sup_{s \le t} N_s}$ where N is a continuous local $\mathbb{F}$-martingale which goes to 0 at infinity, the authors show that $\mathbb{P}(g > t|\mathcal{F}_t) = Z_t$, where $g = \sup\{t \ : Z_t = 1\}$.

Open problem in our setting: characterize Z so that τ can be constructed on Ω

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Open problem in our setting: characterize Z so that τ can be constructed on Ω

Particular case: $Z = e^{i\,\Lambda}$.

In that case a solution is $\mathbb{Q} = \mathbb{Q}^C$ where, for $A \in \mathcal{F}_1$:

$$\mathbb{Q}^C(A \cap \{s < \tau \leq t\}) = \mathbb{P}\left(\mathbb{1}_A \int_s^t e^{i\,\Lambda_u} d\Lambda_u\right)$$

so that (as in the classical Cox process model)

$$\mathbb{Q}^C(\tau > \theta | \mathcal{F}_t) = e^{i\,\Lambda_\theta}, \text{ for } t \geq \theta$$

Outline of the talk

- Increasing families of martingales
- Semi-martingale decompositions
- Predictable Representation Theorem
- Exemple

The link between the supermartingale Z and the conditional law $\mathbb{Q}(\tau \in du|\mathcal{F}_t)$ for $u \leq t$ is: Let $M_t^u = \mathbb{Q}(\tau \leq u|\mathcal{F}_t)$, then M is increasing w.r.t. u and

$$\begin{aligned} M_u^u &= 1 - Z_u \\ M_t^u &\leq M_t^t = 1 - Z_t \end{aligned}$$

Note that, for $t < u$, $M_t^u = \mathbb{E}(1 - Z_u|\mathcal{F}_t)$.

Solving the problem ($\star$) is equivalent to find a family M^u. The solution of problem ($\star$) is not unique.

Given an iM_Z, let $d_u M^u_1$ be the random measure on $(0,\infty)$ associated with the increasing map $u \to M^u_1$. The following probability measure $\mathbb{Q}$ is a solution of the problem ($\star$)

$$\mathbb{Q}(F) := \mathbb{P}\left(\int_{[0,1]} F(u,\cdot)\big(M^0_1\,\delta_0(du) + d_u M^u_1 + (1 - M^1_1)\delta_1(du)\big)\right)$$

The two properties for $\mathbb{Q}$:

- **Restriction condition**: For $B \in \mathcal{F}_1$,

$$\mathbb{Q}(B) = \mathbb{P}\left(\mathbb{I}_B \int_{[0,1]} (M^0_1\,\delta_0(du) + d_u M^u_1 + (1 - M^1_1)\delta_1(du))\right) = \mathbb{P}[B]$$

- **Projection condition:** For $0 \le t < \infty$, $A \in \mathcal{F}_t$,

$$\mathbb{Q}[A \cap \{\tau \le t\}] = \mathbb{P}[\mathbb{I}_A M^t_1] = \mathbb{P}[\mathbb{I}_A M^t_t] = \mathbb{Q}[\mathbb{I}_A(1 - Z_t)]$$

are satisfied.

Family iM_Z

An increasing family of positive martingales bounded by $1 - Z$ (in short iM_Z) is a family of processes $(M^u : 0 < u < \infty)$ satisfying the following conditions:

1. Each M^u is a càdlàg $\mathbb{P}$-$\mathbb{F}$ **martingale** on $[u, \infty]$.
2. For any u, the martingale M^u is positive and closed by $M^u_\infty = \lim_{t\to\infty} M^u_t$.
3. For each fixed t, $0 < t \leq \infty$, $u \in [0, t] \to M^u_t$ **is a right continuous increasing map**.
4. $M^u_u = 1 - Z_u$ and $M^u_t \leq M^t_t = 1 - Z_t$ for $u \leq t \leq \infty$.

$M^u_t = \mathbb{Q}(\tau < u | \mathcal{F}_t), \quad Z_t = 1 - M^t_t = N_t e^{-\Lambda_t}$

Constructions of iM_Z

Hypothesis (✠) For all $0 < t < \infty$, $0 \leq Z_t < 1, 0 \leq Z_{t-} < 1$.

The simplest iM_Z

Under conditions (✠), the family

$$M_t^u := (1 - Z_t) \exp\left(- \int_u^t \frac{Z_s}{1 - Z_s} d\Lambda_s \right) \quad 0 < u < \infty, u \leq t \leq \infty,$$

defines an iM_Z, called **basic solution**. We note that

$$dM_t^u = -M_{t-}^u \frac{e^{-\Lambda_t}}{1 - Z_{t-}} dN_t, \; 0 < u \leq t < \infty.$$

Let us recall that, to construct an iM_Z, we should respect four constraints :

$i.\ M^u_u = (1 - Z_u)$

$ii.\ 0 \leq M^u$

$iii.\ M^u \leq 1 - Z$

$iv.\ M^u \leq M^v$ for $u < v$

These constraints are particularly "easy" to handle if M^u are solutions of a SDE:
The constraint i indicates the initial condition;
the constraint ii means that we must take an exponential SDE;
the constraint iv is a comparison theorem for one dimensional SDE,
the constraint iii can be handled by local time as described in the following result :

Let m be a $(\mathbb{P}, \mathbb{F})$-local martingale such that $m_u \leq 1 - Z_u$. Then, $m_t \leq (1 - Z_t)$ on $t \in [u, \infty)$ if and only if the local time at zero of $m - (1 - Z)$ on $[u, \infty)$ is identically null.

Other solutions when $1 - Z > 0$

Hypothesis (✠✠):

1. For all $0 < t < \infty$, $0 \leq Z_t < 1, 0 \leq Z_{t\,¡}\ < 1$.
2. All $\mathbb{P}$-$\mathbb{F}$ martingales are continuous.

Assume (✠✠). Let Y be a $(\mathbb{P}, \mathbb{F})$ local martingale and f be a bounded Lipschitz function with $f(0) = 0$. For any $0 \leq u < \infty$, we consider the equation

$$(\heartsuit_u) \left\{ \begin{array}{rcl} dX_t & = & X_t \left(-\frac{e^{i\,\Lambda_t}}{1 - Z_t} dN_t + f(X_t - (1 - Z_t)) dY_t \right), \; u \leq t < \infty \\ X_u & = & x \end{array} \right.$$

Hypothesis (✠✠):

1. For all $0 < t < \infty$, $0 \leq Z_t < 1, 0 \leq Z_{t\,\text{¡}} < 1$ (strictly smaller than 1).
2. All $\mathbb{P}$-$\mathbb{F}$ martingales are continuous.

Assume (✠✠). Let Y be a $(\mathbb{P}, \mathbb{F})$ local martingale and f be a bounded Lipschitz function with $f(0) = 0$. For any $0 \leq u < \infty$, we consider the equation

$$(\heartsuit_u) \begin{cases} dX_t &= X_t \left(-\frac{e^{\text{¡}\,\Lambda_t}}{1 - Z_t} dN_t + f(X_t - (1 - Z_t))dY_t \right), \; u \leq t < \infty \\ X_u &= x \end{cases}$$

Let M^u be the solution on $[u, \infty)$ of the equation $(\heartsuit_u)$ with initial condition $M^u_u = 1 - Z_u$. Then, $(M^u, u \leq t < \infty)$ defines an iM_Z.

$M^u_t = \mathbb{Q}(\tau < u|\mathcal{F}_t), \quad Z_t = 1 - M^t_t = N_t e^{-\Lambda_t}$

Proof

• Inequality $M^u \leq 1 - Z$ on $[u, \infty)$ is satisfied if the local time of $\Delta = M^u - (1 - Z)$ at zero is null. This is the consequence of the following estimation:

$$\begin{aligned}
d\langle\Delta\rangle_t &= \Delta_t^2 \left(\frac{e^{i\,\Lambda_t}}{1 - Z_t}\right)^2 d\langle N\rangle_t + M_t^2 f^2(\Delta_t) d\langle Y\rangle_t - 2\Delta_t \frac{e^{i\,\Lambda_t}}{1 - Z_t} M_t f(\Delta_t) d\langle N, Y\rangle_t \\
&\leq 2\Delta_t^2 \left(\frac{e^{i\,\Lambda_t}}{1 - Z_t}\right)^2 d\langle N\rangle_t + 2M_t^2 f^2(\Delta_t) d\langle Y\rangle_t \\
&\leq 2\Delta_t^2 \left(\frac{e^{i\,\Lambda_t}}{1 - Z_t}\right)^2 d\langle N\rangle_t + 2M_t^2 K^2 \Delta_t^2 d\langle Y\rangle_t
\end{aligned}$$

From this, we can write

$$\int_0^t \mathbb{I}_{\{0<\Delta_s<\epsilon\}} \frac{1}{\Delta_s^2} d\langle\Delta\rangle_s < \infty, \; 0 < \epsilon, 0 < t < \infty$$

and get the result according to Revuz-Yor.

• Inequality $M^u \leq M^v$ on $[v, \infty)$ when $u < v$. The comparison theorem holds for SDE(♡). We note also that M^u and M^v satisfy the same SDE(♡) on $[v, \infty)$. So, since $M_v^u \leq (1 - Z_v) = M_v^v$, $M_t^u \leq M_t^v$ for all $t \in [v, \infty)$.

A specific case is when $N = 1$, so that $Z_t = e^{i\,\Lambda_t}$ is decreasing. The basic solution is the standard one, but for a general solution (for example $f(x) = x$), we obtain a non standard construction. The random time τ is then a **pseudo-stopping time**, and for any bounded $\mathbb{F}$ martingale

$$\mathbb{E}(m_\tau) = m_0$$

(note that, in general, τ is NOT a stopping time)

Case when $1 - Z$ can reach zero

We introduce $\mathcal{Z} = \{s : 1 - Z_s = 0\}$ and, for $t \in (0, \infty)$, the random time

$$g_t \quad := \quad \sup\{0 \le s \le t : s \in \mathcal{Z}\}$$

Hypothesis($\mathcal{Z}$) The set $\mathcal{Z}$ is not empty and is closed.
The measure $d\Lambda$ has a decomposition $d\Lambda_s = dV_s + dA_s$ where V, A are continuous increasing processes such that dV charges only $\mathcal{Z}$ while dA charges its complementary $\mathcal{Z}^c$.

Let, for $0 < u \le t \le \infty$

$$M_t^u = (1 - Z_u) - \int_u^t \mathbb{I}_{\{g_s \cdot u\}} \exp\left(-\int_u^s \frac{Z_v}{1 - Z_v} dA_v\right) e^{i\,\Lambda_s} dN_s$$

The family $(M^u : 0 \le u < \infty)$ defines an iM_Z.

Proof indication

We introduce

$$M_t^u = \mathbb{I}_{\{g_t \cdot u\}} \exp\left(-\int_u^t \frac{Z_s}{1-Z_s} dA_s\right)(1-Z_t), \; 0 < u < \infty, u \leq t \leq \infty.$$

(Balayage Formula.) Let Y be a continuous semi-martingale and define

$$g_t = \sup\{s \leq t : Y_s = 0\},$$

with the convention $\sup\{\emptyset\} = 0$. Then

$$h_{g_t} Y_t = h_0 Y_0 + \int_0^t h_{g_s} dY_s$$

for every predictable, locally bounded process h.

We need only to prove that each M^u satisfies the above equation, and therefore, that M^u is a local $\mathbb{P}$-$\mathbb{F}$ martingale. Let

$$E_t^u = \exp\left(-\int_u^t \frac{Z_s}{1-Z_s} dA_s\right)$$

Then,

$$d\left(E_t^u(1-Z_t)\right) = E_t^u\left(-e^{\,i\,\Lambda_t} dN_t + Z_t dV_t\right)$$

We apply the balayage formula and we obtain

$$\begin{aligned} M_t^u &= \mathbb{I}_{f g_t \cdot u g} E_t^u (1-Z_t) \\ &= \mathbb{I}_{f g_t \cdot u g}(1-Z_u) + \int_u^t \mathbb{I}_{f g_s \cdot u g} E_s^u\left(-e^{\,i\,\Lambda_s} dN_s + Z_s dV_s\right) \\ &= (1-Z_u) - \int_u^t \mathbb{I}_{f g_s \cdot u g} E_s^u e^{\,i\,\Lambda_s} dN_s \end{aligned}$$

Semimartingale decomposition formula for the models constructed with SDE(♡), in the case $1 - Z > 0$

We suppose **Hy(✠✠)**, $Z_{1} = 0$ and that the map $u \to M_t^u$ is continuous on $[0, t]$, where M^u is solution of the generating equation $(\heartsuit)$: $0 \le u < \infty$,

$$(\heartsuit_u)\left\{\begin{array}{rcl} dM_t &=& M_t\left(-\frac{e^{-\Lambda_t}}{1 - Z_t}dN_t + f(M_t - (1 - Z_t))dY_t\right), \quad u \le t < \infty \\ M_u &=& 1 - Z_u \end{array}\right.$$

Let X be a $\mathbb{P}$-$\mathbb{F}$ local martingale. Then the process

$$\widetilde{X}_t = X_t - \int_0^t \mathbb{1}_{\{s \le \tau\}} \frac{e^{-\Lambda_s}}{Z_s} d\langle N, X\rangle_s + \int_0^t \mathbb{1}_{\{\tau < s\}} \frac{e^{-\Lambda_s}}{1 - Z_s} d\langle N, X\rangle_s$$

$$- \int_0^t \mathbb{1}_{\{\tau < s\}} (f(M_s^\tau - (1 - Z_s)) + M_s^\tau f'(M_s^\tau - (1 - Z_s)))d\langle Y, X\rangle_s$$

is a $\mathbb{Q}$-$\mathbb{G}$-local martingale.

$M_t^u = \mathbb{Q}(\tau < u | \mathcal{F}_t), \quad Z_t = 1 - M_t^t = N_t e^{-\Lambda_t}$

Semimartingale decomposition formula in the case of eventual $1 - Z = 0$

We suppose $\mathbf{Hy}(\mathcal{Z})$. We consider the iM_Z constructed above and its associated probability measure $\mathbb{Q}$ on $[0, \infty] \times \Omega$. Let $g = lim_{t \to \infty} g_t$.

Let X be a $(\mathbb{P}, \mathbb{F})$-local martingale. Then

$$X_t - \int_0^t \mathbb{1}_{\{s \le g \vee \tau\}} \frac{e^{-\Lambda_s}}{Z_{s-}} d\langle N, X\rangle_s + \int_0^t \mathbb{1}_{\{g \vee \tau < s\}} \frac{e^{-\Lambda_s}\ d\langle N, X\rangle_s}{1 - Z_{s-}},\ 0 \le t < \infty,$$

is a $(\mathbb{Q}, \mathbb{G})$-local martingale.

Predictable Representation Property

Assume ✠✠ and that

1. there exists an $(\mathbb{P}, \mathbb{F})$-martingale m which admits the $(\mathbb{P}, \mathbb{F})$-Predictable Representation Property
2. The martingales N and Y are orthogonal

Let $\tilde{m}$ be the $(\mathbb{P}, \mathbb{G})$-martingale part of the $(\mathbb{P}, \mathbb{G})$-semimartingale m.
Then, $(\tilde{m}, M)$ enjoys the $(\mathbb{Q}, \mathbb{G})$-Predictable Representation Property where
$M_t = 1\!\!1_{\tau \cdot t} - \Lambda_{t \wedge \tau}$.

Example

Let φ is the standard Gaussian density and Φ the Gaussian cumulative function, $\mathbb{F}$ generated by a Brownian motion B.

Let $X = \int_0^1 f(s)dB_s$ where f is a deterministic, square-integrable function and $Y = \psi(X)$ where ψ is a positive and strictly increasing function. Then,

$$\mathbb{P}(Y \leq u|\mathcal{F}_t) = \mathbb{P}\Big(\int_t^1 f(s)dB_s \leq \psi^{-1}(u) - m_t|\mathcal{F}_t\Big)$$

where $m_t = \int_0^t f(s)dB_s$ is $\mathcal{F}_t$-measurable. It follows that

$$M_t^u := \mathbb{P}(Y \leq u|\mathcal{F}_t) = \Phi\Big(\frac{\psi^{-1}(u) - m_t}{\sigma(t)}\Big)$$

The family M_t^u is then a family of iM_Z martingales which satisfies

$$dM_t^u = -\varphi\left(\Phi^{-1}(M_t^u)\right)\frac{f(t)}{\sigma(t)}dB_t$$

The multiplicative decomposition of $Z_t = N_t \exp\left(-\int_0^t \lambda_s ds\right)$ where

$$
\begin{aligned}
dN_t &= N_t \frac{\varphi(Y_t)}{\sigma(t)\Phi(Y_t)} dm_t, \quad \lambda_t = \frac{h^0(t)\,\varphi(Y_t)}{\sigma(t)\,\Phi(Y_t)} \\
Y_t &= \frac{m_t - \psi^{i\,1}(t)}{\sigma(t)}
\end{aligned}
$$

The basic martingale satisfies

$$dM_t^u = -M_t^u \frac{f(t)\varphi(Y_t)}{\sigma(t)\Phi(-Y_t)} dB_t.$$

Jeanblanc, M. and Song, S. (2010)
Explicit Model of Default Time with given Survival Probability. *Stochastic Processes and their Applications*
Default times with given survival probability and their $\mathbb{F}$-martingale decomposition formula. *Stochastic Processes and their Applications*

Li, L. and Rutkowski, M. (2010) Constructing Random Times Through Multiplicative Systems, Preprint.

In that paper, the authors give a solution to the problem $(\star)$, based on **Meyer, P.A. (1967)**: On the multiplicative decomposition of positive supermartingales. In: *Markov Processes and Potential Theory*, J. Chover, ed., J. Wiley, New York, pp. 103–116.

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Lewis Carroll, Alice's Adventures in Wonderland

Thank you for your attention