Markovian sports: Tennis vs. Volleyball

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The model

In this talk I will present a simple model for both the score of a tennis and a volleyball match.

I will assume that the probability that a player wins each point is **constant** during the match, **independent** from the other points played and **depends** just on the fact that the player serves or returns the serve.

So, calling the two players A and B, we will define two parameters p_A and p_B which represents, respectively, the probabilities of winning a rally when the player A or B serves.

To avoid trivial cases, we will always assume that $0 < p_A < 1$ and $0 < p_B < 1$.

The score in a tennis game is divided into two levels: the games and the sets.

In each game just one player serves and the other player serves in the following one. The score is: 0-15-30-40-deuce-win.

A player wins a set if he/she wins 6 games before the other player wins more then 4 games, or wins 7 games and the other 5, or, if they arrive to 6 to 6 the set is assigned by a final “long” deciding game, called tie break.

A player wins a match in most of the tournaments if he/she is the first who wins 2 sets, or, in the grand slam tournament for the men, if he is the first to win 3 sets.

By the independence assumption, we are able to consider independently the games forming the set, since the probability to win a set is equal to the product of the probabilities to win the needed games and the same for the sets in the match.

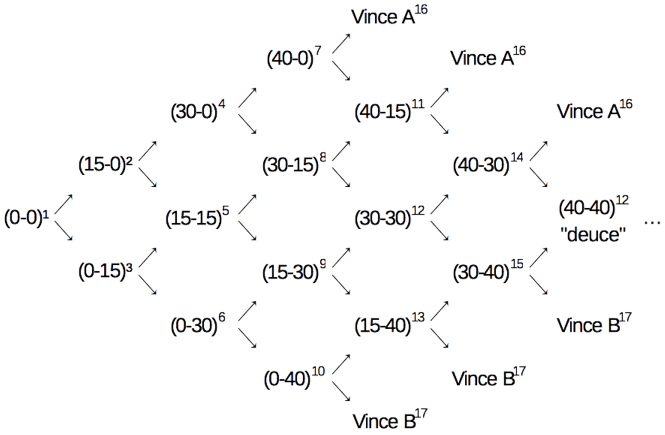
Winning probabilities: Game

Let as assume that player A serves.

By the independence assumption we can model the score of the game with a discrete-time Markov chain X_n with state space

$$S = \Big\{(i,j) : i \in \{0,15,30,40\}, j \in \{0,15,30,40\}\Big\} \cup$$

$$\cup \Big\{Win\ A, Win\ B\Big\} \setminus \Big\{(40,40)\Big\}$$


Vince A16
(40-0)7
Vince A16
(30-0)4
(40-15)11
Vince A16
(15-0)2
(30-15)8
(40-30)14
(0-0)1
(15-15)5
(30-30)12
(40-40)12
"deuce"
...
(0-15)3
(15-30)9
(30-40)15
(0-30)6
(15-40)13
Vince B17
(0-40)10
Vince B17
Vince B17

G =

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	0	p	q	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	p	q	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	p	q	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	p	q	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	p	q	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	p	q	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	q	0	0	0	0	p	0
8	0	0	0	0	0	0	0	0	0	0	p	q	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	p	q	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	p	0	0	0	q
11	0	0	0	0	0	0	0	0	0	0	0	0	0	q	0	p	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	p	q	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	p	0	q
14	0	0	0	0	0	0	0	0	0	0	0	q	0	0	0	p	0
15	0	0	0	0	0	0	0	0	0	0	0	p	0	0	0	0	q
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

In order to compute the probability that player A wins the game (remember we are assuming that he/she is serving in this game) we can evaluate the probability that the Markov Chain X_n starting from the state (0,0) arrives to the (absorbing) state Win A.

This can be done applying the well known results that follow.

Let C be a subset of the state space S, the hitting time of C is the random variable

$$H^C(\omega) = \inf\left\{n \geq 0 : X_n(\omega) \in C\right\}$$

The probability starting from $i \in S$ that X_n ever hits C is then

$$h_i^C = P_i(H^C < \infty).$$

When C is a closed class (or an absorbing state), h_i^C is called the absorption probability.

In the preset case we have therefore to evaluate

$$h_{(0,0)}^{\{Win\ \ A\}}$$

This can be done by applying the following well known result:

Proposition: The vector of the hitting probabilities $h^C = (h_i^C, i \in S)$ is the minimal non-negative solution to the system of linear equations

$$\begin{cases} h_i^C = 1 & \text{for } i \in C \\ h_i^C = \sum_{j \in S} p_{ij} h_j^C & \text{for } i \notin C \end{cases}$$

Even for the case of the game, this is not so simple to apply the previous results, since in this case the matrix P has 17×17 entries.

A more direct approach is the following one (see [2]): denote by 0,1,2,3, 4 the scores 0, 15, 30, 40, Win A define by p_A^G the probability that A wins a serving game and by $p_A^G(i,j)$ the probability that this game arrives to the score (i,j).

It easy to see that

$$p_A^G = \sum_{j=0}^{2} p_A^G(4,j) + p_A^G(3,3)p_{Adv}^G$$

where p_{Adv}^G denotes the probability that player A wins the final tie break of the game.

By simple combinatorial computations, one gets

$$p_{Adv}^{G} = p_A^2[1 - 2p_A(1 - p_A)]^{-1}$$

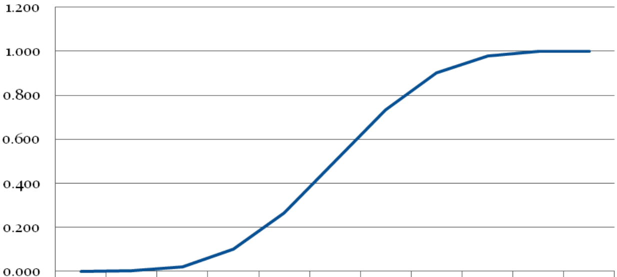
$$p_A^G(4,0) = p_A^4 \quad , \quad p_A^G(4,1) = 4p_A^4(1 - p_A)$$

$$p_A^G(4,2) = 10p_A^4(1 - p_A)^2 \quad , \quad p_A^G(3,3) = 20p_A^3(1 - p_A)^3 \quad ,$$

which leads to the formula

$$p_A^G = p_A^4[1+4(1-p_A)+10(1-p_A)^2]+20p_A^3(1-p_A)^3 p_A^2[1-2p_A(1-p_A)]^{-1}$$

Winning probabilities: GAME



	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Series1	0.000	0.001	0.022	0.099	0.264	0.500	0.736	0.901	0.978	0.999	1.000

Expected duration of a Game

By the Markov chain theory, we are able, again at least theoretically, to evaluate the mean hitting (absorbing) times, which correspond in the present setting to evaluate the mean duration of a game.

Denoting, accordingly to the previous notation, by

$$k_i^C = E_i[H^C)$$

the mean hitting time starting form the state i, a well known result says:

Proposition: The vector of mean hitting times $k^C = (k_i^C, i \in S)$ is the minimal non-negative solution to the system of linear equations

$$\begin{cases} k_i^C = 0 & \text{for } i \in C \\ k_i^C = 1 + \sum_{j \in S \setminus C} p_{ij} k_j^C & \text{for } i \notin C \end{cases}$$

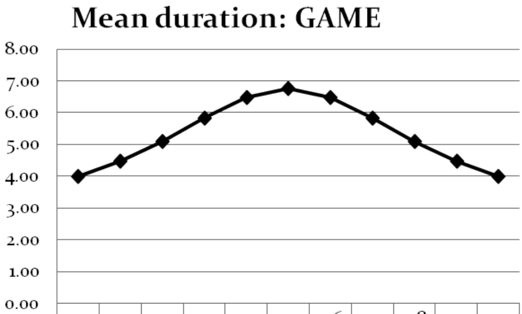
Denoting by $\mathbf{P}$ the sub matrix of P obtained by the entries corresponding to the states in $S \setminus C$ and by $\mathbf{K}$ the vector of the mean hitting times for these states, we get

$$\mathbf{K} = (Id - \mathbf{P})^{-1}\mathbf{1}$$

Applying this result and thanks to the help of Mathematica, one can obtain that the mean duration of a game when A serves is equal to

$$\frac{4(1-p+p^2+6p^3-18p4+18p^5-6p^6)}{1-2p-2p^2}$$

where $p = p_A$.



	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
n° scambi	4.00	4.46	5.09	5.83	6.48	6.75	6.48	5.83	5.09	4.46	4.00

Set

Turno di battuta nel game

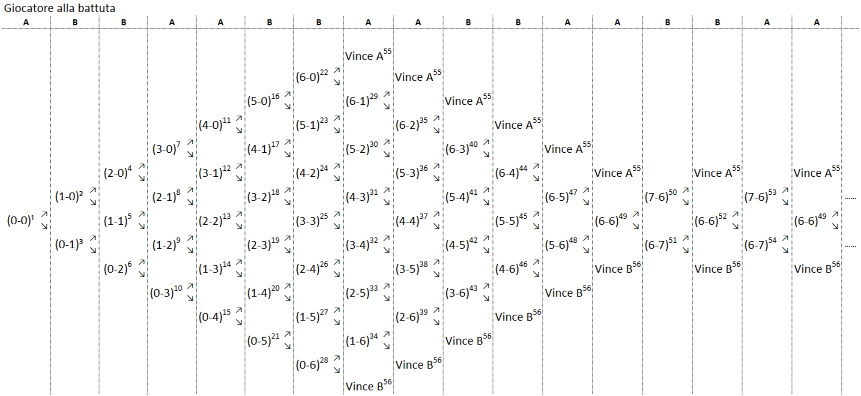
A	B	A	B	A	B	A	B	A	B	A	B	A tie-break	
						Vince A^{40}							
					(5-0)^{16} ↗↘		Vince A^{40}						
				(4-0)^{11} ↗↘		(5-1)^{22} ↗↘		Vince A^{40}					
			(3-0)^{7} ↗↘		(4-1)^{17} ↗↘		(5-2)^{27} ↗↘		Vince A^{40}				
		(2-0)^{4} ↗↘		(3-1)^{12} ↗↘		(4-2)^{23} ↗↘		(5-3)^{31} ↗↘		Vince A^{40}		Vince A^{40}	
	(1-0)^{2} ↗↘		(2-1)^{8} ↗↘		(3-2)^{18} ↗↘		(4-3)^{28} ↗↘		(5-4)^{34} ↗↘		(6-5)^{37} ↗↘		Vince A^{40}
(0-0)^{1} ↗↘		(1-1)^{5} ↗↘		(2-2)^{13} ↗↘		(3-3)^{24} ↗↘		(4-4)^{32} ↗↘		(5-5)^{36} ↗↘		(6-6)^{39} ↗↘	
	(0-1)^{3} ↗↘		(1-2)^{9} ↗↘		(2-3)^{19} ↗↘		(3-4)^{29} ↗↘		(4-5)^{35} ↗↘		(5-6)^{38} ↗↘		Vince B^{41}
		(0-2)^{6} ↗↘		(1-3)^{14} ↗↘		(2-4)^{25} ↗↘		(3-5)^{33} ↗↘		Vince B^{41}		Vince B^{41}	
			(0-3)^{10} ↗↘		(1-4)^{20} ↗↘		(2-5)^{30} ↗↘		Vince B^{41}				
				(0-4)^{15} ↗↘		(1-5)^{26} ↗↘		Vince B^{41}					
					(0-5)^{21} ↗↘		Vince B^{41}						
						Vince B^{41}							

Set

$S =$

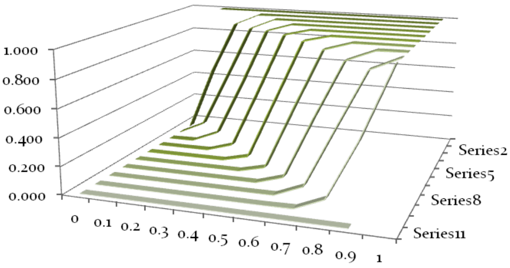
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
1	0	pAG	qAG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	qBG	pBG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	qBG	pBG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	pAG	qAG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	pAG	qAG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	pAG	qAG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	qBG	pBG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	qBG	pBG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	qBG	pBG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	qBG	pBG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pAG	qAG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pAG	qAG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pAG	qAG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pAG	qAG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pAG	qAG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pBG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	qBG	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	qBG	pBG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	qBG	pBG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	qBG	pBG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	qBG	pBG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	qBG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pBG
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	qAG	0	0	0	0	0	0	0	0	0	0	0	0	pAG	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pAG	qAG	0	0	0	0	0	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pAG	qAG	0	0	0	0	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pAG	qAG	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pAG	0	0	0	0	0	0	0	0	0	0	qAG
27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pBG	0	0	0	0	0	0	0	0	qBG	0
28	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	qBG	pBG	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	qBG	pBG	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	qBG	0	0	0	0	0	0	0	pBG
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	qAG	0	0	0	0	0	pAG	0
32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pAG	qAG	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pAG	0	0	0	0	0	qAG
34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pBG	0	0	0	qBG	0
35	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	qBG	0	0	0	0	pBG
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pAG	qAG	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pBG	qBG	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	qBG	0	pBG
39	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pAT	qAT
40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
41	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Tiebreak


Giocatore alla battuta
A B B A A B B A A B B A A B B A A
(0-0)^{1}
(1-0)^{2} (0-1)^{3}
(2-0)^{4} (1-1)^{5} (0-2)^{6}
(3-0)^{7} (2-1)^{8} (1-2)^{9} (0-3)^{10}
(4-0)^{11} (3-1)^{12} (2-2)^{13} (1-3)^{14} (0-4)^{15}
(5-0)^{16} (4-1)^{17} (3-2)^{18} (2-3)^{19} (1-4)^{20} (0-5)^{21}
(6-0)^{22} (5-1)^{23} (4-2)^{24} (3-3)^{25} (2-4)^{26} (1-5)^{27} (0-6)^{28}
Vince A^{55} (6-1)^{29} (5-2)^{30} (4-3)^{31} (3-4)^{32} (2-5)^{33} (1-6)^{34} Vince B^{56}
Vince A^{55} (6-2)^{35} (5-3)^{36} (4-4)^{37} (3-5)^{38} (2-6)^{39} Vince B^{56}
Vince A^{55} (6-3)^{40} (5-4)^{41} (4-5)^{42} (3-6)^{43} Vince B^{56}
Vince A^{55} (6-4)^{44} (5-5)^{45} (4-6)^{46} Vince B^{56}
Vince A^{55} (6-5)^{47} (5-6)^{48} Vince B^{56}
Vince A^{55} (6-6)^{49} Vince B^{56}
(7-6)^{50} (6-7)^{51}
Vince A^{55} (6-6)^{52} Vince B^{56}
(7-6)^{53} (6-7)^{54}
Vince A^{55} (6-6)^{49} Vince B^{56}
......
......

Winning probabilities of A: SET

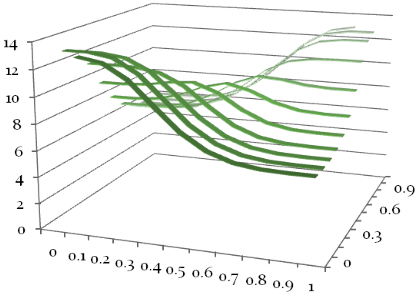
p_B \ p_A	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
0		1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000
0,1	0,000	0,500	0,762	0,922	0,988	0,999	1,000	1,000	1,000	1,000	1,000
0,2	0,000	0,238	0,500	0,768	0,941	0,993	1,000	1,000	1,000	1,000	1,000
0,3	0,000	0,078	0,232	0,500	0,795	0,958	0,996	1,000	1,000	1,000	1,000
0,4	0,000	0,012	0,059	0,205	0,500	0,813	0,963	0,996	1,000	1,000	1,000
0,5	0,000	0,001	0,007	0,042	0,187	0,500	0,813	0,958	0,993	0,999	1,000
0,6	0,000	0,000	0,000	0,004	0,037	0,187	0,500	0,795	0,941	0,988	1,000
0,7	0,000	0,000	0,000	0,000	0,004	0,042	0,205	0,500	0,768	0,922	1,000
0,8	0,000	0,000	0,000	0,000	0,000	0,007	0,059	0,232	0,500	0,762	1,000
0,9	0,000	0,000	0,000	0,000	0,000	0,001	0,012	0,078	0,238	0,500	1,000
1	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	


1.000
0.800
0.600
0.400
0.200
0.000
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Series2
Series5
Series8
Series11

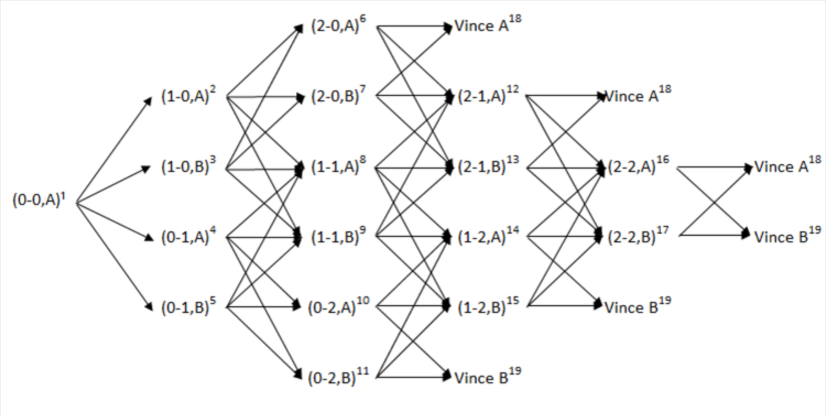
Mean duration: Set

$$
\begin{aligned}
k_1^{S(40,41)} =\ & 13 - 20{p_B^G} + 32{p_B^G}^2 - 35{p_B^G}^3 + 23{p_B^G}^4 - 8{p_B^G}^5 + {p_B^G}^6 + \\
& p_A^G\left(-16 + 182p_B^G - 544{p_B^G}^2 + 783{p_B^G}^3 - 596{p_B^G}^4 + 226{p_B^G}^5 - 32{p_B^G}^6\right) + \\
& {p_A^G}^2\left(19 - 496p_B^G + 2230{p_B^G}^2 - 4159{p_B^G}^3 + 3799{p_B^G}^4 - 1660{p_B^G}^5 + 270{p_B^G}^6\right) + \\
& {p_A^G}^3\left(-17 + 667p_B^G - 3983{p_B^G}^2 + 9226{p_B^G}^3 - 10056{p_B^G}^4 + 5140{p_B^G}^5 - 980{p_B^G}^6\right) + \\
& {p_A^G}^4\left(10 - 488p_B^G + 3571{p_B^G}^2 - 9916{p_B^G}^3 + 12778{p_B^G}^4 - 7700{p_B^G}^5 + 1750{p_B^G}^6\right) + \\
& -2{p_A^G}^5\left(2 - 95p_B^G + 788{p_B^G}^2 - 2542{p_B^G}^3 + 3850{p_B^G}^4 - 2758{p_B^G}^5 + 756{p_B^G}^6\right) + \\
& {p_A^G}^6(1 - 2p_B^G)^2\left(1 - 28p_B^G + 154{p_B^G}^2 - 252{p_B^G}^3 + 126{p_B^G}^4\right)
\end{aligned}
$$

	p_A	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
p_B	$p_B(G)\,|\,p_A(G)$	0,000	0,001	0,022	0,099	0,264	0,500	0,736	0,901	0,978	0,999	1,000
0	0,000		12,98	12,66	11,58	9,83	8,14	6,97	6,32	6,07	6,00	6,00
0,1	0,001	12,97	12,95	12,64	11,57	9,83	8,15	6,97	6,33	6,07	6,01	6,00
0,2	0,022	12,58	12,56	12,32	11,45	9,90	8,26	7,06	6,40	6,13	6,07	6,07
0,3	0,099	11,30	11,29	11,24	10,94	10,06	8,64	7,38	6,66	6,38	6,31	6,30
0,4	0,264	9,41	9,41	9,53	9,84	9,97	9,29	8,13	7,29	6,94	6,85	6,84
0,5	0,500	7,83	7,84	7,96	8,42	9,20	9,66	9,20	8,42	7,96	7,84	7,83
0,6	0,736	6,84	6,85	6,94	7,29	8,13	9,29	9,97	9,84	9,53	9,41	9,41
0,7	0,901	6,30	6,31	6,38	6,66	7,38	8,64	10,06	10,94	11,24	11,29	11,30
0,8	0,978	6,07	6,07	6,13	6,40	7,06	8,26	9,90	11,45	12,32	12,56	12,58
0,9	0,999	6,00	6,01	6,07	6,33	6,97	8,15	9,83	11,57	12,64	12,95	12,97
1	1,000	6,00	6,00	6,07	6,32	6,97	8,14	9,83	11,58	12,66	12,98	


14
12
10
8
6
4
2
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
0
0.3
0.6
0.9

Match


(0-0,A)1
(1-0,A)2
(1-0,B)3
(0-1,A)4
(0-1,B)5
(2-0,A)6
(2-0,B)7
(1-1,A)8
(1-1,B)9
(0-2,A)10
(0-2,B)11
Vince A18
(2-1,A)12
(2-1,B)13
(1-2,A)14
(1-2,B)15
Vince B19
Vince A18
(2-2,A)16
(2-2,B)17
Vince B19
Vince A18
Vince B19

$M =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	0	pABS	pAAS	qABS	qAAS	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	pABS	pAAS	qABS	qAAS	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	qBBS	qBAS	pBBS	pBAS	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	pABS	pAAS	qABS	qAAS	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	qBBS	qBAS	pBBS	pBAS	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	qABS	qAAS	0	0	0	0	pAS	0
7	0	0	0	0	0	0	0	0	0	0	0	pBBS	pBAS	0	0	0	0	qBS	0
8	0	0	0	0	0	0	0	0	0	0	0	pABS	pAAS	qABS	qAAS	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	qBBS	qBAS	pBBS	pBAS	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	pABS	pAAS	0	0	0	qAS
11	0	0	0	0	0	0	0	0	0	0	0	0	0	qBBS	qBAS	0	0	0	pBS
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	qABS	qAAS	pAS	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pBBS	pBAS	qBS	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pABS	pAAS	0	qAS
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	qBBS	qBAS	0	pBS
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	pAS	qAS
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	qBS	pBS
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

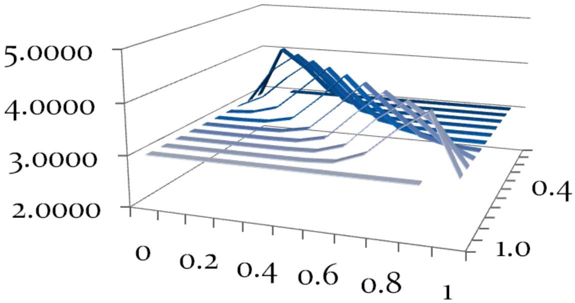
Winning probabilities of A: MATCH

p_B \ p_A	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
0		1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000
0,1	0,000	0,500	0,908	0,996	1,000	1,000	1,000	1,000	1,000	1,000	1,000
0,2	0,000	0,092	0,500	0,914	0,998	1,000	1,000	1,000	1,000	1,000	1,000
0,3	0,000	0,004	0,086	0,500	0,938	0,999	1,000	1,000	1,000	1,000	1,000
0,4	0,000	0,000	0,002	0,062	0,500	0,951	1,000	1,000	1,000	1,000	1,000
0,5	0,000	0,000	0,000	0,001	0,049	0,500	0,951	0,999	1,000	1,000	1,000
0,6	0,000	0,000	0,000	0,000	0,000	0,049	0,500	0,938	0,998	1,000	1,000
0,7	0,000	0,000	0,000	0,000	0,000	0,001	0,062	0,500	0,914	0,996	1,000
0,8	0,000	0,000	0,000	0,000	0,000	0,000	0,002	0,086	0,500	0,908	1,000
0,9	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,004	0,092	0,500	1,000
1	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	

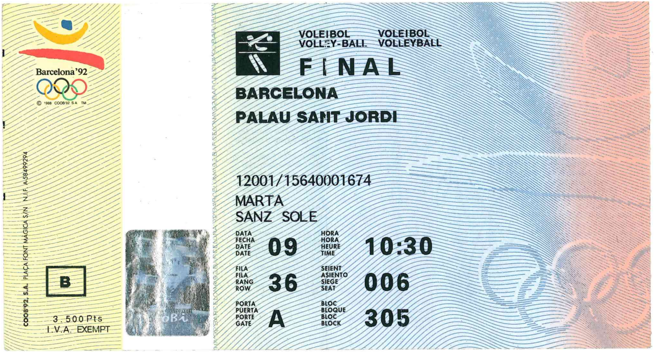
Mean duration: Match

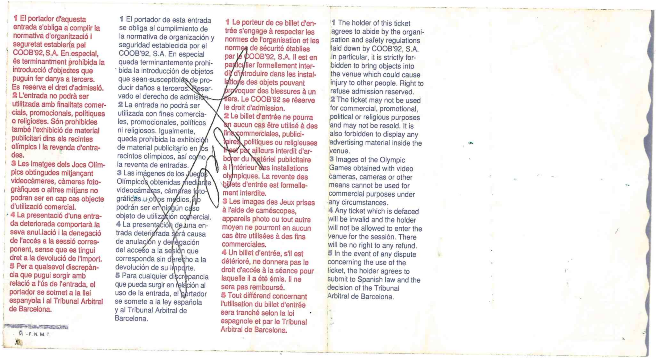
Mean duration: MATCH

p_B \ p_A	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
0		3,0000	3,0000	3,0000	3,0000	3,0000	3,0000	3,0000	3,0000	3,0000	3,0000
0,1	3,0000	4,1250	3,7424	3,2468	3,0376	3,0022	3,0000	3,0000	3,0000	3,0000	3,0000
0,2	3,0000	3,7424	4,1250	3,7258	3,1861	3,0202	3,0009	3,0000	3,0000	3,0000	3,0000
0,3	3,0000	3,2468	3,7258	4,1250	3,6487	3,1319	3,0118	3,0005	3,0000	3,0000	3,0000
0,4	3,0000	3,0376	3,1861	3,6487	4,1250	3,5959	3,1131	3,0118	3,0009	3,0000	3,0000
0,5	3,0000	3,0022	3,0202	3,1319	3,5959	4,1250	3,5959	3,1319	3,0202	3,0022	3,0000
0,6	3,0000	3,0000	3,0009	3,0118	3,1131	3,5959	4,1250	3,6487	3,1861	3,0376	3,0000
0,7	3,0000	3,0000	3,0000	3,0005	3,0118	3,1319	3,6487	4,1250	3,7258	3,2468	3,0000
0,8	3,0000	3,0000	3,0000	3,0000	3,0009	3,0202	3,1861	3,7258	4,1250	3,7424	3,0000
0,9	3,0000	3,0000	3,0000	3,0000	3,0000	3,0022	3,0376	3,2468	3,7424	4,1250	3,0000
1	3,0000	3,0000	3,0000	3,0000	3,0000	3,0000	3,0000	3,0000	3,0000	3,0000	


5.0000
4.0000
3.0000
2.0000
0
0.2
0.4
0.6
0.8
1
0.4
1.0

Volleyball Memorabilia





1 El portador d'aquesta entrada s'obliga a complir la normativa d'organització i seguretat establerta pel COOB'92, S.A. En especial, és terminantment prohibida la introducció d'objectes que puguin fer danys a tercers. Es reserva el dret d'admissió.
2 L'entrada no podrà ser utilitzada amb finalitats comercials, promocionals, polítiques o religioses. Són prohibides també l'exhibició de material publicitari dins els recintes olímpics i la revenda d'entrades.
3 Les imatges dels Jocs Olímpics obtingudes mitjançant videocàmeres, càmeres fotogràfiques o altres mitjans no podran ser en cap cas objecte d'utilització comercial.
4 La presentació d'una entrada deteriorada comportarà la seva anul.lació i la denegació de l'accés a la sessió corresponent, sense que es tingui dret a la devolució de l'import.
5 Per a qualsevol discrepància que pugui sorgir amb relació a l'ús de l'entrada, el portador se sotmet a la llei espanyola i al Tribunal Arbitral de Barcelona.

1 El portador de esta entrada se obliga al cumplimiento de la normativa de organización y seguridad establecida por el COOB'92, S.A. En especial queda terminantemente prohibida la introducción de objetos que sean susceptibles de producir daños a terceros. Reservado el derecho de admisión.
2 La entrada no podrá ser utilizada con fines comerciales, promocionales, políticos ni religiosos. Igualmente, queda prohibida la exhibición de material publicitario en los recintos olímpicos, así como la reventa de entradas.
3 Las imágenes de los Juegos Olímpicos obtenidas mediante videocámaras, cámaras fotográficas u otros medios, no podrán ser en ningún caso objeto de utilización comercial.
4 La presentación de una entrada deteriorada será causa de anulación y denegación del acceso a la sesión que corresponda sin derecho a la devolución de su importe.
5 Para cualquier discrepancia que pueda surgir en relación al uso de la entrada, el portador se somete a la ley española y al Tribunal Arbitral de Barcelona.

1 Le porteur de ce billet d'entrée s'engage à respecter les normes de l'organisation et les normes de sécurité établies par le COOB'92, S.A. Il est en particulier formellement interdit d'introduire dans les installations des objets pouvant provoquer des blessures à un tiers. Le COOB'92 se réserve le droit d'admission.
2 Le billet d'entrée ne pourra en aucun cas être utilisé à des fins commerciales, publicitaires, politiques ou religieuses. Il est par ailleurs interdit d'arborer du matériel publicitaire à l'intérieur des installations olympiques. La revente des billets d'entrée est formellement interdite.
3 Les images des Jeux prises à l'aide de caméscopes, appareils photo ou tout autre moyen ne pourront en aucun cas être utilisées à des fins commerciales.
4 Un billet d'entrée, s'il est détérioré, ne donnera pas le droit d'accès à la séance pour laquelle il a été émis. Il ne sera pas remboursé.
5 Tout différend concernant l'utilisation du billet d'entrée sera tranché selon la loi espagnole et par le Tribunal Arbitral de Barcelona.

1 The holder of this ticket agrees to abide by the organisation and safety regulations laid down by COOB'92, S.A. In particular, it is strictly forbidden to bring objects into the venue which could cause injury to other people. Right to refuse admission reserved.
2 The ticket may not be used for commercial, promotional, political or religious purposes and may not be resold. It is also forbidden to display any advertising material inside the venue.
3 Images of the Olympic Games obtained with video cameras, cameras or other means cannot be used for commercial purposes under any circumstances.
4 Any ticket which is defaced will be invalid and the holder will not be allowed to enter the venue for the session. There will be no right to any refund.
5 In the event of any dispute concerning the use of the ticket, the holder agrees to submit to Spanish law and the decision of the Tribunal Arbitral de Barcelona.

FNMT

The model

Let us consider the following model for volleyball.

The probability that a team wins each point is **constant** during the match, **independent** from the other points played and **depends** just on the fact that the team serves or returns the serve.

So, calling the two teams 0 and 1, we will define two parameters p_0 and p_1 which represents, respectively, the probabilities of winning a rally when the team 0 or 1 serves.

To avoid trivial cases, we will always assume that $0 < p_0 < 1$ and $0 < p_1 < 1$.

Furthermore and in contrast to the similar model for the tennis given before, it will be here reasonable to consider both this numbers **less** then 0.5

In order to analyze the probability of winning a set (and a match) under these assumptions, we recognize that the score can be thought as the realization of a discrete-time Markov chain, whose transition matrix will be specified in the sequel.

Since the scoring system has recently changed, we will consider separately the two cases, starting from the present **rally point scoring system**.

Winning probabilities of a set: Rally point scoring system

Let us start considering the actual rally point scoring system.

We can define the set S of the states of the Markov chain that describes the evolution of a volleyball set under the rally point scoring system.

$$S := \{(i, j, s) : i \in \{0, 1, \dots, 24, Ad, W\}, j \in \{0, 1, \dots, 24\}, s \in \{0, 1\}\}$$

where the first number represents the score of the serving team, the states Ad and W in the first position stand for Advantage and Winning of the serving team, and similarly for the numbers in second position relative to the returning team, while the last number indicates which team serves.

The transition probabilities are defined as follows: when $\max\{i, j\} < 24$ then

$$(i, j, s) \longrightarrow (i + 1, j, s) \quad \text{with probability} \quad p_s$$

$$(i, j, s) \longrightarrow (j + 1, i, 1 - s) \quad \text{with probability} \quad 1 - p_s$$

$$(23, 24, s) \longrightarrow (24, 24, s) \quad \text{with probability} \quad p_s$$

$$(23, 24, s) \longrightarrow (W, 23, 1-s) \quad \text{with probability} \quad 1-p_s$$

$$(24, 23, s) \longrightarrow (W, 23, s) \quad \text{with probability} \quad p_s$$

$$(24, 23, s) \longrightarrow (24, 24, 1-s) \quad \text{with probability} \quad 1-p_s$$

$$
\begin{aligned}
(24, 24, s) &\longrightarrow (Ad, 24, s) \quad \text{with probability} \quad p_s \\
(24, 24, s) &\longrightarrow (Ad, 24, 1-s) \quad \text{with probability} \quad 1 - p_s \\
(Ad, 24, s) &\longrightarrow (W, 24, s) \quad \text{with probability} \quad p_s \\
(Ad, 24, s) &\longrightarrow (24, 24, 1-s) \quad \text{with probability} \quad 1 - p_s
\end{aligned}
$$

Let us now compute the conditional probability that the team who starts serving, wins the set.

We have to evaluate the probabilities that the Markov chain starting from the state $(0, 0, s)$ reaches one of the states $(W, 0, s), (W, 1, s), \dots, (W, 23, s), (W, 24, s)$.

One possible approach would be to consider the whole Markov chain and to compute the absorbing probabilities of these states starting from $(0, 0, s)$. Although this is theoretically correct, it is not viable in practice, since the Markov chain representing a volleyball set can be described by a huge 1265×1265 transition matrix, not suitable for any, at least simple, computation.

As an alternative (see [1]), we can consider directly the computation of this probability.

$$\mathbb{P}[s \text{ wins a set serving first}] = \sum_{l=0}^{23} p_{(W,l,s)} + p_{(24,24,s)} p_{Adv,s} +$$

$$+ p_{(24,24,1-s)}(1 - p_{Adv,1-s})$$

where $p_{(W,l,s)}$ denotes the probability that team s wins the set while team $1-s$ scores exactly l points, while $p_{(24,24,s)}$ is the probability that team s reaches the score $(24,24)$ serving next and $p_{Adv,s}$ that team s wins the tiebreak at the end of the set.

A simple computation gives first that

$$p_{(W,0,s)} = p_s^{25}$$

since the team s has to win all the played rallies.

The situation becomes slightly more complicated once the loosing team scores points itself. In this case to evaluate the value of $p_{(W,l,s)}$ we have to take into account all the possible breaks (changes in the serving team) that happened during the set and their relative position in the set.

This computation leads to this formula

$$p_{(W,l,s)} = \sum_{k=1}^{l} A(k,25,l)\ p_s^{25-k} p_{1-s}^{l-k} (1-p_s)^k (1-p_{1-s})^k$$

for $l \geq 1$, where for positive integers k, m, l, with $k \leq l$,

$$A(k,m,l) = C((k,l-k))C((k+1,m-k)) \ ,$$

and $C((n,k))$ denotes the number of combinations with repetitions of k objects from a set of cardinality n, which is equal to

$$C((n,k)) = \binom{n+k-1}{k} \ .$$

Note that the term $A(k,m,l)$ counts all the possible sequences of consecutive points won by the serving team, between two breaks.

If the set arrives to the score $(24,24)$ we have to consider the probability of winning the final tiebreak.

By the Markov property, we can first compute the probability to reach the score $(24,24,s)$ or $(24,24,1-s)$ and multiply it by the probability, respectively, that team s wins the tiebreak serving first or that team $1-s$ loose the tiebreak serving first

$$p_{(24,24,s)}p_{Adv,s}+p_{(24,24,1-s)}(1-p_{Adv,1-s})$$

Proceeding as before, we obtain that

$$p_{(24,24,s)} = \sum_{k=1}^{24} A(k,24,24)\ p_s^{24-k} p_{1-s}^{24-k} (1-p_s)^k (1-p_{1-s})^k \ ;$$

$$p_{(24,24,1-s)} = \sum_{k=1}^{23} B(k+1,25,24)\ p_s^{24-k} p_{1-s}^{23-k} (1-p_s)^{k+1} (1-p_{1-s})^k \ ,$$

where

$$B(k,m,l) = C((k,m-k))C((k,l-k)) \ .$$

In order to compute the probability $p_{Adv,s}$, let us consider the sub Markov chain consisting only of the states $\{(24,24,0),(24,24,1),(Ad,24,0),(Ad,24,1),(W,24,0),(W,24,1)\}$.

The computation of the absorbing probability, starting from $(24,24,s)$ of the state $(W,24,s)$, gives

$$p_{Adv,s} = \frac{p_s^2}{p_s^2 + p_{1-s}^2 + p_s p_{1-s} - p_s^2 p_{1-s} - p_s p_{1-s}^2} .$$

p_0	p_1 0.1	0.2	0.3	0.4	0.5	0.6	0.7
0.1	0.42650	0.10414	0.01844	0.00233	0.00020	0.00001	0.00000
0.2	0.83438	0.45763	0.17102	0.04462	0.00789	0.00086	0.00005
0.3	0.96807	0.78324	0.47516	0.20804	0.06349	0.01258	0.00140
0.4	0.99581	0.94163	0.76498	0.48834	0.22934	0.07361	0.01424
0.5	0.99965	0.98970	0.92862	0.76175	0.50000	0.24006	0.07476
0.6	0.99998	0.99891	0.98637	0.92639	0.76890	0.51172	0.24125
0.7	1.00000	0.99994	0.99860	0.98685	0.93345	0.78633	0.52511

Table: Rally point scoring system. Probability of winning a set by team 0 when it serves first.

Winning probabilities of a set: Side-out scoring system

The side-out scoring system can be modeled in the Markov chain framework as follow:

let us define the set S of the states as

$$S := \{(i, j, s) : i \in \{0, 1, \ldots, 14, Ad, W\}, j \in \{0, 1, \ldots, 14\}, s \in \{0, 1\}\}$$

where the first number represents the score of the serving team, the states Ad and W in the first position stand for Advantage and Winning of the serving team, and similarly for the numbers in second position relative to the returning team, while the last number indicates which team serves next.

The definition of the transition probabilities here is more delicate: As before, p_s denotes the probability that the team s wins a rally when it is serving. In this scoring system, we have to compute also the probability pp_s, that denotes the probability that team s starts serving and scores a point.
This can be easily preformed by defining a four states Markov chain, with state space $\{A0, A1, W0, W1\}$, where $A0$, respectively $A1$, stands for team 0, resp. 1, serves, while $W0$, resp. $W1$, stands for team 0, resp. team 1, marks the point, and transition probability matrix

$$\begin{bmatrix} 0 & 1-p_0 & p_0 & 0 \\ 1-p_1 & 0 & 0 & p_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The probability pp_s is equal to the absorbing probability of state Ws starting from As, which is equal to

$$pp_s = \frac{p_s}{p_s + p_{1-s} - p_s p_{1-s}} \ . \tag{1}$$

Remark: It is worth noting that if the probabilities $p_0 = p_1 = 1/2$, in the side-out scoring system it is no more true that the probability of scoring a point is independent from the event of who is serving first. Indeed, from (1) we get that in this case the above probability is equal to $pp_0 = pp_1 = 2/3$. It is easy to prove that in general $pp_s \geq p_s$ and that $pp_0 = 1/2$ if

$$p_0 = \frac{p_1}{1 + p_1} \ .$$

Proceeding as before (see also [3] , it is easy to see that if the first serving team is s, then the transition probabilities are defined as follows: when $0 \leq i, j \leq 13$, then

$$(i,j,s) \longrightarrow (i+1,j,s) \quad \text{with probability} \quad pp_s$$

$$(i,j,s) \longrightarrow (j+1,i,1-s) \quad \text{with probability} \quad 1-pp_s$$

and similarly for the other cases.

The winning probability for the set in this case is

$$\mathbb{P}[s \text{ wins a set serving first}] = \sum_{l=0}^{13} p_{(W,l,s)} + p_{(14,14,s)} pp_{Adv,s} +$$

$$+ p_{(14,14,1-s)}(1 - pp_{Adv,1-s})$$

where:

$$p_{(W,0,s)} = pp_s^{15} \ ;$$

$$p_{(W,l,s)} = \sum_{k=1}^{l} A(k,15,l)\ pp_s^{15-k} pp_{1-s}^{l-k} (1 - pp_s)^k (1 - pp_{1-s})^k$$

for $l \geq 1$.

In order to compute the remaining terms, we get

$$p_{(14,14,s)} = \sum_{k=1}^{14} A(k,14,14)\ pp_s^{14-k} pp_{1-s}^{14-k} (1-pp_s)^k (1-pp_{1-s})^k$$

$$p_{(14,14,1-s)} = \sum_{k=1}^{14} B(k,15,14)\ pp_s^{15-k} pp_{1-s}^{14-k} (1-pp_s)^k (1-pp_{1-s})^{k-1}.$$

and

$$pp_{Adv,s} = \frac{pp_s^2}{pp_s^2 + pp_{1-s}^2 + pp_s pp_{1-s} - pp_s^2 pp_{1-s} - pp_s pp_{1-s}^2}\ .$$

p_0	p_1 0.1	0.2	0.3	0.4	0.5	0.6	0.7
0.1	0.50394	0.02017	0.00056	0.00002	0.00000	0.00000	0.00000
0.2	0.98125	0.50837	0.10241	0.01321	0.00139	0.00013	0.00001
0.3	0.99951	0.90690	0.51344	0.17214	0.03974	0.00693	0.00092
0.4	0.99999	0.98890	0.84788	0.51938	0.21975	0.06659	0.01459
0.5	1.00000	0.99894	0.96793	0.81260	0.52658	0.25127	0.08614
0.6	1.00000	0.99991	0.99501	0.94912	0.79556	0.53574	0.27064
0.7	1.00000	0.99999	0.99943	0.99038	0.93915	0.79399	0.54832

Table: Side-out scoring system. Probability of winning a set by team 0 when it serves first.

It is now interesting to compare the winning probabilities in the two scoring systems for the same parameters p_0 and p_1.

Comparison of Table 1. and 2. shows that the introduction of the rally point system increased the difficulty of winning a set for the first serving team, for every choice of probabilities such that $p_0 \geq p_1$.

On the other hand, if $p_1 > p_0$ and the difference $p_1 - p_0$ is substantial, then team 0 (that serves first in the set) has more chances to win the set. Hence, the change in the scoring system facilitated the weaker teams and introduced a source of randomness in the outcomes of the sets (and therefore of the matches).

Winning probabilities: Match

Let us now compute the winning probabilities, in both the present- and former scoring systems.

Who serves first in the first set, then serves first in the third set, while the other team starts serving in the second and in the (possible) fourth set.

If the teams play the deciding fifth set, a toss is carried out to determine who starts serving.

The fifth, deciding set, in the rally point scoring system as in the side-out scoring system, corresponds to a rally point set ending with 15 points.

By the Markovian assumption, we get that the probability to win a match is equal to the product of the probabilities for the two teams to win the single sets.

Let us denote by

$$p_{(W,0)} = \mathbb{P}[0 \text{ wins a set serving first}]$$

$$p_{(W,1)} = \mathbb{P}[1 \text{ wins a set serving first}] \ ,$$

while

$$p_{(T,0)} = \mathbb{P}[0 \text{ wins the deciding set serving first}]$$

$$p_{(T,1)} = \mathbb{P}[1 \text{ wins the deciding set serving first}] \ .$$

Since a toss is carried out to determine who first serves the deciding set, the probability that team 0 wins this set will be equal to

$$p_T = \frac{1}{2} p_{(T,0)} + \frac{1}{2}(1 - p_{(T,1)})$$

A simple computation gives in the rally point scoring system:

$$\mathbb{P}[0 \text{ wins } (3,0)] = p^2_{(W,0)}(1 - p_{(W,1)})$$

$$\mathbb{P}[0 \text{ wins } (3,1)] = 2(1 - p_{(W,0)})p_{(W,0)}(1 - p_{(W,1)})^2 + \\ + p^2_{(W,0)}p_{(W,1)}(1 - p_{(W,1)})$$

$$\mathbb{P}[0 \text{ wins } (3,2)] = \left[p^2_{(W,0)}p^2_{(W,1)} + (1 - p_{(W,0)})^2(1 - p_{(W,1)})^2 + \\ + 4p_{(W,0)}p_{(W,1)}(1 - p_{(W,0)})(1 - p_{(W,1)})\right] p_T \ .$$

Therefore, the probability that team 0 wins a match when starts serving in the first set is equal to

$$\mathbb{P}[0 \text{ wins } (3,0)] + \mathbb{P}[0 \text{ wins } (3,1)] + \mathbb{P}[0 \text{ wins } (3,2)].$$

Expected duration of a set

Let us consider the expected duration of a set, measured in number of rallies.

We shall assume again that the probabilities to win a rally could be different for the two teams, but constant along the set and independent of the previous rallies. Moreover, we shall assume that team 0 starts serving.

From the Markov chain theory, it is possible to solve this problem since this is equivalent to determine the expected number of steps that the chain takes to arrive for the first time to a given state or subset of states C.

The problem is that this solution is **finite**, and therefore useful, just when the subset of states C includes all the closed classes of the Markov chain.

In the present case, it is possible to determine the expected number of rallies needed to finish a given set, but not the expected number of rallies needed to play a set won by team 0.

This problem can be overcome in the rally point scoring system, but not in the side-out scoring system.

Rally point scoring system

In this case the computation is simple, since a point is scored at the end of each rally.

If the probability that team 0 or team 1 wins a set with a final score $(25, l)$, $l \in \{0, \ldots, 23\}$, we get that the contribution of this outcome to the expected duration of the set is equal to

$$(25 + l) \times \left(p_{(W,l,0)} + p_{(W,l,1)}\right) .$$

Slightly more complicated is the case when the score reaches $(24, 24)$.

In this case we have to compute the expected number of rallies that one team needs to end the set, conditional to the fact that we arrive to the tie break after exactly 48 rallies.

This can be easily computed thanks to the Markov chain theory if we define a suitable sub Markov chain.

As before, we have to consider separately the cases that we arrive to the score $(24, 24, 0)$ or $(24, 24, 1)$, since the expected length of the tie break is generally different.

Let us consider a Markov chain defined on the state space $S := \{(24, 24, 0), (24, 24, 1), (25, 24, 0), (25, 24, 1), (26, 24, 0), (26, 24, 1)\}$ with transition matrix

$$P = \begin{bmatrix} 0 & 0 & p_0 & 1-p_0 & 0 & 0 \\ 0 & 0 & 1-p_1 & p_1 & 0 & 0 \\ 0 & 1-p_0 & 0 & 0 & p_0 & 0 \\ 1-p_1 & 0 & 0 & 0 & 0 & p_1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Denoting by $E = \{(26, 24, 0), (26, 24, 1)\}$ the set of the absorbing states, an easy computation allows us to obtain the mean absorbing times to the set E starting form the states $(24, 24, 0), (24, 24, 1), (25, 24, 0), (25, 24, 1)$ as the (minimal) nonnegative solution k of the linear system

$$\begin{cases} k_i = \sum_{j=1}^{4} P_{i,j} k_j \quad , \quad \text{for } i = 1, \ldots, 4 \\ k_5 = k_6 = 0 \end{cases} \tag{2}$$

where we renames the states, in the same order as before, as $\{1, 2, 3, 4, 5, 6\}$.

Solving this system, we obtain that the mean duration of the tie break starting by $(24, 24, 0)$ is equal to k_1, where

$$k_1 = \frac{2(p_0 + p_1 - p_0 p_1) + 2p_0(1 - p_0)}{(p_0 + p_1 - p_0 p_1)^2 - p_0 p_1 (1 - p_0)(1 - p_1)} \qquad (3)$$

while the mean duration of the tie break starting by $(24, 24, 1)$ is equal to k_2, where

$$k_2 = \frac{2 + p_1(1 - p_1) \times k_1}{p_0 + p_1 - p_0 p_1}. \qquad (4)$$

Therefore, conditioning on the fact that the set reaches the $(24, 24, 0)$ or $(24, 24, 1)$ scores, respectively, the expected duration of such a set is equal to

$$k_{TB} = p_{(24,24,0)} \times (48 + k_1) + p_{(24,24,1)} \times (48 + k_2)\ .$$

Collecting all these terms, we obtain that the expected duration of a set under the rally point scoring system is equal to

$$\sum_{l=0}^{23} (25 + l)(p_{(W,l,0)} + p_{(W,l,1)}) + k_{TB}$$

p_0	p_1 0.1	0.2	0.3	0.4	0.5	0.6	0.7
0.1	54.05265 (14.91289)	47.80129 (7.18016)	44.01562 (3.93678)	41.13975 (3.14317)	38.44668 (3.01603)	35.77786 (2.90388)	33.11061 (2.68593)
0.2	48.87572 (7.89125)	48.85170 (6.74883)	46.35838 (5.02883)	43.30936 (4.12290)	40.26262 (3.74588)	37.25137 (3.51311)	34.24906 (3.23776)
0.3	44.78652 (4.32246)	46.83601 (5.15053)	47.02174 (4.70905)	45.26924 (4.41072)	42.40624 (4.36950)	39.11653 (4.23204)	35.71018 (3.90302)
0.4	41.71137 (3.25019)	43.79625 (4.18764)	45.52038 (4.37762)	45.80878 (4.16734)	44.26045 (4.39654)	41.29771 (4.74957)	37.60288 (4.67248)
0.5	38.89249 (3.06057)	40.63788 (3.77841)	42.65790 (4.36223)	44.34614 (4.37096)	44.72994 (4.23089)	43.18093 (4.68663)	39.87165 (5.22015)
0.6	36.11127 (2.94329)	37.50153 (3.54051)	39.25614 (4.24623)	41.29776 (4.74215)	43.08409 (4.72003)	43.58975 (4.59817)	41.88578 (5.19371)
0.7	33.33334 (2.72073)	34.37508 (3.24971)	35.71170 (3.90737)	37.44397 (4.65775)	39.54475 (5.25218)	41.52885 (5.31985)	42.21003 (5.21359)

Table: Rally point scoring system. Expected duration of a set (and standard deviation, estimated by 1,000,000 replicates of played sets).

Side-out point scoring system

This case is more complicated, since the side-out scoring system needs a “small tie break” to decide if a team scores a single point.

Thanks to the Markov chain theory, described above, we are able to compute the expected duration of any such “small tie break”. However, this duration depends on who is serving first and so it will not be sufficient to know the expected duration of the “small tie break”, but we should know the duration of a “small tie break” won by team s

This is not easy to compute using the classical Markov chain approach and so we have two alternatives.

1) we can consider the whole set as a Markov chain and evaluate directly the expected duration solving the linear system recalled before;

2) we can simulate a large number of sets and estimate the expected duration of the set along with its standard deviation in a very simple and fast way.

The first approach is "complicate" in practice, even if theoretically feasible, since for the rally point scoring system this is equivalent to solve a linear system with 1254 equations or, which is equivalent, define and invert a 1254 square matrix, while for the side-out scoring system these numbers fall to 510.

The second approach is much easier and one can obtain the results that are summarized in the following Table, where the simulated durations of 1,000,000 sets have been obtained for some given pairs of the parameters (p_0, p_1)

p_0	p_1 0.1	0.2	0.3	0.4	0.5	0.6	0.7
0.1	258.89002 (59.80855)	142.48141 (36.34303)	90.00662 (22.90847)	63.61648 (15.94363)	47.78484 (11.64220)	37.20972 (8.69363)	29.67830 (6.46740)
0.2	141.33413 (36.27911)	128.11294 (28.99328)	93.00187 (22.69687)	66.74642 (16.88541)	50.00907 (12.52573)	38.74286 (9.37190)	30.71503 (6.99569)
0.3	88.87538 (22.90984)	91.99717 (22.77603)	84.42871 (18.73650)	67.93844 (15.90992)	52.37300 (13.03704)	40.64645 (10.14068)	32.03526 (7.64992)
0.4	62.53499 (15.92535)	65.47384 (16.93181)	67.00376 (16.11341)	62.46464 (13.65832)	52.90798 (12.06242)	42.52611 (10.40962)	33.67196 (8.32781)
0.5	46.68319 (11.64442)	48.76056 (12.49328)	51.06549 (13.13281)	51.95234 (12.30704)	49.13692 (10.68072)	42.86594 (9.62438)	35.23850 (8.52962)
0.6	36.11113 (8.69699)	37.49909 (9.35017)	39.23885 (10.14523)	41.08895 (10.55737)	41.84133 (9.92346)	40.05917 (8.76973)	35.53665 (7.98580)
0.7	28.56746 (6.45749)	29.46810 (6.97537)	30.60417 (7.61331)	32.05322 (8.35032)	33.59113 (8.73049)	34.41184 (8.34485)	33.33850 (7.50407)

p_0	p_1						
	0.1	0.2	0.3	0.4	0.5	0.6	0.7
0.1	258.89002 **54.05265**	142.48141 **47.80129**	90.00662 **44.01562**	63.61648 **41.13975**	47.78484 **38.44668**	37.20972 **35.77786**	29.67830 **33.11061**
0.2	141.33413 **48.87572**	128.11294 **48.85170**	93.00187 **46.35838**	66.74642 **43.30936**	50.00907 **40.26262**	38.74286 **37.25137**	30.71503 **34.24906**
0.3	88.87538 **44.78652**	91.99717 **46.83601**	84.42871 **47.02174**	67.93844 **45.26924**	52.37300 **42.40624**	40.64645 **39.11653**	32.03526 **35.71018**
0.4	62.53499 **41.71137**	65.47384 **43.79625**	67.00376 **45.52038**	62.46464 **45.80878**	52.90798 **44.26045**	42.52611 **41.29771**	33.67196 **37.60288**
0.5	46.68319 **38.89249**	48.76056 **40.63788**	51.06549 **42.65790**	51.95234 **44.34614**	49.13692 **44.72994**	42.86594 **43.18093**	35.23850 **39.87165**
0.6	36.11113 **36.11127**	37.49909 **37.50153**	39.23885 **39.25614**	41.08895 **41.29776**	41.84133 **43.08409**	40.05917 **43.58975**	35.53665 **41.88578**
0.7	28.56746 **33.33334**	29.46810 **34.37508**	30.60417 **35.71170**	32.05322 **37.44397**	33.59113 **39.54475**	34.41184 **41.52885**	33.33850 **42.21003**

The mean durations are lower in the rally point system as long as $p_0 \leq 0.5$ and $p_1 \leq 0.5$.

Outside this range the mean durations are, generally higher in the rally point system (except for $p_1 = 0.6$ and $p_0 \leq 0.4$).

Probably, this is due to the fact that, as outlined before, in the rally point system it is more probable for the weaker team to reach higher scores (and possibly win the set).

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