Existence of densities for non smooth SDEs.

Arnaud Debussche

ENS Cachan Bretagne.

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Collaborators: Nicolas Fournier and Marco Romito

Motivations

We consider SDEs with non smooth coefficients and want to prove existence of the densities for the law of the solutions.

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- SDEs driven by Levy noise with irregular coefficients in $\mathbb{R}^d$:

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  with b and σ only Hölder (or even only bounded for b) $\rightsquigarrow$ no Malliavin calculus.
  If b and σ are smooth, Malliavin calculus can be developped (Bitcheler-Gravereaux-Jacod, 1987, Picard, 1996, ...)

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- application to process used in the probabilistic representation of the homogeneous Boltzmann equation: when $d = 3$, the collision Kernel is too singular.
- develop a method to prove existence of densities without Malliavin calculus $\rightarrow$ application to SPDEs for instance: the application we have in mind is the $3D$ Navier-Stokes equations where it is not known whether the solutions are Malliavin differentiable.

Idea of the method: the non degenerate Brownian case

$$X_t = x + \int_0^t \sigma(X_s)dW_s + \int_0^t b(X_s)ds \qquad (2)$$

Now, W is a d-dimensional brownian motion.

1) Take $\epsilon > 0$ small and define

$$X_t^\epsilon = X_{t-\epsilon} + \sigma(X_{t-\epsilon})(W(t) - W(t-\epsilon)) + \epsilon b(X_{t-\epsilon})$$

2) If σ is non degenerate, X_t^ϵ has a smooth density
3) Use the fact that X^ϵ and X are close to obtain the existence of a density for X_t

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First possibility: Use Characteristic functions (Fournier and Printems)

$$\begin{aligned} \mathbb{E}(e^{i\langle \xi, X_t^\epsilon\rangle}) &= \mathbb{E}\left(\mathbb{E}\left(e^{i\langle \xi, X_t^\epsilon\rangle} | \mathcal{F}_{t-\epsilon}\right)\right) \\ &= \mathbb{E}\left(e^{i\langle X_{t-\epsilon} + \epsilon b(X_{t-\epsilon}), \xi\rangle - \frac{\epsilon}{2}|\sigma(X_{t-\epsilon})\xi|^2}\right) \end{aligned}$$

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$$\mathbb{E}(e^{i\langle \xi, X_t\rangle}) \leq Ce^{-\alpha\epsilon\xi^2} + \left|\mathbb{E}(e^{i\langle \xi, X_t^\epsilon\rangle} - e^{i\langle \xi, X_t\rangle})\right|$$

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&\leq Ce^{-\alpha\epsilon\xi^2} + |\xi|\mathbb{E}\left|X_t^\epsilon - X_t\right| \\
&\leq Ce^{-\alpha\epsilon\xi^2} + \widetilde{C}|\xi|\epsilon^{\frac{1+\gamma}{2}}
\end{aligned}$$

If b is bounded, σ is γ Hölder and elliptic.

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$$\mathbb{E}(e^{i\langle \xi, X_t \rangle}) \leq Ce^{-\alpha\epsilon\xi^2} + \widetilde{C}|\xi|\epsilon^{\frac{1+\gamma}{2}}$$

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• If $\gamma > 1/2$, the method gives a little more: since $|\xi|^\eta \mathbb{E}(e^{i\langle \xi, X_t \rangle})$ is in $L^2(\mathbb{R})$, the density has some extra regularity.

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We keep the first ingredient (approximation of X_t by X_t^ϵ) but instead of using the characteristic function, we go bak to the basic idea of Malliavin: use an integration by part.

Since we do not expect to have very smooth density, we use discrete derivatives $\rightsquigarrow$ try to estimate:

$$\begin{aligned}\mathbb{E}(\varphi(X_t+h)-\varphi(X_t)) &= \int_{\mathbb{R}^d}(\varphi(x+h)-\varphi(x))f_{X_t}(x)dx \\ &= \int_{\mathbb{R}^d}\varphi(x)(f_{X_t}(x-h)-f_{X_t}(x))dx.\end{aligned}$$

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If we are able to prove

$$|\mathbb{E}(\varphi(X_t+h)-\varphi(X_t))| \le C\|\varphi\|_\infty\|h\|^\eta.$$

This (formally) says that

$$\int_{\mathbb{R}^d}|f_{X_t}(x+h)-f_{X_t}(x)|dx \le C\|h\|^\eta.$$

The Besov space $B^s_{1,\infty}$

The Besov space $B^s_{1,\infty}$ can be characterized in terms of finite differences: define

$$(\Delta^1_h f)(x) = f(x+h) - f(x),$$

then, for $s < 1$,

$$\|f\|_{B^s_{1,\infty}} = \|f\|_{L^1} + \sup_{|h|\leq 1} \frac{\|\Delta^1_h f\|_{L^1}}{|h|^s},$$

is an equivalent norm of $B^s_{1,\infty}(\mathbb{R}^d)$. Moreover $B^s_{1,\infty}(\mathbb{R}^d)$ can be defined as the set of $L^1(\mathbb{R}^d)$ functions such that these quantities are finite.

It is well know that we have $s > \tilde{s}$, $p \in [1, d/(d-\tilde{s})]$:

$$B^s_{1,\infty}(\mathbb{R}^d) \subset B^{\tilde{s}}_{1,1}(\mathbb{R}^d) = W^{\tilde{s},1}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d).$$

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$\rightsquigarrow$ $f_{X_t} \in B^\eta_{1,\infty}(\mathbb{R}^d)$ and thus in $L^p(\mathbb{R}^d)$ for some $p > 1$.

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We again use X_t^ϵ and write:

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\begin{aligned}
\mathbb{E}(\varphi(X_t+h)-\varphi(X_t))| &\le |\mathbb{E}(\varphi(X_t^\epsilon+h)-\varphi^\epsilon(X_t^\epsilon))| \\
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with $\epsilon = \|h\|^{\frac{2}{\eta(1+\gamma)+1}}$.

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We thus obtain the weaker inequality:

$$\mathbb{E}(\varphi(X_t + h) - \varphi(X_t))| \leq C\|h\|^{\frac{\eta(1+\gamma)}{\eta(1+\gamma)+1}}\|\varphi\|_{C^\eta}$$

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Lemma Let $g \in \mathcal{M}(\mathbb{R}^d)$. Assume that there are $0 < \eta < a < 1$ and a constant K such that for all $\phi \in C^\eta(\mathbb{R}^d)$, all $h \in \mathbb{R}^d$ with $|h| \leq 1$,

$$\left|\int_{\mathbb{R}^d} \Delta_h^1 \phi(x) g(dx)\right| \leq K||\phi||_{C^\eta(\mathbb{R}^d)}|h|^a. \quad (3)$$

Then, for any $\gamma \in (0, a-\eta)$, g has a density in $B^\gamma_{1,\infty}$.

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Then, for any $\gamma \in (0, a-\eta)$, g has a density in $B^\gamma_{1,\infty}$.

$\rightsquigarrow$ We obtain a density with Besov regularity under the assumption $\gamma > 0$.

Remarks

- The above argument can be slighltly improved and a Besov regularity of order $< \gamma$ can be obtained
- Also, we do not need $\gamma > 1/2$.
- The obtained regularity is low and not optimal at all ! By PDE argument, much more regularity can be obtained.
- In the Brownian case, when σ is invertible, Girsanov formula can be used. However, this method can be applied in situations where Girsanov formula does not apply. For instance if X_t is the solution of a SPDE.

Application to Lévy driven SDEs

We consider, on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, a pure jump d-dimensional Lévy process $(Z_t)_{t\geq 0}$ with Lévy measure m.

Denote by f_{Z_t} the law of Z_t and recall that for $\xi \in \mathbb{R}^d$

$$\begin{cases} \widehat{f_{Z_t}}(\xi) := \mathbb{E}[\exp(i \langle \xi, Z_t \rangle)] = \exp(-t\Psi(\xi)), \\ where \quad \Psi(\xi) = \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, z\rangle} + i \langle \xi, z\rangle \mathbb{I}_{\{|z|\leq 1\}}\right) m(dz). \end{cases}$$

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Let $(X_t)_{t\geq 0}$ be a $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg solution to

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We introduce $X_t^\epsilon = X_{t-\epsilon} + \sigma(X_{t-\epsilon})(Z_t - Z_{t-\epsilon}) + \epsilon b(X_{t-\epsilon})$

We need two ingredients:

- Prove that X_t^ϵ has a smooth density and measure its smoothness in terms of ϵ
- Estimate precisely how X_t and X_t^ϵ are close.

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We consider stable like processes:

$$\begin{cases} \text{(i)} \, \forall \, \beta \in [0,\alpha), \; \int_{\{|z|\geq 1\}} |z|^\beta m(dz) < \infty, \\ \\ \text{(ii)} \, \exists \, C > 0, \; \forall \, a \in (0,1], \; \int_{\{|z|\leq a\}} |z|^2 m(dz) \leq C a^{2-\alpha}, \\ \\ \text{(iii)} \, \exists \, c > 0, \, \exists \, r > 0, \, \forall \, |\xi| \geq r, \; \int_{\mathbb{R}^d} (1 - \cos(\langle \xi, z \rangle)) m(dz) \geq c|\xi|^\alpha. \end{cases}$$

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$\rightsquigarrow c|\xi|^\alpha \leq \Re\, \Psi(\xi) \leq C|\xi|^\alpha$, $\hat{f}_{Z_t}$ decays very fast and f_{Z_t} is smooth.

Schilling, Sztonyk, Wang have proved that this implies that

$$\|\partial^\beta f_{Z_t}\|_{L^1(\mathbb{R}^d)} \leq C(m) t^{-m/\alpha}$$

for $|\beta| = m$.

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We deduce that X_t^ϵ has a density $f_{X_t^\epsilon}$ such that

$$\|f_{X_t^\epsilon}\|_{B^m_{\infty,1}(\mathbb{R}^d)} \leq C(\delta)\|f_{X_t^\epsilon}\|_{B^m_{1,1}(\mathbb{R}^d)} = \|f_{X_t^\epsilon}\|_{W^{m,1}(\mathbb{R}^d)} \leq C(m,\delta)\epsilon^{-m/\alpha}.$$

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$$(\Delta_h^1 f)(x) = f(x+h) - f(x),$$

$$(\Delta_h^n f)(x) = \Delta_h^1(\Delta_h^{n-1} f)(x) = \sum_{j=0}^n (-1)^{n-j}\binom{n}{j} f(x+jh)$$

then, for n integer such that $s < n$, we can take

$$\|f\|_{B^s_{1,\infty}} = \|f\|_{L^1} + \sup_{|h|\leq 1} \frac{\|\Delta_h^n f\|_{L^1}}{|h|^s}.$$

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$$|\mathbb{E}(\Delta_h^n \varphi(X_t))| \leq |\mathbb{E}(\Delta_h^n \varphi(X_t^\epsilon))| + |\mathbb{E}(\Delta_h^n \varphi(X_t) - \Delta_h^n \varphi(X_t^\epsilon))|$$

Write:

$$|\mathbb{E}(\Delta_h^n \varphi(X_t^\epsilon))| = |\int_{\mathbb{R}^d} \Delta_h^n \varphi(x) f_{X_t^\epsilon}(x)dx| = |\int_{\mathbb{R}^d} \varphi(x) \Delta_{-h}^n f_{X_t^\epsilon}(x)dx|.$$

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$$|\mathbb{E}(\Delta_h^n \varphi(X_t))| \leq |\mathbb{E}(\Delta_h^n \varphi(X_t^\epsilon))| + |\mathbb{E}(\Delta_h^n \varphi(X_t) - \Delta_h^n \varphi(X_t^\epsilon))|$$

Write:

$$|\mathbb{E}(\Delta_h^n \varphi(X_t^\epsilon))| = |\int_{\mathbb{R}^d} \Delta_h^n \varphi(x) f_{X_t^\epsilon}(x)dx| = |\int_{\mathbb{R}^d} \varphi(x) \Delta_{-h}^n f_{X_t^\epsilon}(x)dx|.$$

Then

$$\|f_{X_t^\epsilon}\|_{B^n_{\infty,1}(\mathbb{R}^d)} = \|f_{X_t^\epsilon}\|_{L^1(\mathbb{R}^d)} + \sup_{|h|\leq 1} \frac{\|\Delta_h^n f_{X_t^\epsilon}\|_{L^1}}{|h|^{n-\delta}} \leq C(n,\delta)\epsilon^{-n/\alpha}.$$

Deduce:

$$\begin{aligned} |\mathbb{E}(\Delta_h^n \varphi(X_t))| &\leq C\|\varphi\|_\infty \epsilon^{-n/\alpha}\|h\|^n + |\mathbb{E}(\Delta_h^n \varphi(X_t) - \Delta_h^n \varphi(X_t^\epsilon))| \\ &\leq C\|\varphi\|_\infty \epsilon^{-n/\alpha}\|h\|^n + C\|\varphi\|_{C^\eta}\mathbb{E}\|X_t - X_t^\epsilon\|^\eta. \end{aligned}$$

Estimate of $\mathbb{E}\|X_t - X_t^\epsilon\|^\eta$ and conclusion:

For $\alpha \in [1,2)$, this is estimated by classical stochastic calculus:

$$\mathbb{E}\|X_t - X_t^\epsilon\|^\beta \le C_\beta \left(\epsilon^{\beta(1+\theta_1)/\alpha} + \epsilon^{\beta(1+\theta_2/\alpha)} \right)$$

for all $\beta \in (0,\alpha)$. If b is θ_2 Hölder (or bounded with $\theta_2 = 0$) and σ is θ_1 Hölder. Set $\kappa = \min\{1+\theta_1, \alpha+\theta_2\} \rightsquigarrow$

$$\begin{aligned} |\mathbb{E}(\Delta_h^n \varphi(X_t))| &\le C\|\varphi\|_{C^\eta} \left(\epsilon^{-n/\alpha}\|h\|^n + \epsilon^{\eta\kappa/\alpha} \right) \\ &\le C\|\varphi\|_{C^\eta}\|h\|^{\lambda_n}. \end{aligned}$$

with $\lambda_n \to \kappa\eta$, when $n \to \infty$.
Use a generalization of the above Lemma $\rightsquigarrow$ X_t has a density in $B^s_{1,\infty}$ for any $s < \kappa - 1$.

Estimate of $\mathbb{E}\|X_t - X_t^\epsilon\|^\eta$ and conclusion:

- The case $\alpha \in (0,1)$ is slightly more complicated: $X_t^\epsilon = X_{t-\epsilon} + \sigma(X_{t-\epsilon})(Z_t - Z_{t-\epsilon}) + \epsilon b(X_{t-\epsilon})$ is not a good approximation of X_t.

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- Set $Y_t = Z_t + t\int_{\|z\|\le 1} zm(dz)$, $\tilde{b}(x) = b(x) - \sigma(x)\int_{\|z\|\le 1} zm(dz)$ and rewrite the equation as

$$X_t = x + \int_0^t \sigma(X_{s-})dY_s + \int_0^t \tilde{b}(X_s)ds$$

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- We take V_t^ϵ which satisfies

$$V_t^\epsilon = X_{t-\epsilon} + \int_{t-\epsilon}^t \tilde{b}(V_s^\epsilon)ds$$

and set

$$X_t^\epsilon = V_t^\epsilon + \sigma(X_{t-\epsilon})(Y_t - Y_{t-\epsilon})$$

- Then, for $\beta < \alpha$,

$$\mathbb{E}\|X_t - X_t^\epsilon\|^\beta \le C_\beta\left(\epsilon^{\beta(1/\alpha+\theta_1)} + \epsilon^{\beta(1+\theta_2/\alpha)} + \epsilon^{\beta/(1-\theta_2)}\right)$$

Estimate of $\mathbb{E}\|X_t - X_t^\epsilon\|^\eta$ and conclusion:

We again have: $X_t^\epsilon = V_t^\epsilon + \sigma(X_{t-\epsilon})(Y_t - Y_{t-\epsilon})$ with V_t^ϵ which is $\mathcal{F}_{t-\epsilon}$-measurable and such that, for $\beta < \alpha$,

$$\mathbb{E}\|X_t - X_t^\epsilon\|^\beta \leq C_\beta \left(\epsilon^{\beta(1/\alpha+\theta_1)} + \epsilon^{\beta(1+\theta_2/\alpha)} + \epsilon^{\beta/(1-\theta_2)}\right)$$

Similar argument as in the case $\alpha \in [1, 2)$ give a density in $B^s_{1,\infty}$ for any $s < (\kappa - 1)\alpha$.

Refinement when σ is not invertible everywhere:

Write:

$$\begin{aligned}\left|\mathbb{E}\left(\frac{\Delta_h^n\phi(X_t)}{|\sigma^{-1}(X_t)|}\right)\right| &\leq \left|\mathbb{E}\left(\Delta_h^n\phi(X_t)\left[\frac{1}{|\sigma^{-1}(X_t)|}-\frac{1}{|\sigma^{-1}(X_{t-\epsilon})|}\right]\right)\right| \\ &+ \left|\mathbb{E}\left([\Delta_h^n\phi(X_t)-\Delta_h^n\phi(X_t^\epsilon)]\frac{1}{|\sigma^{-1}(X_{t-\epsilon})|}\right)\right| \\ &+ \left|\mathbb{E}\left([\Delta_h^n\phi(X_t^\epsilon)]\frac{1}{|\sigma^{-1}(X_{t-\epsilon})|}\right)\right|.\end{aligned}$$

Since

$$\mathbb{E}\left(\frac{\Delta_h^n\phi(X_t)}{|\sigma^{-1}(X_t)|}\right)=\int_{\mathbb{R}^d}\Delta_h^n\phi(x)\frac{1}{|\sigma^{-1}(x)|}f_{X_t}(dx)$$

Use the same ideas to obtain that $f_{X_t}(dx)/|\sigma^{-1}(x)|$ has a density in a Besov space.

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Use the same ideas to obtain that $f_{X_t}(dx)/|\sigma^{-1}(x)|$ has a density in a Besov space. This shows that f_{X_t} has a density on the set $\{x\in\mathbb{R}^d:\ \sigma(x) \text{ is invertible}\}$.

The result for $\alpha \in [1,2)$:

Let $\sigma : \mathbb{R}^d \mapsto M_{d\times d}(\mathbb{R})$ and $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ be measurable and bounded. Consider a $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg solution $(X_t)_{t\geq 0}$ to

$$X_t = x + \int_0^t \sigma(X_{s-})dZ_s + \int_0^t b(X_s)ds$$

where $(Z_t)_{t\geq 0}$ is a Lévy process with Lévy measure m satisfying (H_α) for some $\alpha \in [1,2)$.
Assume that $\sigma \in C^{\theta_1}(\mathbb{R}^d)$ for some $\theta_1 \in (0,1)$, that b is measurable (then set $\theta_2 = 0$) or that $b \in C^{\theta_2}(\mathbb{R}^d)$ for some $\theta_2 \in (0,1)$ and that $\kappa = \min\{1+\theta_1, \alpha+\theta_2\} > 1$.
Then for all $t > 0$, the law f_{X_t} of X_t has a density on the set $\{y \in \mathbb{R}^d : \sigma(y) \text{ invertible}\}$. Furthermore, for all $\gamma \in (0, \kappa - 1)$, $|\sigma^{-1}|^{-1} f_{X_t} \in B^{\gamma}_{1,\infty}(\mathbb{R}^d)$.

The result for $\alpha \in (0,1)$:

Let $\sigma : \mathbb{R}^d \mapsto M_{d\times d}(\mathbb{R})$ and $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ be measurable and bounded. Consider a $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg solution $(X_t)_{t\geq 0}$ to

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Assume that $\sigma \in C^{\theta_1}(\mathbb{R}^d)$ for some $\theta_1 \in (0,1)$, that $\tilde{b} \in C^{\theta_2}(\mathbb{R}^d)$ for some $\theta_2 \in (1-\alpha, 1)$, where
$\tilde{b}(x) := b(x) - \sigma(x)\int_{\{|z|\leq 1\}} zm(dz)$ is the true drift coefficient and set $\kappa = \min\{1+\alpha\theta_1, \alpha+\theta_2, \alpha/(1-\theta_2)\} > 1$.
Then for all $t > 0$, the law f_{X_t} of X_t has a density on the set $\{y \in \mathbb{R}^d : \sigma(y) \text{ invertible}\}$. Furthermore, for all $\gamma \in (0, (\kappa-1)\alpha)$, $|\sigma^{-1}|^{-1} f_{X_t} \in B^{\gamma}_{1,\infty}(\mathbb{R}^d)$.

Comments on the assumptions on m:

We have assumed

$$\begin{cases} \text{(i)}\ \forall\ \beta \in [0,\alpha),\ \int_{\{|z|\geq 1\}} |z|^\beta m(dz) < \infty, \\ \\ \text{(ii)}\ \exists\ C>0,\ \forall\ a \in (0,1],\ \int_{\{|z|\leq a\}} |z|^2 m(dz) \leq Ca^{2-\alpha}, \\ \\ \text{(iii)}\ \exists\ c>0,\ \exists\ r>0,\ \forall\ |\xi| \geq r,\ \int_{\mathbb{R}^d} (1-\cos(\langle \xi, z\rangle)) m(dz) \geq c|\xi|^\alpha. \end{cases}$$

- (iii) is equivalent to $\exists\ c>0,\ \forall a \in (0,1],\ \forall |\zeta| = 1$:

$$\int_{\{|z|\leq a\}} \langle \zeta, z\rangle^2\, m(dz) \geq ca^{2-\alpha}.$$

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- This can be satisfies by very rough measures. For instance:

$$m(A) = \int_0^\infty \mu(dr) \int_{S^{d-1}} \mathbf{1}_A(r\sigma)\lambda(d\sigma)$$

for λ nonnegative finite measure on S^{d-1} whose support contains a basis of $\mathbb{R}^d$ and $\mu = \sum_{n\geq 1} n^{\alpha-1}\delta_{1/n}$

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- If we consider a α-stable process $(Y_t)_{t\geq 0}$ with $\alpha \in (1,2)$,

$$\begin{cases} \widehat{f_{Y_t}}(\xi) := \mathbb{E}[\exp(i \langle \xi, Y_t \rangle)] = \exp(-t\Psi(\xi)), \\ where \quad \Psi(\xi) = \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, z\rangle} + i \langle \xi, z \rangle\right) m(dz), \end{cases}$$

  where $m(A) = \int_0^{\infty} r^{-\alpha-1} dr \int_{\mathbb{S}^{d-1}} \mathbb{I}_A(r\sigma)\lambda(d\sigma)$, (H_α)-(i)-(ii) clearly hold. If the support of λ contains a basis of $\mathbb{R}^d$, then (H_α)-(iii) is also OK.
- $Z_t = Y_t - t \int_{\|x\|\geq 1} x m(dx)$ satisfies all our assumptions.

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- (i) is not essential and can be replaced by

$$\int_{\{|z|\geq 1\}} m(dz) < \infty.$$

The conclusion is the same: existence of a density where σ is invertible but we loose the Besov smoothness

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- It would more satisfactory to prove a result with (ii) and (iii) satisfied with possibly different values of α. This could be studied but the computations would be much longer.

The stochastic Navier-Stokes equations in dimension 3

Let u, p be the velocity and pressure of an incompressible fluid in a domain O:

$$\begin{cases} du + (-\nu\Delta u + \nabla p + (u \cdot \nabla)u)dt = fdt + d\eta, \ t \geq 0, \ x \in O, \\ \text{div } u = 0, \ t \geq 0, \ x \in O, \\ u = 0, \quad \text{on } \partial O, \\ u(0) = u_0, \ x \in O. \end{cases}$$

- ν is the viscosity and we take it equal to 1.
- The exterior forcing has two component. A deterministic one f, we take $f = 0$ and a random one of white noise type: $\eta = Q^{1/2} \sum_{i \in \mathbb{N}} \beta_i e_i = Q^{1/2} W$.
- The covariance operator describres the spatial smoothness of the noise

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- Project the equation on $H = \{u \in (L^2(O))^3; \quad \text{div } u = 0\}$.
- Define $A = \Delta u$, $D(A) = H^2(O) \cap H_0^1(0) \cap H$.
- $\mathcal{P}$ is the projector onto H and $b(u) = \mathcal{P}$.

$\rightarrow$

$$\begin{cases} du & = (Au + b(u))dt + \sqrt{Q}dW, \\ u(0) & = u_0 \in H. \end{cases}$$

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- It is difficult to go beyond this except for very non degenerate noises.

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- open problem. Very difficult, even for $d = 2$
- Try to prove that finite dimensional projections of u have densities with respect to the Lebesgue measure.

The stochastic Navier-Stokes equations in dimension 3

$$\begin{cases} du & = (Au + b(u))dt + \sqrt{Q}dW, \\ u(0) & = u_0 \in H. \end{cases}$$

- It seems difficult to use Malliavin calculus. Indeed $D_s^h u(t) = \eta(t)$ where η is the solution of

$$\begin{cases} \frac{d}{dt}\eta & = A\eta + b'(u) \cdot \eta, \\ \eta(0) & = \sqrt{Q}h. \end{cases}$$

- We have no control on the Malliavin derivative of u.
- If the noise is sufficiently non degenerate, it is possible to use Malliavin on a truncated form of the Navier-Stokes equation and to obtain densities for the finite dimensional projections.
- Can we obtain something for degenerate noise ?

The stochastic Navier-Stokes equations in dimension 3

$$\begin{cases} du & = (Au + b(u))dt + \sqrt{Q}dW, \\ \\ u(0) & = u_0 \in H. \end{cases}$$

- Let F be a finite dimensional subspace of H and π_F be the projector onto F.

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$\int_F \Delta_n^h \varphi(z) d\nu_F(z) = \mathbb{E}(\Delta_n^h \varphi(\pi_F u(1)))$
- Introduce u^ϵ:

$$\begin{cases} u^\epsilon(t) = u(t), t \le 1 - \epsilon, \\ (I - \pi_F)u^\epsilon(t) = (I - \pi_F)u(t), t \ge 1 - \epsilon, \\ d\pi_F u^\epsilon = \pi_F A u^\epsilon + \pi_F \sqrt{Q}dW, t \ge 1 - \epsilon. \end{cases}$$

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$$u(1) = e^{\epsilon A}u(1-\epsilon) + \int_{1-\epsilon}^{1} e^{A(1-s)} b(u(s))ds + \int_{1-\epsilon}^{1} e^{A(1-s)} \sqrt{Q}dW(s)$$

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$$\begin{cases} du &= (Au + b(u))dt + \sqrt{Q}dW, \\ u(0) &= u_0 \in H. \end{cases}$$

$$\begin{cases} u^\epsilon(t) = u(t), t \leq 1-\epsilon, \\ (I-\pi_F)u^\epsilon(t) = (I-\pi_F)u(t), t \geq 1-\epsilon, \\ d\pi_F u^\epsilon = \pi_F A u^\epsilon + \pi_F\sqrt{Q}dW, t \geq 1-\epsilon. \end{cases}$$

$$u(1) = e^{\epsilon A}u(1-\epsilon) + \int_{1-\epsilon}^{1} e^{A(1-s)}b(u(s))ds + \int_{1-\epsilon}^{1} e^{A(1-s)}\sqrt{Q}dW(s)$$

$$u^\epsilon(1) = e^{\epsilon A}u(1-\epsilon) + \int_{1-\epsilon}^{1} e^{A(1-s)}\sqrt{Q}dW(s)$$

- Since $\mathbb{E}(|\pi_F b(u)|)$ is bounded. It is easy to check that $\mathbb{E}(|\pi_F u(1) - \pi_F u^\epsilon(1)|) \leq C_1\epsilon$.

The stochastic Navier-Stokes equations in dimension 3

We then write, assuming that $\ker Q = \{0\}$:

$$
\begin{aligned}
\mathbb{E}(\Delta_n^h \varphi(u(1))) &= \mathbb{E}(\Delta_n^h \varphi(u(1)) - \Delta_n^h \varphi(u^\epsilon(1))) \\
&\quad + \mathbb{E}(\Delta_n^h \varphi(u^\epsilon(1) + h) - \Delta_n^h \varphi(u^\epsilon(1)))
\end{aligned}
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&\quad + \mathbb{E}(\Delta_n^h \varphi(u^\epsilon(1) + h) - \Delta_n^h \varphi(u^\epsilon(1))) \\
&\leq |\varphi|_{C^\alpha(F)} \mathbb{E}(|\pi_F(u(1) - u^\epsilon(1)|)^\alpha \\
&\quad + C_{F,Q} \|\varphi\|_\infty |h|^n \, \epsilon^{-n/2}
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&\quad + C_{F,Q} \|\varphi\|_\infty |h|^n \, \epsilon^{-n/2} \\
&\leq C_1^\alpha |\varphi|_{C^\alpha(F)} \epsilon^\alpha + C_2 \|\varphi\|_\infty |h|^n \, \epsilon^{-n/2}
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&\leq |\varphi|_{C^\alpha(F)} \mathbb{E}(|\pi_F(u(1) - u^\epsilon(1)|)^\alpha \\
&\quad + C_{F,Q} \|\varphi\|_\infty |h|^n \, \epsilon^{-n/2} \\
&\leq C_1^\alpha |\varphi|_{C^\alpha(F)} \epsilon^\alpha + C_2 \|\varphi\|_\infty |h|^n \, \epsilon^{-n/2} \\
&\leq C_4 |\varphi|_{C^\alpha(F)} |h|^{\frac{2\alpha n}{2\alpha + n}}
\end{aligned}$$

for $\epsilon = |h|^{\frac{2n}{2\alpha+n}}$.

The stochastic Navier-Stokes equations in dimension 3

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&\leq C_1^\alpha |\varphi|_{C^\alpha(F)} \epsilon^\alpha + C_2 \|\varphi\|_\infty |h|^n \, \epsilon^{-n/2} \\
&\leq C_4 |\varphi|_{C^\alpha(F)} |h|^{\frac{2\alpha n}{2\alpha + n}}
\end{aligned}$$

for $\epsilon = |h|^{\frac{2n}{2\alpha+n}}$. We deduce that $u(1)$ has a density in $B^s_{1,\infty}(F)$ for $s < \frac{2\alpha n}{2\alpha+n} - \alpha \to \alpha$ when $n \to \infty$.

The stochastic Navier-Stokes equations in dimension 3

Theorem: Consider the stochastic Navier-Stokes equations, assume that $ker\ Q = \{0\}$ then for any finite dimensional space $F \subset H$ and any solution u of the martingale problem (limit of some Galerkin approximation), $\pi_F u(1)$ has a density with respect to the Lebesgue measure in $B^{\gamma}_{1,\infty}(F)$ for any $\gamma < 1$ and in $L^p(F)$ for any $1 \leq p < d/d - 1$.

The stochastic Navier-Stokes equations in dimension 3

A stationary solution has more regularity property (Flandoli & Romito):

$$\mathbb{E}\left(|\nabla u_S|^2\right) < \infty$$

This allow to improve the approximation of u by u^ϵ:

$$\begin{aligned} u^\epsilon(1) = \quad & e^{\epsilon A} u(1-\epsilon) + \int_{1-\epsilon}^{1} e^{A(1-s)} b(e^{A(s-1+\epsilon)} u(1-\epsilon)) ds \\ & + \int_{1-\epsilon}^{1} e^{A(1-s)} \sqrt{Q} dW(s) \end{aligned}$$

We can prove

Theorem: Consider the stochastic Navier-Stokes equations, assume that $ker\ Q = \{0\}$ then for any finite dimensional space $F \subset H$ and any stationary solution u of the martingale problem (limit of some Galerkin approximation), $\pi_F u(1)$ has a density with respect to the Lebesgue measure in $B^{\gamma}_{1,\infty}(F)$ for any $\gamma < 2$ and in $W^{1,p}(F)$ for any $1 \leq p < d/d-1$or in $L^p(F)$ for any $1 \leq p < d/d-2$.

The stochastic Navier-Stokes equations in dimension 3

Remark

• This together with a result of Shirikyan implies that the densities are positive.

• Can we extend this result to the hypoelliptic case ?

• Can we obtain more regularity ?