# An Osgood criterion for stochastic differential equations with an additive noise

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Barcelona, 2011

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# Content

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# Osgood criterion

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We define the explosion time as

$$T = \sup\{t > 0 : |v(t)| < \infty\}.$$

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**(H1)** $a \in \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}_+$ is a positive, non-decreasing and locally Lipschitz function.

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**(H1)** $a \in \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}_+$ is a positive, non-decreasing and locally Lipschitz function.

## Remark

*Under (**H1**), there exists a maximal interval on which above equation has a unique solution.*

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Observe that

$$\frac{v'(t)}{b(v(t))} = 1 \Rightarrow \int_0^t \frac{v'(s)}{b(v(s))} ds = \int_0^t 1 ds = t.$$

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Hence

$$\int_a^{v(t)} \frac{dy}{b(y)} = \int_{v(0)}^{v(t)} \frac{dy}{b(y)} = t.$$

# Osgood criterion

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Define

$$B(x) = \int_a^x \frac{dy}{b(y)}, \ x \geq a.$$

Therefore $B(v(t)) = t$. Thus

$$v(t) = B^{-1}(t), \ 0 < t < B(\infty).$$

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$$v(t) = B^{-1}(t), \; 0 < t < B(\infty).$$

In this case the explosion time is

$$B(\infty) = \int_a^\infty \frac{ds}{b(s)}.$$

# A basic example

Consider the following non-linear ordinary differential equation

$$\begin{cases} \dfrac{dv(t)}{dt} = (v(t))^2, \quad t > 0, \\ v(0) = a. \end{cases}$$

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The number $T = 1/a$ is the explotion time.

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Consider the following non-linear ordinary differential equation

$$\begin{cases} \dfrac{dv(t)}{dt} = (v(t))^2, \quad t > 0, \\ v(0) = a. \end{cases}$$

For $a < 0$, there is a unique solution $v$, in the interval $0 < t < 1/a$ :

$$v(t) = -\frac{1}{t - 1/a}.$$

In this case the explosion time is $T = \infty$.

# A basic example

Consider the following non-linear ordinary differential equation

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## Remark

*Note that*

$$\int_a^\infty \frac{dx}{x^2}$$

*is finite if and only if* $a > 0$.

# Main result

Suppose that $g$ is a measurable function such that

$$\limsup_{t\to\infty}\left(\inf_{0\le h\le 1} g(t+h)\right)=\infty,$$

and $b$ is a positive and nondecreasing function. The solution of the integral equation

$$X_t = a + \int_0^t b(X_s)ds + g(t), \quad t \ge 0,$$

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explodes in finite time if and only if $\int_{\cdot}^{\infty} \frac{ds}{b(s)} < \infty$.

As an example, we see that $g$ can represent the paths of some stochastic processes.

# Contents

# Comparison Theorem

## Lemma

*Assume that b satisfies Hypothesis (**H1**),* $a_1 > a_2$ *and* $T > 0$*. Also assume that u and v are two measurable functions on* $[0, T]$ *such that*

$$v(t) \geq a_1 + \int_0^t b(v(s))ds, \quad t \in [0, T],$$

*and*

$$u(t) = a_2 + \int_0^t b(u(s))ds, \quad t \in [0, T].$$

*Then,* $v \geq u$ *on* $[0, T]$*.*

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*Assume that $b$ satisfies Hypothesis (**H1**), $a_1 > a_2$ and $T > 0$. Also assume that $u$ and $v$ are two measurable functions on $[0, T]$ such that*

$$v(t) \geq a_1 + \int_0^t b(v(s))ds, \quad t \in [0, T],$$

*and*

$$u(t) = a_2 + \int_0^t b(u(s))ds, \quad t \in [0, T].$$

*Then, $v \geq u$ on $[0, T]$.*

**Proof :** Note that the facts that $a_1 > a_2$ and $u$ is continuous yield

$$\{t \in [0, T] : v(s) \geq u(s), \; s \in [0, t]\} \neq \emptyset.$$

# Proof

Thus, we only need to show

$$\tilde{T} = \sup\{t \in (0, T] : v(s) \geq u(s),\ s \in [0, t]\} = T.$$

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But, if it is not so, then Hypothesis (**H1**) and the continuity of the integral lead to write

$$\begin{aligned} v(\tilde{T}+t) - u(\tilde{T}+t) &\geq a_1 - a_2 + \int_0^{\tilde{T}+t} [b(v(s)) - b(u(s))]\, ds \\ &\geq a_1 - a_2 + \int_{\tilde{T}}^{\tilde{T}+t} [b(v(s)) - b(u(s))]\, ds \\ &\geq \frac{a_1 - a_2}{2} > 0, \end{aligned}$$

for $t$ small enough. Therefore $\tilde{T}$ cannot be the supremum. □

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# Main result

Now we study the explosion in finite time of the solution to

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$$X_t = a + \int_0^t b(X_s)ds + g(t), \quad t \geq 0.$$

Here $a \in \mathbb{R}$, $b$ satisfies Hypothesis (**H1**) and

(**H2**) $g : [0, \infty) \to \mathbb{R}$ is a function with left and right-limits such that

$$\limsup_{t \to \infty} \left( \inf_{0 \leq h \leq 1} g(t+h) \right) = \infty.$$

# Main result

## Theorem

*Let Hypotheses (**H1**) and (**H2**) hold. Then, the solution of equation*

$$X_t = a + \int_0^t b(X_s)ds + g(t), \quad t \geq 0.$$

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**Proof :** *Necessity :* Suppose that $X$ exploits at the time $T_e < \infty$. Set $M := \sup_{0 \leq t \leq T_e} |g(t)|$. Then,

$$X_t \leq (a + M) + \int_0^t b(X_s)ds, \quad t \in [0, T_e].$$

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Hence, our comparison result implies that the solution of

$$u(t) = (a + M + 1) + \int_0^t b(u(s))ds, \quad t \geq 0,$$

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Hence, our comparison result implies that the solution of

$$u(t) = (a + M + 1) + \int_0^t b(u(s))ds, \quad t \ge 0,$$

exploits in the interval $[0, T_e]$, which allows to conclude that

$$\int_{\cdot}^{\infty} \frac{ds}{b(s)} < \infty$$

because of Osgood criterion.

# Main result

**Theorem**

*Let Hypotheses (**H1**) and (**H2**) hold. Then, the solution of equation*

$$X_t = a + \int_0^t b(X_s)ds + g(t), \quad t \geq 0.$$

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**Proof :** *Sufficiency :* Now assume that the solution $X$ of does not exploit in finite time.

# Proof

Now assume that the solution $X$ of does not exploit in finite time. So, using Hypothesis (**H2**), we can find an increasing sequence $\{t_n : n \in \mathbb{N}\}$ such that $t_n \uparrow \infty$ and

$$\inf_{0 \leq h \leq 1} g(t_n + h) \uparrow \infty, \quad \text{as } n \to \infty.$$

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On the other hand, Hypothesis (**H1**) yields that

$$\begin{aligned} X_{t+t_n} &\geq a + \int_{t_n}^{t+t_n} b(X_s)ds + g(t+t_n) \\ &\geq a + \int_0^t b(X_{s+t_n})ds + \inf_{0 \leq h \leq 1} g(t_n + h), \quad t \in [0,1]. \end{aligned}$$

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Then, the solution of the equation

$$u(t) = \frac{1}{2}\left(a + \inf_{0 \le h \le 1} g(t_n + h)\right) + \int_0^t b(u(s))ds, \quad t \ge 0,$$

cannot exploit in the interval $[0, 1]$ due to comparison Lemma.

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$$\int_{2^{-1}(a + \inf_{0 \le h \le 1} g(t_n + h))}^{\infty} \frac{ds}{b(s)}$$

is bigger than 1.

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Then, the solution of the equation

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$$\int_{2^{-1}(a+\inf_{0\leq h\leq 1} g(t_n+h))}^{\infty} \frac{ds}{b(s)}$$

of this equation is bigger than 1.
Finally, (1) gives $\int_{\cdot}^{\infty} \frac{ds}{b(s)} = \infty$. □

# Contents

# Main result

Here we show two classes of processes $Z$, whose paths satisfy Hypothesis (**H2**), with probability 1. Consequently, we can analyze the explosion in finite time of the solution to

$$X_t = a + \int_0^t b(X_s)ds + Z_t, \quad t \geq 0,$$

where $a \in \mathbb{R}$.

# Bifractional Brownian motion

The bifractional Brownian motion (bBm) with parameters $H \in (0,1)$ and $K \in (0,1]$ was introduced by Houdré and Villa (2003). It is a centered Gaussian process $B_{H,K} = \{B_{H,K}(t) : t \geq 0\}$ with covariance function

$$R_{H,K}(t,s) = \frac{1}{2^K} \left\{ \left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK} \right\}$$

such that, for $s, t \geq 0$, the inequalities

$$2^{-K}|t-s|^{2HK} \leq E\left[\left(B_{H,K}(t) - B_{H,K}(s)\right)^2\right] \leq 2^{1-K}|t-s|^{2HK} \quad (2)$$

hold.
$B_H := B_{H,1}$ is the fractional Brownian motion with Hurst parameter $H$ and $W := B_{\frac{1}{2},1}$ is the Brownian motion.

# Bifractional Brownian motion : The law of the iterated logarithm

## Lemma

*With probability 1,*

$$\limsup_{t\longrightarrow\infty}\frac{B_{H,K}(t)}{\psi_{H,K}(t)}=1,$$

*with*

$$\psi_{H,K}(t):=\sqrt{2t^{2HK}\log\log t},\quad t>e.$$

# The law of the iterated logarithm

**Proof :** In order to see that the result is true, we proceed as in the Brownian case. That is, we only need to observe that the definition of $B_{H,K}$ yields that the process

$$\widetilde{B}_{H,K}(t) = \begin{cases} 0, & t = 0, \\ t^{2HK} B_{H,K}(\frac{1}{t}), & t > 0 \end{cases}$$

is also a bBm with parameters $H$ and $K$, and that

$$\limsup_{t \longrightarrow \infty} \frac{B_{H,K}(t)}{\psi_{H,K}(t)} = \limsup_{h \downarrow 0} \frac{\widetilde{B}_{H,K}(h)}{\sqrt{2h^{2HK} \log \log h^{-1}}} = 1.$$

The last equality is an immediate consequence of Arcones (1995) (Corollary 3.1). □

# Bifractional Brownian motion

## Lemma

*With probability 1,*

$$\sup_{s,t\in[n,n+2]} \frac{|B_{H,K}(t) - B_{H,K}(s)|}{\psi_{H,K}(n)} \to 0, \quad \text{as } n \to \infty,$$

*where*

$$\psi_{H,K}(t) := \sqrt{2t^{2HK} \log\log t}, \quad t > e.$$

# Proof

For each $n \in \mathbb{N}$ consider the centered Gaussian process $\{B_{H,K}(t+n) - B_{H,K}(n) : t \in [0,2]\}$. Then, from inequality (2) and Carmona et al. (2003) (Lemma 5.2), we have that for $p \geq 1/HK$, there exists a constant $C > 0$, that only depends on $H$, $K$ and $p$, such that

$$E\left[\left(\sup_{s,t\in[n,n+2]} |B_{H,K}(t) - B_{H,K}(s)|\right)^p\right] \leq C2^{pHK}.$$

Hence,

$$E\left[\sum_{n=1}^{\infty}\left(\sup_{s,t\in[n,n+2]} \frac{|B_{H,K}(t) - B_{H,K}(s)|}{\psi_{H,K}(n)}\right)^p\right] \leq \sum_{n=1}^{\infty} \frac{C2^{pHK}}{\psi_{H,K}(n)^p} < \infty.$$

□

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**Theorem**

*Assume that Hypothesis (**H1**) is satisfied and that the process $Z$ is the bifractional Brownian motion $B_{H,K}$. Then, the solution of above equation explodes in finite time with probability 1 if and only if* $\int_{\cdot}^{\infty} \frac{ds}{b(s)} < \infty$.

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Proof : we only need to see that the paths of $B_{H,K}$ are as in Hypothesis (**H2**) with probability 1.

# Proof

Toward this end, let $\omega_0 \in \Omega$. Then,

$$\begin{aligned} & \inf_{0\le h\le 1} B_{H,K}(\omega_0, t+h) \\ = \quad & B_{H,K}(\omega_0, t) + \inf_{0\le h\le 1} \left(B_{H,K}(\omega_0, t+h) - B_{H,K}(\omega_0, t)\right) \end{aligned}$$

# Proof

Toward this end, let $\omega_0 \in \Omega$. Then,

$$\begin{aligned}
&\inf_{0\le h\le 1} B_{H,K}(\omega_0, t+h) \\
&\quad = \; B_{H,K}(\omega_0, t) + \inf_{0\le h\le 1} \left(B_{H,K}(\omega_0, t+h) - B_{H,K}(\omega_0, t)\right) \\
&\quad \ge \; B_{H,K}(\omega_0, t) + \inf_{0\le h\le 1} \left(-\left|B_{H,K}(\omega_0, t+h) - B_{H,K}(\omega_0, t)\right|\right)
\end{aligned}$$

# Proof

Toward this end, let $\omega_0 \in \Omega$. Then,

$$\begin{aligned}
&\inf_{0\leq h\leq 1} B_{H,K}(\omega_0, t+h) \\
&= B_{H,K}(\omega_0, t) + \inf_{0\leq h\leq 1} \left(B_{H,K}(\omega_0, t+h) - B_{H,K}(\omega_0, t)\right) \\
&\geq B_{H,K}(\omega_0, t) + \inf_{0\leq h\leq 1} \left(-\left|B_{H,K}(\omega_0, t+h) - B_{H,K}(\omega_0, t)\right|\right) \\
&= B_{H,K}(\omega_0, t) - \left(\sup_{0\leq h\leq 1} \frac{\left|B_{H,K}(\omega_0, t+h) - B_{H,K}(\omega_0, t)\right|}{\psi_{H,K}([t])}\right)\psi_{H,K}([t]),
\end{aligned}$$

where $[t]$ is the integer part of $t$.

# Proof

Toward this end, let $\omega_0 \in \Omega$. Then,

$$\begin{aligned} &\inf_{0\le h\le 1} B_{H,K}(\omega_0, t+h) \\ &\geq \quad B_{H,K}(\omega_0, t) - \left( \sup_{0\le h\le 1} \frac{|B_{H,K}(\omega_0, t+h) - B_{H,K}(\omega_0, t)}{\psi_{H,K}([t])} \right) \psi_{H,K}([t]), \end{aligned}$$

## Lemma

*With probability 1,*

$$\sup_{s,t\in[n,n+2]} \frac{|B_{H,K}(t) - B_{H,K}(s)|}{\psi_{H,K}(n)} \to 0, \quad \text{as } n \to \infty,$$

*where*

$$\psi_{H,K}(t) := \sqrt{2t^{2HK} \log\log t}, \quad t > e.$$

# Proof

Therefore, for $t$ large enough,

$$\inf_{0\leq h\leq 1} B_{H,K}(\omega_0, t+h) \geq B_{H,K}(\omega_0, t) - \frac{1}{4}\psi_{H,K}([t]).$$

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Therefore, for $t$ large enough,

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# Proof

Therefore, for $t$ large enough,

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$$= \ \frac{B_{H,K}(\omega_0, t)}{\psi_{H,K}(t)}\psi_{H,K}(t) - \frac{1}{4}\psi_{H,K}([t]).$$

## Lemma

*With probability 1,*

$$\limsup_{t\longrightarrow\infty} \frac{B_{H,K}(t)}{\psi_{H,K}(t)} = 1,$$

*with*

$$\psi_{H,K}(t) := \sqrt{2t^{2HK}\log\log t}, \quad t > e.$$

# Proof

Hence, there exists a sequence $0 < t_n \uparrow \infty$ such that

$$\begin{aligned} \inf_{0\leq h\leq 1} B_{H,K}(\omega_0, t_n + h) &\geq \frac{1}{2}\psi_{H,K}(t_n) - \frac{1}{4}\psi_{H,K}([t_n]) \\ &\geq \frac{1}{4}\psi_{H,K}([t_n]) \to \infty. \end{aligned}$$

# Proof

Hence, there exists a sequence $0 < t_n \uparrow \infty$ such that

$$\begin{aligned}\inf_{0\le h\le 1} B_{H,K}(\omega_0, t_n + h) &\ge \frac{1}{2}\psi_{H,K}(t_n) - \frac{1}{4}\psi_{H,K}([t_n]) \\ &\ge \frac{1}{4}\psi_{H,K}([t_n]) \to \infty.\end{aligned}$$

Thus,

$$\limsup_{t\to\infty}\left(\inf_{0\le h\le 1} B_{H,K}(\omega_0, t + h)\right) = \infty.$$

□

# Self-similar processes

Let $\beta \in (0,1)$. Now we consider an increasing $1/\beta$-self similar process $Z$, whose trajectories satisfies Hypothesis (**H2**). The reader can consult Rivero (2003) (and references therein) for details.

# Self-similar processes

Let $\xi = \{\xi_t : t \geq 0\}$ be a Lévy process. Set

$$A_t = \int_0^t \exp\left(\frac{\xi_s}{\alpha}\right) ds, \quad t \geq 0,$$

where $\alpha > 0$, and

$$\tau(t) = \inf\{s : A_s > t\},$$

the time change related to $A$.

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where $\alpha > 0$, and

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the time change related to $A$. Then, the process

$$Z_t = x \exp\left(\xi_{\tau(tx^{-1/\alpha})}\right), \quad t \geq 0,$$

with $x > 0$, is an $\alpha$-self similar Markov process.

# Self-similar processes

It is well-known that the law of $\xi$ is characterized by its Laplace transform

$$E\left(\exp(-\lambda\xi_t)\right) = \exp(-t\phi(\lambda)), \quad t \geq 0 \text{ and } \lambda \geq 0.$$

# Self-similar processes

It is well-known that the law of $\xi$ is characterized by its Laplace transform

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Here, by the Lévy-Khintchine's formula, $\phi$ has the form

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Now consider $\beta \in (0,1)$ and the measure

$$\Pi(dx) = \frac{\beta \exp(x)}{\Gamma(1-\beta)(\exp(x)-1)^{1+\beta}} dx. \tag{3}$$

# Self-similar processes

## Theorem

*Let $Z$ be the $1/\beta$-self similar Markov process related to a subordinator with zero drift and Lévy measure (3). Then, under Hypothesis (**H1**), the solution of equation*

$$X_t = a + \int_0^t b(X_s)ds + Z_t, \quad t \geq 0,$$

*explodes in finite time with probability 1 if and only if*

$$\int_{\cdot}^{\infty} \frac{ds}{b(s)} < \infty.$$

# Proof

The result follows from Theorem M and from the fact that

$$\liminf_{t\to\infty} \frac{Z_t}{t^{1/\beta}(\log\log t)^{(\beta-1)/\beta}} = \beta(1-\beta)^{(1-\beta)/\beta},$$

which is proven in Rivero (2003) (see p. 469). □

# Contents

# Feller test

Let $W = \{W_t : t \geq 0\}$ be a Brownian motion. We can use the Feller test to see if the solution of a stochastic differential equation of the form

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dW_t, \quad t > 0, \\ X_0 &= a, \end{aligned}$$

explodes in finite time, with probability 1, knowing only the coefficients $b$ and $\sigma$.

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explodes in finite time, with probability 1, knowing only the coefficients $b$ and $\sigma$.
In our case (i.e., $\sigma \equiv 1$), this test can be expressed as follows :

# Feller test

Let $\rho(x) = \int_0^x \exp\left(-2\int_0^s b(r)dr\right) ds$ and $v(x) = 2\int_0^x \frac{\rho(x)-\rho(y)}{\rho'(y)} dy$.

## Proposition (Feller test)

*The explosion time $T_e$ of the solution $X$ of the equation*

$$\begin{aligned} dX_t &= b(X_t)dt + dW_t, \quad t > 0, \\ X_0 &= a, \end{aligned}$$

*is finite with probability 1 if and only if, one of the following conditions holds :* $(i)$ $v(\infty) < \infty$ *and* $v(-\infty) < \infty$,
$(ii)$ $v(\infty) < \infty$ *and* $\rho(-\infty) = -\infty$,
$(iii)$ $v(-\infty) < \infty$ *and* $\rho(\infty) = \infty$.

# Example

Let us consider

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$$\begin{aligned} \rho(x) &= \int_0^x exp\left(-2\int_0^s r^2 dr\right) ds \\ &= \int_0^x e^{-\frac{2}{3}s^3} ds. \end{aligned}$$

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Hence

$$v(x) = 2\int_0^x \int_0^y e^{\frac{2}{3}(z^3-y^3)} dzdy \Rightarrow v(\infty) = 2\int_0^\infty \int_0^y e^{\frac{2}{3}(z^3-y^3)} dzdy.$$

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So we study

$$\int_0^\infty \int_0^y e^{\frac{2}{3}(z^3-y^3)} dzdy = \left(\int_0^1 + \int_1^\infty\right)\left(\int_0^y e^{\frac{2}{3}(z^3-y^3)} dz\right) dy.$$

# Example

$$\int_1^\infty \int_0^y e^{\frac{2}{3}(z^3-y^3)} dzdy = \int_0^1 e^{\frac{2}{3}z^3} dz \int_1^\infty e^{-\frac{2}{3}y^3} dy + \int_1^\infty e^{\frac{2}{3}z^3} \left( \int_z^\infty e^{-\frac{2}{3}y^3} dy \right) dz.$$

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On the other hand

$$\int_z^\infty e^{-\frac{2}{3}y^3} dy \leq \int_z^\infty \frac{y^2}{z^2} e^{-\frac{2}{3}y^3} dy = \frac{1}{z^2} \int_z^\infty y^2 e^{-\frac{2}{3}y^3} dy = \frac{1}{z^2} \left( -\frac{e^{-\frac{2}{3}y^3}}{2} \right) \Bigg|_z^\infty = \frac{e^{-\frac{2}{3}z^3}}{2z^3}.$$

Using this,

$$\int_1^\infty e^{\frac{2}{3}z^3} \left( \int_z^\infty e^{-\frac{2}{3}y^3} dy \right) dz \leq \int_1^\infty e^{\frac{2}{3}z^3} \left( \frac{e^{-\frac{2}{3}z^3}}{2z^2} \right) dz = \frac{1}{2} < \infty.$$

# Example

This implies $v(\infty) < \infty$. Moreover

$$\rho(-\infty) = \int_0^{-\infty} e^{-\frac{2}{3}s^3} ds = -\infty.$$

By Feller test, $P(T_e < \infty) = 1$.

# Feller test

## Theorem

*Let $b$ satisfy Hypothesis (**H1**). Then we have*

(i) $\rho(-\infty) = -\infty$.

(ii) $\int_0^\infty \frac{ds}{b(s)} < \infty$ *if and only if* $v(\infty) < \infty$.

# Feller test

## Theorem

*Let $b$ satisfy Hypothesis (**H1**). Then we have*

(i) $\rho(-\infty) = -\infty$.

(ii) $\int_0^\infty \frac{ds}{b(s)} < \infty$ *if and only if* $v(\infty) < \infty$.

**Remark :** The Feller test is proven using the Itô's calculus. However, when $Z \; B_{H,K}$, we cannot use this important tool because, in general, $B_{H,K}$ is not a semimartingale.

# Proof

$(i)$ Observe that

$$\begin{aligned}\rho(-\infty) &= -\int_{-\infty}^{0} \exp\left(2\int_{s}^{0} b(r)dr\right) ds \\ &\leq -\int_{-\infty}^{0} \exp\left(2b(s)(-s)\right) ds \\ &\leq -\int_{-\infty}^{0} \exp\left(0\right) ds = -\infty.\end{aligned}$$

# Proof

$(ii)$ Suppose $\int_0^\infty \frac{ds}{b(s)} < \infty$. Then

$$\begin{aligned} v(\infty) &\le 2\int_0^\infty \int_y^\infty \frac{b(s)}{b(y)} \exp\left(-2\int_0^s b(r)dr\right) \exp\left(2\int_0^y b(t)dt\right) dsdy \\ &= 2\int_0^\infty \frac{1}{b(y)} \exp\left(2\int_0^y b(t)dt\right) \left(\int_y^\infty b(s) \exp\left(-2\int_0^s b(r)dr\right) ds\right) dy \\ &= \int_0^\infty \frac{ds}{b(s)}, \end{aligned}$$

where, in order to evaluate the integral, we used that

$$\int_0^\infty b(r)dr \ge b(0) \int_0^\infty dr = \infty.$$

# Proof

Conversely, now let us assume that $v(\infty) < \infty$. We first note that

$$\begin{aligned}
&\int_0^\infty \frac{1}{b(s)} \exp\left(-2\int_0^s b(r)dr\right) ds \\
&\leq \int_0^\infty \frac{1}{b(s)} \exp\left(-2b(0)s\right) ds \\
&\leq \frac{1}{b(0)} \int_0^\infty \exp\left(-2b(0)s\right) ds < \infty.
\end{aligned}$$

# Proof

Conversely, now let us assume that $v(\infty) < \infty$. We first note that

$$\int_0^\infty \frac{1}{b(s)} \exp\left(-2\int_0^s b(r)dr\right) ds \leq \frac{1}{b(0)} \int_0^\infty \exp\left(-2b(0)s\right) ds < \infty. \tag{4}$$

On the other hand, Fubini theorem yields

$$\begin{aligned} v(\infty) &= 2\int_0^\infty \int_0^s \exp\left(-2\int_0^s b(r)dr\right) \exp\left(2\int_0^y b(t)dt\right) dyds \\ &\geq \int_0^\infty \frac{1}{b(s)} \exp\left(-2\int_0^s b(r)dr\right) \left[\exp\left(2\int_0^s b(r)dr\right) - 1\right] ds \\ &= \int_0^\infty \frac{1}{b(s)} \left[1 - \exp\left(-2\int_0^s b(r)dr\right)\right] ds. \end{aligned}$$

Hence, (4) implies that $\int_0^\infty \frac{ds}{b(s)} < \infty$. Thus the proof is complete. □

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# Gracias por la Atención