

Asymptotic Behavior of Stock Price Distribution Densities in Stochastic Volatility Models

Archil Gulisashvili
Ohio University

Universitat de Barcelona, April 25, 2012

Stock Price Processes

- In general stochastic stock price models, a random behavior of the stock price is modeled by a positive adapted stochastic process X defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}^*)$.
- It is assumed that the following conditions hold:
 1. For every $t > 0$, the stock price X_t is an unbounded random variable.
 2. $\mathbb{E}^* [X_t] < \infty, t > 0$.
 3. $X_0 = x_0$ \mathbb{P}^* -a.s. for some $x_0 > 0$.
 4. \mathbb{P}^* is a risk-neutral measure. This means that the discounted stock price process $\{e^{-rt}X_t\}_{t \geq 0}$ is a martingale. Here $r \geq 0$ is the interest rate.

Pricing Functions

- The pricing function for a European call option at time $t = 0$ is defined by

$$C(T, K) = e^{-rT} \mathbb{E}^* [(X_T - K)^+]$$

where $K > 0$ is the strike price and $T > 0$ is the maturity.

- The pricing function for a European put option at time t is defined by

$$P(T, K) = e^{-rT} \mathbb{E}^* [(K - X_T)^+].$$

- The functions C and P satisfy the put-call parity condition

$$C(T, K) = P(T, K) + x_0 - e^{-rT} K.$$

Black-Scholes Call Pricing Function

- In the Black-Scholes model, the stock price process is a geometric Brownian motion, satisfying the following stochastic differential equation:

$$dX_t = rX_t dt + \sigma X_t dW_t^*,$$

where $r \geq 0$ is the interest rate, $\sigma > 0$ is the volatility of the stock, and W^* is a standard Brownian motion under the risk-free measure \mathbb{P}^* .

- The stock price process X in the Black-Scholes model is given by

$$X_t = x_0 \exp \left\{ \left(r - \frac{\sigma^2}{2} \right) t + \sigma W_t^* \right\}$$

where $x_0 > 0$ is the initial price.

- Black and Scholes found an explicit formula for the pricing function C_{BS} :

$$C_{BS}(T, K, \sigma) = x_0 N(d_1(K, \sigma)) - K e^{-rT} N(d_2(K, \sigma)),$$

where

$$d_1(K, \sigma) = \frac{\log x_0 - \log K + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}},$$

$$d_2(K, \sigma) = d_1(K, \sigma) - \sigma\sqrt{T},$$

and

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp\left\{-\frac{y^2}{2}\right\} dy.$$

Implied Volatility

Let C be a call pricing function. The implied volatility

$$I = I(T, K), \quad (T, K) \in (0, \infty)^2,$$

associated with the pricing function C , is a function of two variables satisfying the following condition:

$$C_{BS}(T, K, I(T, K)) = C(T, K).$$

Special Stochastic Volatility Models and Asymptotic

Behavior of Stock Price Densities

Stock price models with stochastic volatility have been developed in the last decades to improve pricing and hedging performance of the classical Black-Scholes model and to account for certain imperfections in it. The main shortcoming of the Black-Scholes model is its constant volatility assumption. Statistical analysis of stock market data shows that the volatility of a stock is a time-dependent quantity. Moreover, it exhibits various random features. Stochastic volatility models address this randomness by assuming that both the stock price and the volatility are stochastic processes affected by different sources of risk.

- Hull-White model The stock price process X and the volatility process Y in the Hull-White model satisfy the following system of stochastic differential equations:

$$\begin{cases} dX_t = \mu X_t dt + Y_t X_t dW_t \\ dY_t = \nu Y_t dt + \xi Y_t dZ_t, \end{cases}$$

where $\mu, \nu \in \mathbb{R}$ and $\xi > 0$.

- It is assumed in the previous equations that standard Brownian motions W and Z are such that

$$d\langle W, Z \rangle_t = \rho dt$$

with $\rho \in [-1, 1]$. The initial conditions for the processes X and Y are denoted by x_0 and y_0 , respectively. The volatility process in the Hull-White model is a geometric Brownian motion.

A Sharp Asymptotic Formula for the Stock Price Density

The following formula holds for the stock price density in the uncorrelated Hull-White model (see [6]):

$$D_t(x) = Cx^{-2}(\log x)^{\frac{c_2-1}{2}} (\log \log x)^{c_3}$$

$$\exp \left\{ -\frac{1}{2t\xi^2} \left(\log \left[\frac{1}{y_0} \sqrt{\frac{2}{t}} \log \frac{x}{x_0 e^{\mu t}} \right] + \frac{1}{2} \log \log \left[\frac{1}{y_0} \sqrt{\frac{2}{t}} \log \frac{x}{x_0 e^{\mu t}} \right] \right)^2 \right\}$$

$$\left(1 + O \left((\log \log x)^{-\frac{1}{2}} \right) \right)$$

as $x \rightarrow \infty$.

A Short Sketch of the Proof of the Asymptotic Formula

- Step 1. Define the realized volatility for the Hull-White model as follows:

$$\alpha_t = \left\{ \frac{1}{t} \int_0^t Y_s^2 ds \right\}^{\frac{1}{2}}.$$

The distribution of the realized volatility is called the mixing distribution. This distribution admits a density m_t , which is called the mixing distribution density.

- Step 2. For a geometric Brownian motion with $\nu = \frac{1}{2}$, $\xi = 1$, and $y_0 = 1$, the following formula is known (Alili-Gruet, an alternative proof can be found in Stein-A.G. [6]):

$$m_t \left(y; \frac{1}{2}, 1, 1 \right) = \frac{1}{\pi t} \exp \left\{ \frac{\pi^2}{8t} \right\} y^{-2}$$

$$\int_{-\infty}^{\infty} \exp \left\{ -\frac{\cosh^2 u}{2ty^2} \right\} \cosh u \exp \left\{ -\frac{u^2}{2t} \right\} \exp \left\{ \frac{i\pi u}{2t} \right\} du.$$

- Step 3. Prove the following asymptotic formula for the mixing density:

$$m_t(y; \nu, \xi, y_0) = c_1 y^{c_2} (\log y)^{c_3}$$

$$\exp \left\{ -\frac{1}{2t\xi^2} \left(\log \frac{y}{y_0} + \frac{1}{2} \log \log \frac{y}{y_0} \right)^2 \right\} \left(1 + O \left((\log y)^{-\frac{1}{2}} \right) \right)$$

as $y \rightarrow \infty$.

- The previous asymptotic formula is first established in the special case where $\nu = \frac{1}{2}$, $\xi = 1$, and $y_0 = 1$, using the explicit formula for the mixing density in the Hull-White model.
- In order to understand the behavior of the integral, representing the mixing density, the following oscillating integral is studied:

$$\begin{aligned} I(\varepsilon) &= \int_{-\infty}^{\infty} \exp \left\{ -\varepsilon (\cosh u)^2 \right\} \exp \left\{ -\frac{u^2}{2t} \right\} \cosh u \exp \left\{ \frac{i\pi u}{2t} \right\} du \end{aligned}$$

where $\varepsilon > 0$.

- We have

$$I(\varepsilon) = I_0(\varepsilon) \left(1 + O \left(\log \frac{1}{\varepsilon} \right)^{-\frac{1}{2}} \right)$$

as $\varepsilon \rightarrow 0$, where

$$I_0(\varepsilon) = \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \exp \left\{ -\frac{\pi^2}{8t} \right\} \exp \left\{ -\frac{N_\varepsilon^2}{2t} + \frac{N_\varepsilon}{2t} \right\} \varepsilon^{-\frac{1}{2}},$$

and N_ε is the solution of the equation

$$\varepsilon \sinh (2N_\varepsilon) = \frac{N_\varepsilon}{t}.$$

- The proof of the previous formula requires that we deform the contour of integration for I_ε (the real one) into the complex u -plane, where the principal contributions are then given on the segments $[N_\varepsilon, N_\varepsilon + i\pi]$ and $[-N_\varepsilon, -N_\varepsilon + i\pi]$.

- After completing the proof of the asymptotic formula in the case where $\nu = \frac{1}{2}$, $\xi = 1$, and $y_0 = 1$, we can use the corresponding asymptotics for

$$\left(\frac{d}{d\varepsilon}\right)^k I(\varepsilon)$$

to obtain the required result for $\nu = 2k + \frac{1}{2}$, where k is a positive integer.

- In the case where $\nu \neq 2k + \frac{1}{2}$, we use Dufresne's recurrence formula which allows to navigate between Hull-White models with different values of the model parameters.
- Finally, we drop the restriction $\xi = 1$, $y_0 = 1$, using special scaling properties of the mixing density.
- Step 4. Represent the stock price density D_t as a log-normal integral operator applied to the mixing density:

$$\begin{aligned} D_t(x) &= \frac{1}{\sqrt{2\pi t}} \int_0^\infty y^{-1} \exp \left\{ - \left[\frac{\log^2 x}{2ty^2} + \frac{ty^2}{8} \right] \right\} m_t(x) dx. \end{aligned}$$

- Step 5. Use the asymptotic formula for the Hull-White mixing density, and the following Abelian theorem obtained in [6] to establish the asymptotic formula for the stock price density.

An Abelian theorem

Let A , ζ , and b be positive Borel functions on $[0, \infty)$, and suppose the following conditions hold:

1. The functions A and ζ are integrable over any finite subinterval of $[0, \infty)$.
2. The function b is bounded and $\lim_{y \rightarrow \infty} b(y) = 0$.
3. There exist $y_1 > 0$, $c > 0$, and γ with $0 < \gamma \leq 1$ such that ζ and b are differentiable on $[y_1, \infty)$, and in addition,

$$|\zeta'(y)| \leq cy^{-\gamma}\zeta(y) \quad \text{and} \quad |b'(y)| \leq cy^{-\gamma}b(y)$$

for all $y \geq y_1$.

4. For every $a > 0$, there exists $y_a > 0$ such that

$$b(y)\zeta(y) \geq \exp \{ -a y^4 \}$$

for all $y > y_a$.

5. There exists a real number l such that

$$A(y) = e^{ly}\zeta(y)(1 + O(b(y)))$$

as $y \rightarrow \infty$.

Then, for every fixed $k > 0$ and $w \rightarrow \infty$,

$$\begin{aligned} & \int_0^\infty A(y) \exp \left\{ - \left(\frac{w^2}{y^2} + k^2 y^2 \right) \right\} dy \\ &= \frac{\sqrt{\pi}}{2k} \exp \left\{ \frac{l^2}{16k^2} \right\} \zeta \left(k^{-\frac{1}{2}} w^{\frac{1}{2}} \right) \exp \left\{ l k^{-\frac{1}{2}} w^{\frac{1}{2}} \right\} e^{-2kw} \\ & \left[1 + O \left(w^{-\frac{\gamma}{2}} \right) + O \left(b \left(k^{-\frac{1}{2}} w^{\frac{1}{2}} \right) \right) \right]. \end{aligned}$$

- Heston model The stock price process X and the variance process Y in the Heston model satisfy the following system of stochastic differential equations:

$$\begin{cases} dX_t = \mu X_t dt + \sqrt{Y_t} X_t dW_t \\ dY_t = q(m - Y_t) dt + c\sqrt{Y_t} dZ_t, \end{cases}$$

where $q \geq 0$, $m \geq 0$, and $c > 0$.

- The standard Brownian motions W and Z are correlated with

$$d\langle W, Z \rangle_t = \rho dt$$

The correlation coefficient ρ satisfies $\rho \in [-1, 1]$. The initial conditions for the processes X and Y are denoted by x_0 and y_0 , respectively. The variance process in the Heston model is called the Cox-Ingersoll-Ross process (the Feller process).

- In terms of the log-price process $\tilde{X} = \log X$ and the variance process Y , the Heston model can be rewritten as follows:

$$\begin{cases} d\tilde{X}_t = (\mu - \frac{1}{2}Y_t) dt + \sqrt{Y_t} dW_t \\ dY_t = q(m - Y_t) dt + c\sqrt{Y_t} dZ_t. \end{cases}$$

A Sharp Asymptotic Formula for the Stock Price Density

For every $t > 0$, the following formula holds for the distribution density D_t of the stock price X_t in the Heston model with $-1 < \rho \leq 0$:

$$D_t(x) = A_1 x^{-A_3} e^{A_2 \sqrt{\log x}} (\log x)^{-\frac{3}{4} + \frac{a}{c^2}} \left(1 + O((\log x)^{-\frac{1}{2}})\right) \quad (1)$$

as $x \rightarrow \infty$.

- The previous formula was obtained in the case where $\rho = 0$ in a joint paper [7] of E. M. Stein and A. G. For the correlated Heston model, the formula was established by P. Friz, S. Gerhold, S. Sturm, and A. G. (see [2]).

The Constants A_1 , A_2 , and A_3

Useful parameters:

- *The Explosion Time for the Moment of Order $s > 1$:*

$$T^*(s) = \sup \{t \geq 0 : \mathbb{E}[X_t^s] < \infty\}.$$

- The Upper Critical Moment:

$$s_+ = s_+(T) = \sup \{s \geq 1 : \mathbb{E}[X_T^s] < \infty\}.$$

- The Upper Critical Slope:

$$\sigma = - \left. \frac{\partial T^*(s)}{\partial s} \right|_{s=s_+}.$$

- The Upper Critical Curvature:

$$\kappa = \left. \frac{\partial^2 T^*(s)}{\partial s^2} \right|_{s=s_+}.$$

- An Explicit Formula for T^* :

$$T^*(s) = \frac{2}{\sqrt{-\Delta(s)}} \left[\arctan \frac{\sqrt{-\Delta(s)}}{\chi(s)} + \pi \right],$$

where $\chi(s) = spc - b$ and $\delta(s) = (spc - b)^2 - c^2(s^2 - s)$.

- Explicit formulas for the constants σ and κ can be obtained by differentiating the functions in the formula for T^* and plugging $s = s_+$ into the resulting formulas.
- The following formulas hold:

$$\begin{aligned}
A_1 &= \frac{1}{\sqrt{\pi}} 2^{-\frac{3}{4} - \frac{a}{c^2}} y_0^{\frac{1}{4} - \frac{a}{c^2}} c^{\frac{2a}{c^2} - \frac{1}{2}} \sigma^{-\frac{a}{c^2} - \frac{1}{4}} \\
&\times \exp \left\{ -y_0 \left(\frac{c\rho s_+ - b}{c^2} + \frac{\kappa}{c^2 \sigma^2} \right) - \frac{aT}{c^2} (c\rho s_+ - b) \right\} \\
&\times \left\{ \frac{2\sqrt{(b - c\rho s_+)^2 + c^2(s_+ - s_+^2)}}{c^2 s_+ (s_+ - 1) \sinh \left[\frac{T}{2} \sqrt{(b - c\rho s_+)^2 + c^2(s_+ - s_+^2)} \right]} \right\}^{\frac{2a}{c^2}},
\end{aligned}$$

$$A_2 = 2 \frac{\sqrt{2y_0}}{c\sqrt{\sigma}}, \quad \text{and} \quad A_3 = s_+ + 1.$$

- The constants A_1 , A_2 , and A_3 can be expressed in terms of the constant s_+ and the Heston model parameters.

Sketches of Two Different Proofs of the Asymptotic Formula

Proof I ([7], the uncorrelated Heston model)

- Step 1. The realized volatility in the Heston model is as follows:

$$\alpha_t = \left\{ \frac{1}{t} \int_0^t Y_s ds \right\}^{\frac{1}{2}}.$$

As before, the mixing distribution density is denoted by m_t .

- Step 2. Compute the exponential functional of the variance process/the Laplace transform of the mixing density given by

$$\mathbb{E} \left[\exp \left\{ -\lambda \int_0^t Y_s ds \right\} \right] = \int_0^\infty e^{-\lambda t y^2} m_t(y) dy,$$

using the Pitman-Yor theorem concerning exponential functionals of Bessel processes, and the fact that the CIR process is a time-changed Bessel process.

- Step 3. Find an asymptotic formula for the mixing distribution, using the following theorem established in [7].

Tauberian Theorem Suppose the Laplace transform,

$$I(\lambda) = \int_0^{\infty} e^{-\lambda y} M(y) dy, \quad \lambda > 0,$$

of a positive function M defined on $(0, \infty)$ satisfies the following six conditions:

1. The function I can be analytically continued from $(0, \infty)$ into the open right half-plane

$$\mathbb{C}_+ = \{\lambda : \operatorname{Re}(\lambda) > 0\}.$$

2. The function I admits the following factorization in \mathbb{C}_+ :

$$I(\lambda) = \lambda^{\gamma_1} G_1(\lambda)^{\gamma_2} G_2(\lambda) e^{F(\lambda)},$$

where γ_1 and γ_2 are non-negative constants.

3. The function G_2 is analytic in the closed half-plane

$$\overline{\mathbb{C}}_+ = \{\lambda : \operatorname{Re}(\lambda) \geq 0\}$$

and satisfies the condition $G_2(0) \neq 0$.

4. The function G_1 is analytic in $\overline{\mathbb{C}}_+$ except for a simple pole at $\lambda = 0$ with residue 1, that is,

$$G_1(\lambda) = \frac{1}{\lambda} + \tilde{G}(\lambda)$$

where \tilde{G} is an analytic function in $\overline{\mathbb{C}}_+$.

5. The function F is analytic in $\overline{\mathbb{C}}_+$ and has a simple pole at $\lambda = 0$ with residue $\alpha > 0$, that is,

$$F(\lambda) = \frac{\alpha}{\lambda} + \tilde{F}(\lambda),$$

where \tilde{F} is an analytic function in $\overline{\mathbb{C}}_+$.

6. The following growth condition is satisfied:

$$\left| G_1(\lambda)^{\gamma_2} G_2(\lambda) e^{F(\lambda)} \right| \leq \exp \{ -|\lambda|^\delta \}$$

as $|\lambda| \rightarrow \infty$ in $\overline{\mathbb{C}}_+$, for some $\delta > 0$.

Then

$$M(y) = \frac{1}{2\sqrt{\pi}} \alpha^{\frac{1}{4} + \frac{\gamma_1 - \gamma_2}{2}} G_2(0) e^{\tilde{F}(0)} y^{-\frac{3}{4} + \frac{\gamma_2 - \gamma_1}{2}} e^{2\sqrt{\alpha}\sqrt{y}} \left(1 + O\left(y^{-\frac{1}{2}}\right)\right)$$

as $y \rightarrow \infty$.

- For the Heston model, the Tauberian theorem gives

$$m_t(y) = A e^{-Cy^2} e^{By} y^{-\frac{1}{2} + \frac{2a}{c^2}} \left(1 + O\left(y^{-1}\right)\right)$$

as $y \rightarrow \infty$.

- Step 4. Represent the stock price density D_t by the log-normal integral.
- Step 5. Use the asymptotic formula for the mixing distribution density and a special Abelian theorem formulated above to establish the asymptotic formula for the stock price density in the uncorrelated Heston model.

Proof II ([2], the general case)

- Step 1. Use affine principles to show that the moment generating function in the Heston model is given by

$$\log \mathbb{E} \left[e^{s\tilde{X}_t} \right] = \varphi(s, t) + y_0\psi(s, t).$$

- The functions φ and ψ in the previous equality satisfy the following Riccati equations:

$$\dot{\phi} = F(s, \psi), \quad \phi(0) = 0,$$

$$\dot{\psi} = R(s, \psi), \quad \psi(0) = 0,$$

with

$$F(s, v) = av \quad \text{and} \quad R(s, v) = \frac{1}{2}(s^2 - s) + \frac{1}{2}c^2v^2 - bv + s\rho cv.$$

- Step 2. Consider the Mellin transform defined by

$$MD_T(u) = \mathbb{E} [X_T^{u-1}] = \mathbb{E} \left[e^{(u-1)\tilde{X}_t} \right].$$

The domain of MD_T is the strip $(s_- + 1, s_+ + 1)$ in the complex plane. Using the Mellin inversion formula, we obtain

$$D_T(x) = \frac{1}{2\pi i} \int_{s-i\infty}^{s+i\infty} e^{-uL + \phi(u-1, T) + y_0\psi(u-1, T)} du. \quad (2)$$

Here $L = \log x$ and $\Re(s) \in (s_-(T), s_+(T))$.

- Step 3. Asymptotic expansion:

$$\begin{aligned} & \phi(u-1, T) + y_0\psi(u-1, T) \\ &= \frac{\beta^2}{u^* - u} + \frac{2a}{c^2} \log \frac{1}{u^* - u} + \Gamma + O(u^* - u) \end{aligned}$$

as $u \rightarrow u^* = s_+ + 1$ with $\Re(u) < u^*$. In the previous formula, Γ and β do not depend on u .

- Step 4. Consider an approximate saddle point equation.

We keep only the dominating term in the previous asymptotic expansion.

Equation:

$$\left[x^{-u} \exp \left\{ \frac{\beta^2}{u^* - u} \right\} \right]' = 0.$$

Solution:

$$\hat{u} = \hat{u}(x) = u^* - \beta L^{-1/2}.$$

- Step 5. Steepest descent method. Shift the contour of integration in the Mellin transform formula to $s = \hat{u}$. This gives

$$D_T(x) = x^{-\hat{u}} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iyL + \phi(\hat{u} + iy - 1, T) + y_0 \psi(\hat{u} + iy - 1, T)} dy.$$

- Step 6. The rest of the proof uses an expansion of the function

$$\phi(\hat{u} - 1 + iy, T) + y_0\psi(\hat{u} - 1 + iy, T)$$

for sufficiently small values of y . This helps to find leading term and prove tail estimates.

- For the uncorrelated Heston model, the constants A_1 , A_2 , and A_3 in the asymptotic formula for the stock price density obtained in the proofs sketched above are given by different expressions. They were reconciled in [2] for the constants A_2 and A_3 . Recently, the author proved that the expressions for the constant A_1 in different proofs are also equal.

- Stein-Stein Model In the absence of correlation between the stock price and the volatility process, the Stein-Stein model can be considered in the following two forms:

$$\begin{cases} dX_t = \mu X_t dt + Y_t X_t dW_t \\ dY_t = q(m - Y_t) dt + \sigma dZ_t, \end{cases}$$

or

$$\begin{cases} dX_t = \mu X_t dt + |Y_t| X_t dW_t \\ dY_t = q(m - Y_t) dt + \sigma dZ_t, \end{cases}$$

where $q \geq 0$, $m \geq 0$, and $\sigma > 0$.

- The volatility process in the first of the previous models is the Ornstein-Uhlenbeck process, while in the second model the process $|Y|$ is used to model the volatility. It is known that the marginal distributions of the stock price process in both models coincide. In the presence of correlation, it is more popular to consider the Stein-Stein model with Y instead of $|Y|$.

A Sharp Asymptotic Formula for the Stock Price Density

For every $t > 0$, the following formula holds for the distribution density D_t of the stock price X_t in the correlated Stein-Stein model with $-1 < \rho \leq 0$:

$$D_t(x) = B_1 x^{-B_3} e^{B_2 \sqrt{\log x}} (\log x)^{-\frac{1}{2}} (1 + O((\log x)^{-\frac{1}{2}}))$$

as $x \rightarrow \infty$.

- The asymptotic formula for the stock price density in the Stein-Stein model formulated above was established for uncorrelated Stein-Stein models by E. M. Stein and A.G. (see [7]). For the correlated Stein-Stein model with the long-run mean m equal to zero, the formula follows from the corresponding formula for the Heston density. For the correlated Stein-Stein model with $m \neq 0$, the asymptotic formula for the stock price density was obtained in a recent paper of J.-D. Deuschel, P. Friz, A. Jacquer, and S. Violante [1].

Asymptotic Behavior of Call Pricing Functions

It is assumed that the restrictions on the models imposed above hold.

- *Hull-White model:* Let C be the call pricing function in the uncorrelated Hull-White model. Then

$$C(K) = 4T\xi^2 C_0 e^{-rT} (\log K)^{\frac{c_2+1}{2}} (\log \log K)^{c_3-1}$$

$$\exp \left\{ -\frac{1}{2T\xi^2} \left(\log \left[\frac{1}{y_0} \sqrt{\frac{2 \log K}{T}} \right] + \frac{1}{2} \log \log \left[\frac{1}{y_0} \sqrt{\frac{2 \log K}{T}} \right] \right)^2 \right\}$$

$$\left(1 + O \left((\log \log K)^{-\frac{1}{2}} \right) \right)$$

as $K \rightarrow \infty$.

- *Heston model:* Let C be the call pricing function in the Heston model. Then

$$C(K) = \widehat{A}_1 (\log K)^{-\frac{3}{4} + \frac{qm}{c^2}} e^{\widehat{A}_2 \sqrt{\log K}} K^{-\widehat{A}_3} \left(1 + O\left((\log K)^{-\frac{1}{2}}\right)\right)$$

as $K \rightarrow \infty$.

- *Stein-Stein model:* Let C be the call pricing function in the Stein-Stein model. Then

$$C(K) = \widehat{B}_1 (\log K)^{-\frac{1}{2}} e^{\widehat{B}_2 \sqrt{\log K}} K^{-\widehat{B}_3} \left(1 + O\left((\log K)^{-\frac{1}{2}}\right)\right)$$

as $K \rightarrow \infty$.

Asymptotic Behavior of the Implied Volatility

- *General Theorems (see [4]).* Let C be a call pricing function, and let \tilde{C} be a positive function such that

$$\tilde{C}(K) \approx C(K)$$

as $K \rightarrow \infty$. Then

$$\begin{aligned} \sqrt{T}I(K) &= \sqrt{2 \log K + 2 \log \frac{1}{\tilde{C}(K)} - \log \log \frac{1}{\tilde{C}(K)}} \\ &\quad - \sqrt{2 \log \frac{1}{\tilde{C}(K)} - \log \log \frac{1}{\tilde{C}(K)}} \\ &\quad + O \left(\left(\log \frac{1}{\tilde{C}(K)} \right)^{-\frac{1}{2}} \right) \end{aligned}$$

as $K \rightarrow \infty$.

- In an important recent paper [3] of K. Gao and R. Lee, the previous formula was generalized. More expansion terms were obtained, but a flexibility of passing from the function C to the function \tilde{C} was lost.
- Let C be a call pricing function, and let P be the corresponding put pricing function. Suppose that

$$P(K) \approx \tilde{P}(K)$$

as $K \rightarrow 0$, where \tilde{P} is a positive function. Then the following asymptotic formula holds:

$$\begin{aligned} \sqrt{TI}(K) &= \sqrt{2 \log \frac{1}{\tilde{P}(K)} - \log \log \frac{K}{\tilde{P}(K)}} \\ &\quad - \sqrt{2 \log \frac{K}{\tilde{P}(K)} - \log \log \frac{K}{\tilde{P}(K)}} \\ &\quad + O \left(\left(\log \frac{K}{\tilde{P}(K)} \right)^{-\frac{1}{2}} \right) \end{aligned}$$

as $K \rightarrow 0$.

- It follows that for any call pricing function C ,

$$\begin{aligned}\sqrt{T}I(K) &= \sqrt{2\log K + 2\log \frac{1}{C(K)}} - \sqrt{2\log \frac{1}{C(K)}} \\ &+ O\left(\left(\log \frac{1}{C(K)}\right)^{-\frac{1}{2}} \log \log \frac{1}{C(K)}\right)\end{aligned}$$

as $K \rightarrow \infty$. Moreover,

$$\begin{aligned}\sqrt{T}I(K) &= \sqrt{2\log \frac{1}{P(K)}} - \sqrt{2\log \frac{K}{P(K)}} \\ &+ O\left(\left(\log \frac{K}{P(K)}\right)^{-\frac{1}{2}} \log \log \frac{K}{P(K)}\right)\end{aligned}$$

as $K \rightarrow 0$.

- Fix the maturity T and consider the implied volatility as a function $k \mapsto I(k)$ of the log-strike $k = \log K$.
- The following asymptotic formula holds for the implied volatility in the Hull-White model (see [5]):

$$\begin{aligned}
I(k) &= \frac{\sqrt{2}}{\sqrt{T}} \sqrt{k} \\
&\quad - \frac{1}{\sqrt{T}} \sqrt{\frac{1}{4T\xi^2} (\log k + \log \log k)^2 + a_1 \log k + a_2 \log \log k} \\
&\quad + O\left(\frac{1}{\log k}\right)
\end{aligned}$$

as $k \rightarrow \infty$.

- The following asymptotic formula holds for the implied volatility in the Heston model (see [7] and [2]):

$$\widehat{I}(k) = \beta_1 k^{\frac{1}{2}} + \beta_2 + \beta_3 \frac{\log k}{k^{\frac{1}{2}}} + O\left(\frac{1}{k^{\frac{1}{2}}}\right)$$

as $k \rightarrow \infty$.

- The following asymptotic formula holds for the implied volatility in the Stein-Stein model (see [7] and [1]):

$$\widehat{I}(k) = \gamma_1 k^{\frac{1}{2}} + \gamma_2 + O\left(\frac{1}{k^{\frac{1}{2}}}\right)$$

as $k \rightarrow \infty$.

- The previous formulas can be strengthened, using the results of K. Gao and R. Lee obtained in [3].

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