Asymptotic Behavior of Implied Volatility at Extreme Strikes in Stochastic Stock Price Models

Archil Gulisashvili
Ohio University

Universitat Autònoma de Barcelona, April 26, 2012

Asset Price Process

- A random behavior of the asset price is modeled by a positive adapted stochastic process X defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}^*)$.

- It is assumed that the following conditions hold:

  1. For every $t > 0$, the stock price X_t is an unbounded random variable.

  2. $\mathbb{E}^*[X_t] < \infty$, $t > 0$.

  3. $X_0 = x_0$ $\mathbb{P}^*$-a.s. for some $x_0 > 0$.

  4. $\mathbb{P}^*$ is a risk-free measure. This means that the discounted stock price process $\{e^{-rt}X_t\}_{t\geq 0}$ is a martingale. Here $r \geq 0$ is the interest rate.

Pricing Functions

- The pricing function for a European call option at time $t = 0$ is defined by

$$C(T, K) = e^{-rT}\mathbb{E}^* \left[(X_T - K)^+\right]$$

where $K > 0$ is the strike price and $T > 0$ is the maturity.

- The pricing function for a European put option at time t is defined by

$$P(T, K) = e^{-rT}\mathbb{E}^* \left[(K - X_T)^+\right].$$

- The functions C and P satisfy the put-call parity condition

$$C(T, K) = P(T, K) + x_0 - e^{-rT}K.$$

Black-Scholes Call Pricing Function

- In the Black-Scholes model, the stock price process is a geometric Brownian motion, satisfying the following stochastic differential equation:

$$dX_t = rX_t dt + \sigma X_t dW_t^*,$$

where $r \geq 0$ is the interest rate, $\sigma > 0$ is the volatility of the stock, and W^* is a standard Brownian motion under the risk-free measure $\mathbb{P}^*$.

- The stock price process X in the Black-Scholes model is given by

$$X_t = x_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t^*\right\}$$

where $x_0 > 0$ is the initial price.

- Black and Scholes found an explicit formula for the pricing function C_{BS}:

$$C_{BS}\left(T,K,\sigma\right)=x_0N\left(d_1(K,\sigma)\right)-Ke^{-rT}N\left(d_2(K,\sigma)\right),$$

  where

$$d_1(K,\sigma)=\frac{\log x_0-\log K+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}},$$

$$d_2(K,\sigma)=d_1(K,\sigma)-\sigma\sqrt{T},$$

  and

$$N(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}\exp\left\{-\frac{y^2}{2}\right\}dy.$$

Implied Volatility

Let C be a call pricing function. The implied volatility

$$I=I(T,K),\quad (T,K)\in(0,\infty)^2,$$

associated with the pricing function C, is a function of two variables satisfying the following condition:

$$C_{BS}(T,K,I(T,K))=C(T,K).$$

Asymptotic Behavior of the Implied Volatility

- Let C be a call pricing function, and let $\widetilde{C}$ be a positive function such that

$$\widetilde{C}(K) \approx C(K)$$

as $K \to \infty$. Then

$$\sqrt{T} I(K) = \sqrt{2\log K + 2\log\frac{1}{\widetilde{C}(K)} - \log\log\frac{1}{\widetilde{C}(K)}}$$

$$- \sqrt{2\log\frac{1}{\widetilde{C}(K)} - \log\log\frac{1}{\widetilde{C}(K)}}$$

$$+ O\left(\left(\log\frac{1}{\widetilde{C}(K)}\right)^{-\frac{1}{2}}\right)$$

as $K \to \infty$.

- The previous formula was obtained in [5]. This formula was generalized in an important recent paper [4] of K. Gao and R. Lee.

- First, we prove the following proposition. Let C be a call pricing function, and fix a positive continuous increasing function ψ, satisfying $\psi(K) \to \infty$ as $K \to \infty$. Suppose ϕ is a positive function such that $\phi(K) \to \infty$ as $K \to \infty$ and

$$C(K) \approx \frac{\psi(K)}{\phi(K)} \exp\left\{-\frac{\phi(K)^2}{2}\right\}.$$

Then we have

$$I(K) = \frac{1}{\sqrt{T}}\left(\sqrt{2\log\frac{K}{x_0 e^{rT}} + \phi(K)^2} - \phi(K)\right)$$

$$+ O\left(\frac{\psi(K)}{\phi(K)}\right)$$

as $K \to \infty$.

- To establish the previous statement, we first show that for every function $\widetilde{I}$ such that

$$0 < \widetilde{I}(K) < I(K), \quad K > K_0,$$

the following asymptotic formula holds:

$$I(K) = \widetilde{I}(K) + O\left(C(K)\exp\left\{\frac{1}{2}d_1\left(K, \widetilde{I}(K)\right)^2\right\}\right)$$

as $K \to \infty$.

- *The choice of* $\widetilde{I}$. The function $\widetilde{I}$ is determined from the equality

$$d_1\left(K, \tilde{I}(K)\right) = -\phi(K), \quad K > K_0.$$

Such a function exists, since for large values of K the function $\sigma \mapsto d_1(K, \sigma)$ increases from $-\infty$ to ∞.

- *Explicit formula for the function* $\widetilde{I}$:

$$\widetilde{I}(K) = \frac{1}{\sqrt{T}}\left(\sqrt{2\log\frac{K}{x_0 e^{rT}} + \phi(K)^2} - \phi(K)\right).$$

- Prove that the function $\widetilde{I}$ defined above satisfies

$$\widetilde{I}(K) \leq I(K)$$

for $K > K_0$, and use the asymptotic formula relating I and $\widetilde{I}$ to establish the statement formulated above.

- To prove the asymptotic formula for the implied volatility, we choose the function ϕ as follows:

$$\phi(K) = \left[2 \log \frac{1}{C(K)} - \log \log \frac{1}{C(K)} + 2 \log \psi(K) \right]^{\frac{1}{2}}.$$

- It is not hard to check that if the function ϕ is defined by the previous formula, then the condition in the proposition formulated above holds. Using this proposition and the mean value theorem, and getting rid of the function ψ in the error term, we obtain the asymptotic formula for the implied volatility.

- Let C be a call pricing function, and let P be the corresponding put pricing function. Suppose that

$$P(K) \approx \widetilde{P}(K)$$

as $K \rightarrow 0$, where $\widetilde{P}$ is a positive function. Then the following asymptotic formula holds:

$$\sqrt{T} I(K) = \sqrt{2 \log \frac{1}{\widetilde{P}(K)} - \log \log \frac{K}{\widetilde{P}(K)}}$$

$$- \sqrt{2 \log \frac{K}{\widetilde{P}(K)} - \log \log \frac{K}{\widetilde{P}(K)}}$$

$$+ O\left(\left(\log \frac{K}{\widetilde{P}(K)}\right)^{-\frac{1}{2}}\right)$$

as $K \rightarrow 0$.

- Using the mean value theorem, we see that for any call pricing function C,

$$\sqrt{T}I(K) = \sqrt{2\log K + 2\log\frac{1}{C(K)}} - \sqrt{2\log\frac{1}{C(K)}}$$

$$+ O\left(\left(\log\frac{1}{C(K)}\right)^{-\frac{1}{2}} \log\log\frac{1}{C(K)}\right)$$

as $K \to \infty$. Moreover,

$$\sqrt{T}I(K) = \sqrt{2\log\frac{1}{P(K)}} - \sqrt{2\log\frac{K}{P(K)}}$$

$$+ O\left(\left(\log\frac{K}{P(K)}\right)^{-\frac{1}{2}} \log\log\frac{K}{P(K)}\right)$$

as $K \to 0$.

Lee's Moment Formulas

The theorems formulated below were obtained by R. Lee (see [10]).

- Let I be the implied volatility associated with a call pricing function C. Define a number $\tilde{p}$ by

$$\tilde{p} = \sup \left\{ p \geq 0 : \mathbb{E}^* \left[X_T^{1+p} \right] < \infty \right\}.$$

Then the following equality holds:

$$\limsup_{K \to \infty} \frac{TI(K)^2}{\log K} = \psi(\tilde{p})$$

where the function ψ is given by

$$\psi(u) = 2 - 4 \left(\sqrt{u^2 + u} - u \right), \quad u \geq 0.$$

- Let I be the implied volatility associated with a call pricing function C. Define a number $\tilde{q}$ by

$$\tilde{q} = sup \left\{ q \geq 0 : \mathbb{E}\left[X_T^{-q} \right] < \infty \right\}.$$

  Then the following formula holds:

$$\limsup_{K \to 0} \frac{T I(K)^2}{\log \frac{1}{K}} = \psi(\tilde{q}).$$

- Lee's moment formulas characterize the behavior of the implied volatility for large and small strikes in terms of the critical orders at which the moments of the stock price distribution explode. An asset price model admits exploding moments if $\tilde{p} < \infty$ and $\tilde{q} < \infty$. The model does not have exploding moments if $\tilde{p} = \tilde{q} = \infty$.

- The moment formulas can be derived from sharp asymptotic formulas for the implied volatility formulated above (see [5]).

Regularly Varying Functions

- Let $\alpha \in \mathbb{R}$, and let f be a positive function defined on some neighborhood of infinity. The function f is called regularly varying with index α if for every $\lambda > 0$,

$$\frac{f(\lambda x)}{f(x)} \to \lambda^{\alpha}$$

  as $x \to \infty$. The class consisting of all regularly varying functions with index α is denoted by R_{α}. Functions belonging to the class R_0 are called slowly varying.

- Let $\alpha \in \mathbb{R}$, and let f be a positive function defined on some neighborhood of infinity. The function f is called smoothly varying with index α if the function

$$h(x) = \log f\left(e^{x}\right)$$

  is infinitely differentiable and the following conditions hold as $x \to \infty$:

$$h'(x) \to \alpha, \quad h^{(n)}(x) \to 0 \quad \text{for all integers} \quad n \geq 2.$$

- An equivalent definition of the class SR_α is as follows:

$$f \in SR_\alpha \Leftrightarrow \lim_{x\to\infty} \frac{x^n f^{(n)}(x)}{f(x)} = \alpha(\alpha-1)\ldots(\alpha-n+1)$$

for all $n \geq 1$.

- The following are examples of smoothly varying functions with index 0:

$$(\log x)^a, \quad a \in \mathbb{R};$$

$$(\log\log x)^a, \quad a \in \mathbb{R};$$

$$\exp\left\{(\log x)^b\right\}, \quad b < 1.$$

It can be shown that all smoothly varying functions with index 0 are slowly varying.

Tail-Wing Formulas of Benaim and Friz

Asymptotic formulas found by Benaim and Friz (see [1]) provide a link between the behavior of the implied volatility at extreme strikes and the tail behavior of the distribution of the stock price. Such formulas are called tail-wing formulas. We will next formulate some of the results obtained by Benaim and Friz, adapting these results our notation. It is assumed that the stock price X_T satisfies the following condition:

$$\mathbb{E}^* \left[X_T^{1+\varepsilon} \right] < \infty \quad \text{for some} \quad \varepsilon > 0.$$

Call pricing function – implied volatility

Suppose that

$$C(K) = \exp\left\{-\eta(\log K)\right\}$$

with $\eta \in R_\alpha$, $\alpha > 0$. Then

$$I(K) \sim \frac{\sqrt{\log K}}{\sqrt{T}} \sqrt{\psi\left(-\frac{\log C(K)}{\log K}\right)}$$

as $K \to \infty$, where

$$\psi(u) = 2 - 4\left(\sqrt{u^2 + u} - u\right).$$

Complementary distribution function – implied volatility

Let $\bar{F}_T$ be the complementary distribution function of the stock price X_T given by

$$\bar{F}_T(y) = \mathbb{P}^* \left(X_T > y \right),$$

and suppose that

$$\bar{F}_T(y) = \exp \left\{ -\rho(\log y) \right\}$$

with $\rho \in R_\alpha$, $\alpha > 0$. Then

$$I(K) \sim \frac{\sqrt{\log K}}{\sqrt{T}} \sqrt{\psi \left(-\frac{\log[K\rho(K)]}{\log K} \right)}$$

as $K \to \infty$.

Stock price density – implied volatility

If the distribution μ_T of the stock price X_T admits a density D_T, and if

$$D_T(x) = \frac{1}{x} \exp \left\{ -h(\log x) \right\}$$

as $x \to \infty$, where $h \in R_\alpha$, $\alpha > 0$, then

$$I(K) \sim \frac{\sqrt{\log K}}{\sqrt{T}} \sqrt{\psi \left(-\frac{\log[K^2 D_T(K)]}{\log K} \right)}$$

as $K \to \infty$.

Tail-Wing Formulas. Improvements

- The following formula is equivalent to one of the sharp asymptotic formulas for the implied volatility formulated above:

$$I(K) = \frac{\sqrt{\log K}}{\sqrt{T}} \sqrt{\psi\left(-\frac{\log C(K)}{\log K}\right)}$$

$$+ O\left(\left(\log \frac{1}{C(K)}\right)^{-\frac{1}{2}} \log \log \frac{1}{C(K)}\right)$$

as $K \to \infty$.

- The previous formula is stronger than a similar formula obtained by Benaim and Friz. Note that there are no restrictions on the call pricing function in the previous formula, and an error term is included.

- The tail-wing formulas due to Benaim and Friz do not contain error terms. We will next discuss tail-wing formulas with error estimates. Smoothly varying functions play an important role in this discussion.

Tail-wing formulas with error estimates

- Let C be a call pricing function and let $\bar{F}$ be the complementary distribution function of the stock price X_T. Suppose that

$$\bar{F}(y) \approx \exp\{-\rho(\log y)\}$$

as $y \to \infty$, where ρ is a function such that either $\rho \in SR_\alpha$ with $\alpha > 1$, or $\rho \in SR_1$ and $\lambda(u) = \rho(u) - u \in R_\beta$ for some $0 < \beta \leq 1$. Then

$$I(K) = \frac{\sqrt{2}}{\sqrt{T}}\left(\sqrt{\rho(\log K)} - \sqrt{\rho(\log K) - \log K}\right)$$

$$+ O\left(\frac{\log\left[\rho(\log K)\right]}{\sqrt{\rho(\log K)}}\right)$$

as $K \to \infty$.

- Let C be a call pricing function and let D_T be the distribution density of the stock price X_T. Suppose that

$$D_T(x) \approx \frac{1}{x} \exp\{-h(\log x)\}$$

as $x \to \infty$, where h is a function such that either $h \in SR_\alpha$ with $\alpha > 1$, or $h \in SR_1$ and $g(u) = h(u) - u \in SR_\beta$ for some $0 < \beta \leq 1$. Then

$$I(K) = \frac{\sqrt{2}}{\sqrt{T}} \left(\sqrt{h(\log K)} - \sqrt{h(\log K) - \log K} \right)$$

$$+ O\left(\frac{\log [h(\log K)]}{\sqrt{h(\log K)}} \right)$$

as $K \to \infty$.

Special Stochastic Volatility Models

Heston model.

- The stock price process S and the variance process Y in the Heston model satisfy the following system of stochastic differential equations:

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{Y_t} S_t dW_t \\ \\ dY_t = q\,(m - Y_t)\,dt + c\sqrt{Y_t} dZ_t, \end{cases}$$

where $q \geq 0$, $m \geq 0$, $c > 0$, and standard Brownian motions W and Z are such that

$$d\langle W, Z\rangle_t = \rho dt \quad \text{with} \quad \rho \in [-1, 1].$$

The initial conditions for the processes X and Y are denoted by x_0 and y_0, respectively. The variance process in the Heston model is called the Cox-Ingersoll-Ross process (the Feller process).

- In terms of the log-price process $X = \log S$ and the variance process Y, the Heston model can be rewritten as follows:

$$\begin{cases} dX_t = \left(\mu - \frac{1}{2} Y_t\right) dt + \sqrt{Y_t} dW_t \\ dY_t = q\,(m - Y_t)\,dt + c\sqrt{Y_t} dZ_t. \end{cases}$$

- For every $t > 0$, the following formula holds for the distribution density D_t of the stock price X_t in the Heston model with $-1 < \rho \leq 0$:

$$D_t(x) = A_1 x^{-A_3} e^{A_2\sqrt{\log x}} (\log x)^{-\frac{3}{4}+\frac{a}{c^2}}$$

$$\left(1 + O((\log x)^{-\frac{1}{2}})\right)$$

as $x \to \infty$.

- The previous formula was obtained in the case where $\rho = 0$ in a joint paper [9] of E. M. Stein and A. G. This result was extended to the correlated Heston model by P. Friz, S. Gerhold, S. Sturm, and A. G. (see [3]).

Stein-Stein model.

- In the absence of correlation between the stock price and the volatility process, the Stein-Stein model can be considered in the following two forms:

$$\begin{cases} dX_t = \mu X_t dt + Y_t X_t dW_t \\ \\ dY_t = q\left(m - Y_t\right) dt + \sigma dZ_t, \end{cases}$$

or

$$\begin{cases} dX_t = \mu X_t dt + \left|Y_t\right| X_t dW_t \\ \\ dY_t = q\left(m - Y_t\right) dt + \sigma dZ_t, \end{cases}$$

where $q \geq 0$, $m \geq 0$, and $\sigma > 0$.

- The volatility process in the first of the previous models is the Ornstein-Uhlenbeck process, while in the second model the process $|Y|$ is used to model the volatility. It is known that the marginal distributions of the stock price process in both models coincide. In the presence of correlation, it is more popular to consider the Stein-Stein model with Y instead of $|Y|$.

- Let D_T be the stock price density in the Stein-Stein model. Then

$$D_T(x) = B_1 x^{-B_3} e^{B_2\sqrt{\log x}} (\log x)^{-\frac{1}{2}} \left(1 + O((\log x)^{-\frac{1}{2}})\right)$$

as $x \to \infty$.

- The previous formula was obtained for the uncorrelated Stein-Stein model by E. M. Stein and A. G. (see [9]). For the correlated model with the long-run mean m equal to zero, the formula follows from the asymptotic formula for the Heston density. In the case where $m \neq 0$ in the correlated Stein-Stein model, the formula also holds. This was shown in a recent paper of J.-D. Deuschel, P. Friz, A. Jacquier, and S. Violante (see [2]).

Asymptotic behavior of the implied volatility

- Fix the maturity T and consider the implied volatility as a function $k \mapsto \widehat{I}(k)$ of the log-strike $k = \log K$.

- *Heston model.* The following asymptotic formula holds for the implied volatility in the Heston model:

$$\widehat{I}(k) = \beta_1 k^{\frac{1}{2}} + \beta_2 + \beta_3 \frac{\log k}{k^{\frac{1}{2}}} + O\left(\frac{1}{k^{\frac{1}{2}}}\right)$$

  as $k \to \infty$, where the constants β_1, β_2, and β_3 depend on T and on the model parameters.

- *Stein-Stein model.* The following asymptotic formula holds for the implied volatility in the Stein-Stein model:

$$\widehat{I}(k) = \gamma_1 k^{\frac{1}{2}} + \gamma_2 + O\left(\frac{1}{k^{\frac{1}{2}}}\right)$$

  as $k \to \infty$, where the constants γ_1 and γ_2 depend on T and on the model parameters.

- The previous formulas can be strengthened, using the recent results obtained in [4].

Models without Moment Explosions

- Suppose that a stochastic model is such that all moments of the stock price are finite. Then Lee's moment formulas do not characterize the asymptotic behavior of the implied volatility. However, our general formulas can be applied to stock price models without moment explosions. The following statements hold (see [6]):

  1. Let $\widetilde{C}$ be a positive function such that

  $$\widetilde{C}(K) \approx C(K) \quad \text{as} \quad K \to \infty.$$

  Suppose also that $\tilde{p} = \infty$. Then

  $$I(K) = \frac{1}{\sqrt{2T}} \frac{\log K}{\sqrt{\log \frac{1}{\widetilde{C}(K)}}} + O\left(\frac{(\log K)^2}{\left(\log \frac{1}{\widetilde{C}(K)}\right)^{\frac{3}{2}}} \right) + O\left(\frac{1}{\sqrt{\log \frac{1}{\widetilde{C}(K)}}} \right)$$

  as $K \to \infty$.

2. Suppose that

$$P(K) \approx \widetilde{P}(K) \quad \text{as} \quad K \to 0$$

where $\widetilde{P}$ is a positive function. Suppose also that $\tilde{q} = \infty$. Then

$$I(K) = \frac{1}{\sqrt{2T}} \frac{\log \frac{1}{K}}{\sqrt{\log \frac{K}{\widetilde{P}(K)}}} + O\left(\frac{(\log \frac{1}{K})^2}{\left(\log \frac{K}{\widetilde{P}(K)}\right)^{\frac{3}{2}}} \right) + O\left(\frac{1}{\sqrt{\log \frac{K}{\widetilde{P}(K)}}} \right)$$

as $K \to 0$.

Examples

- *Constant Elasticity of Variance Model.* The asset price process in the CEV model obeys the stochastic differential equation

$$dS_t = \sigma S_t^{\rho} dW_t.$$

We assume that $0 < \rho < 1$ and $\sigma > 0$. For the CEV model we have $\tilde{p} = \infty$ and $\tilde{q} = 2(1 - \rho)$. Hence, the behavior of the implied volatility as $K \to 0$ is regular, while the case $K \to \infty$ is characterized by a nonstandard behavior.

- The following formulas hold for the implied volatility in the CEV model:

$$I(K) = \sigma(1-\rho)\frac{\log K}{K^{1-\rho}} + O\left(\frac{1}{K^{1-\rho}}\right)$$

as $K \to \infty$, and

$$\frac{\sqrt{T}}{\sqrt{2}} I(K) = \sqrt{(3-2\rho)\log\frac{1}{K} - \frac{1}{2}\log\log\frac{1}{K}} - \sqrt{(2-2\rho)\log\frac{1}{K} - \frac{1}{2}\log\log\frac{1}{K}} + O\left(\left(\log\frac{1}{K}\right)^{-\frac{1}{2}}\right)$$

as $K \to 0$.

- *Rubinstein's Displaced Diffusion Model.* The displaced diffusion model was introduced by M. Rubinstein. The stock price process in this model is a convex combination of a risky asset following a driftless geometric Brownian motion and a riskless asset. If the interest rate satisfies $r = 0$, then the stock price process S in Rubinstein's model is given by

$$S_t = S_0 \left[\eta \exp\left\{-\frac{1}{2}\sigma^2 t + \sigma W_t\right\} + (1-\eta)\right]$$

where $0 \leq \eta \leq 1$, $S_0 > 0$ is the initial price, and $\sigma > 0$ is the volatility parameter.

- In a more general displaced diffusion model, the stock price process is the solution to the following stochastic differential equation:

$$dS_t = \sigma (S_t + a) \, dW_t, \quad S_0 = s_0 \quad \text{a.s.}$$

where $s_0 > 0$, $\sigma > 0$, and $a \neq 0$. It follows that the process $X_t = S_t + a$ is a driftless geometric Brownian motion with the volatility equal to σ and the initial condition given by $x_0 = s_0 + a$. It follows that

$$S_t = (s_0 + a) \exp \left\{ -\frac{1}{2}\sigma^2 t + \sigma W_t \right\} - a.$$

- Since we are considerimg only nonnegative stock price processes, it is natural to suppose that $a < 0$ and $s_0 > |a|$. Then the process S coincides with the stock price process in Rubinstein's model with

$$S_0 = s_0 \quad \text{and} \quad \eta = \frac{s_0 + a}{s_0}.$$

- Let I be the implied volatility in the displaced diffusion model with $a < 0$ and $s_0 > |a|$. Then

$$I(K) = \sigma + O\left(\frac{\log \log K}{\log K}\right)$$

as $K \to \infty$.

- *Finite Moment Log-Stable Model of Carr and Wu.* In this model, α-stable Lévy processes with skew parameter $\beta = -1$ are used in stochastic modeling of log-returns associated with the spot levels of S&P 500 index.

- The index level S in the finite moment log-stable model satisfies the following stochastic differential equation:

$$dS_t = S_t \left(r dt + \sigma d L_t^{\alpha,-1} \right),$$

where $1 < \alpha < 2$, $\sigma > 0$, and $r > 0$ is the interest rate. The process $L^{\alpha,-1}$ driving the equation is the Lévy process such that the random variable $L_t^{\alpha,-1}$ is distributed according to the α-stable law $L_\alpha \left(0, t^{\frac{1}{\alpha}}, -1\right)$.

- The following formulas hold for the implied volatility in the finite moment log-stable model:

$$I(K) = \frac{(\sigma\alpha)^{\frac{\alpha}{2(\alpha-1)}} T^{\frac{1-\alpha}{2\alpha}}}{\sqrt{2(\alpha-1)}} (\log K)^{-\frac{2-\alpha}{2(\alpha-1)}} + O\left((\log K)^{-\frac{\alpha}{2(\alpha-1)}} \right)$$

as $K \to \infty$, and

$$\frac{\sqrt{T}}{\sqrt{2}} I(K) = \sqrt{\log \frac{1}{K} + \alpha \log\log \frac{1}{K} - \frac{1}{2} \log\log\log \frac{1}{K}}$$
$$- \sqrt{\alpha \log\log \frac{1}{K} - \frac{1}{2} \log\log\log \frac{1}{K}} + O\left(\left(\log\log \frac{1}{K} \right)^{-\frac{1}{2}} \right)$$

as $K \to 0$.

- *SV1 Model of Rogers and Veraart.* This model was suggested by L. Rogers and L. Veraart as a less complicated alternative to SABR model. The asset price in SV1 model is given by

$$X = X^{(1)} X^{(2)},$$

where

$$X_t^{(1)} = \sigma_t^{\frac{2}{\gamma}}, \quad X_t^{(2)} = z_t^{\frac{1}{\gamma}}, \quad t > 0,$$

and the processes σ and z solve the following stochastic differential equations:

$$d\sigma_t = \eta \sigma_t dB_t$$

and

$$dz_t = (a_1 - a_2 z_t)\, dt + 2\sigma \sqrt{z_t} dW_t.$$

- *Assumptions:* B and W are independent standard Brownian motions, $\eta > 0$ is fixed, the interest rate r is equal to zero, $1 \leq \gamma < 2$, $a_1 = 2(\gamma - 1)\gamma^{-1}$, and $a_2 = (2 - \gamma)\eta^2\gamma^{-1}$. It was established in the paper of Rogers and Veraart that if the parameters are chosen as above, then the process X is a martingale.

- Let $K \mapsto I(K)$ be the implied volatility in the SV1 model. Then the following asymptotic formulas hold:

$$I(K) \sim 2\eta\gamma^{-1} \quad \text{as} \quad K \to \infty \tag{1}$$

and

$$I(K) \sim \left(\frac{2}{T}\right)^{\frac{1}{2}} \sqrt{\log \frac{1}{K}} \tag{2}$$

as $K \to 0$.

Piterbarg's Conjecture

- Let X be a stock price process for which $\tilde{p} < \infty$ and $\tilde{q} < \infty$. Then, as we already know, a typical behavior of the implied volatility near infinity is described by the function

$$K \mapsto c_1 \sqrt{\log K}$$

and near zero by the function

$$c_2 \sqrt{\log \frac{1}{K}}.$$

However, if $\tilde{p} = \infty$ or $\tilde{q} = \infty$, then the class of approximating functions is wider.

• *Piterbarg's conjecture (see [11]).*

Let w be a positive increasing function on $(0,\infty)$ satisfying $w(y) \to \infty$ as $y \to \infty$ and such that the limit

$$M = \lim_{y\to\infty} \frac{w(y)}{\log y}$$

exists (finite or infinite). Put

$$\tilde{p}_w = \sup \left\{ p \geq 0 : \mathbb{E}^* \left[\exp \left\{ pw \left(X_T \right) \right\} \right] < \infty \right\}.$$

Then, under the condition $\tilde{p} = \infty$,

$$\limsup_{K\to\infty} \frac{I(K)\sqrt{w(K)}}{\log K} = \frac{1}{\sqrt{2T\tilde{p}_w}}$$

• *Remark:* Our notation is different from that in Piterbarg's paper.

- *A modification.* We modify the formula conjectured by Piterbarg as follows. Let $C \in PF_\infty$ be a call pricing function, and suppose $\tilde{p} = \infty$. Then

$$\limsup_{K\to\infty} \frac{I(K)\sqrt{w(K)}}{\log K} = \frac{1}{\sqrt{2T\hat{p}_w}}.$$

where

$$\hat{p}_w = \sup\left\{p \geq 0 : \mathbb{E}^*\left[\int_0^{X_T} e^{pw(y)}dy\right] < \infty\right\},$$

- *Statement.* The modified formula always holds.

References

[1] Benaim, S., Friz, P., Regular variation and smile asymptotics, *Math. Finance* 19 (2009), 1-12.

[2] Deuschel, J.-D., Friz, P. K., Jacquier, A., Violante, S., Marginal density expansions for diffusions and stochastic volatility, preprint, 2011; available at arxiv:112462.

[3] Friz, P., Gerhold, S., Gulisashvili, A., Sturm, S. On refined volatility smile expansion in the Heston model, *Quantitative Finance* 11 (2011), 1151-1164.

[4] Gao, K., Lee, R. Asymptotics of implied volatility to arbitrary order, pre-print.

[5] Gulisashvili, A., Asymptotic formulas with error estimates for call pricing functions and the implied volatility at extreme strikes, *SIAM Journal Math. Finance* 1 (2010), 609-641.

[6] Gulisashvili, A., Asymptotic equivalence in Lee's moment formulas for the implied volatility, asset price models without moment explosions, and Piterbarg's conjecture, to be published in *International Journal of Theoretical and Applied Finance.*

[7] Gulisashvili, A., Stein, E. M., Implied volatility in the Hull-White model, *Mathematical Finance* 19 (2009), 303-327.

[8] Gulisashvili, A., Stein, E. M., Asymptotic behavior of distribution densities in models with stochastic volatility, I, *Math. Finance.* 20 (2010), 447477.

[9] Gulisashvili, A., Stein, E. M., Asymptotic behavior of the stock price distribution density and implied volatility in stochastic volatility models, *Appl. Math. Optim.* 61 (2010), 287315.

[10] Lee, R., The moment formula for implied volatility at extreme strikes, *Math. Finance* 14 (2004), 469-480.

[11] Piterbarg, V., Implied volatility smile asymptotics when all moments are finite,, working paper, 2004.