# Second Quantised Representation of Mehler Semigroups Associated with Banach Space Valued Lévy processes

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Talk at *Seminari de Probabilitats de Barcelona*, *Universitat de Barcelona, Facultat de Matemàtiques.*

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Talk based on joint work with Jan van Neerven (Delft)

# Outline of Talk

- Mehler semigroups arise as transition semigroups of linear SPDEs with additive Lévy noise.
- Szymon Peszat has shown that these semigroups can be expressed functorially using second quantisation.
- Peszat's approach is based on chaotic decomposition formulae due to Last and Penrose.
- We pursue an alternative strategy using vectors related to exponential martingales.

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# Reproducing Kernel Hilbert Space (RKHS)

$E$ is a real separable Banach space, $E^*$ is its dual,

$\langle \cdot, \cdot \rangle$ is pairing $E \times E^* \to \mathbb{R}$.

$T \in \mathcal{L}(E^*, E)$ is

- *symmetric* if for all $a, b \in E^*$, $\langle Ta, b \rangle = \langle Tb, a \rangle$,
- *positive* if for all $a \in E^*$, $\langle Ta, a \rangle \geq 0$.

If $T$ is positive and symmetric, $[\cdot, \cdot]$ is an inner product on $\text{Im}(T)$, where

$$[Ta, Tb] = \langle Ta, b \rangle.$$

RKHS $H_T$ is closure of $\text{Im}(T)$ in associated norm.

Inclusion $\iota_T : \text{Im}(T) \to E$ extends to a continuous injection $\iota_T : H_T \to E$.

$$T = \iota_T \circ \iota_T^*.$$

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# Infinite Divisibility in Banach Spaces

$\mu$ a Borel measure on $E$. Reversed measure $\widetilde{\mu}(E) = \mu(-E)$. $\mu$ *symmetric* if $\widetilde{\mu} = \mu$.

$\mu$ a (Borel) probabililty measure on $E$ Its *Fourier transform/ characteristic function* is the mapping $\widehat{\mu} : E^* \to \mathbb{C}$ defined for $a \in E^*$ by:

$$\widehat{\mu}(a) = \int_E e^{i\langle x,a\rangle}\mu(dx).$$

A measure $\nu \in \mathcal{M}(E)$ is a *symmetric Lévy measure* if it is symmetric and satisfies

(i) $\nu(\{0\}) = 0$,

(ii) The mapping from $E^*$ to $\mathbb{R}$ given by

$$a \to \exp\left\{\int_E [\cos(\langle x,a\rangle) - 1]\nu(dx)\right\}$$

is the characteristic function of a probability measure on $E$.

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$\nu \in \mathcal{M}(E)$ is a *Lévy measure* if $\nu + \tilde{\nu}$ is a symmetric Lévy measure.

If $\nu$ is a Lévy measure on $E$, the mapping from $E^*$ to $\mathbb{C}$ given by

$$a \to \exp\left\{\int_E [e^{i\langle x,a\rangle} - 1 - i\langle x, a\rangle 1_{B_1}(x)]\nu(dx)\right\}$$

is the characteristic function of a probability measure on $E$.

We say that a probability measure $\mu$ on $E$ is *infinitely divisible* if for all $n \in \mathbb{N}$, $\mu$ has a convolution $n$th root $\mu_n$.

Equivalently for all $n \in \mathbb{N}$ there exists a probability measure $\mu_n$ on $E$ such that $\widehat{\mu}(a) = (\widehat{\mu_n}(a))^n$ for all $a \in E^*$.

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## Theorem (Lévy-Khintchine)

*A probability measure $\mu \in \mathcal{M}_1(E)$ is infinitely divisible if and only if there exists $x_0 \in E^*$, a positive symmetric operator $R \in \mathcal{L}(E^*, E)$ and a Lévy measure $\nu$ on $E$ such that for all $a \in E^*$,*

$$\widehat{\mu}(a) = e^{\eta(a)},$$

*where*

$$\begin{aligned} \eta(a) &= i\langle x_0, a\rangle - \frac{1}{2}\langle Ra, a\rangle \\ &+ \int_E (e^{i\langle y, a\rangle} - 1 - i\langle y, a\rangle 1_{B_1}(y))\nu(dy). \end{aligned}$$

The triple $(x_0, R, \nu)$ is called the *characteristics* of the measure $\nu$ and $\eta$ is known as the *characteristic exponent.*

See e.g. W.Linde, *Probability in Banach Spaces - Stable and Infinitely Divisible Distributions*, Wiley-Interscience (1986).

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# Covariance Operators

A probability measure $\mu$ on $E$ has *uniformly weak second order moments* if

$$\sup_{||a|| \leq 1} \int_E |\langle x, a \rangle|^2 \mu(dx) < \infty.$$

In this case, there exists a *covariance operator* $Q \in \mathcal{L}(E^*, E)$ which is positive and symmetric:

$$\langle Qa, b \rangle = \int_E \langle x, a \rangle \langle x, b \rangle \mu(dx) - \left( \int_E \langle x, a \rangle \mu(dx) \right) \left( \int_E \langle x, b \rangle \mu(dx) \right).$$

Associated RKHS is $H_Q$.

If $\mu$ is infinitely divisible with characteristics $(x_0, R, \nu)$ and has uniformly weak second order moments:

$$Qa = Ra + \int_E \langle x, a \rangle x \, \nu(dx).$$

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# Mehler Semigroups

Let $(\mu_t, t \geq 0)$ be a family of probability measures on $E$ with $\mu_0 = \delta_0$ and $(S(t), t \geq 0)$ be a $C_0$-semigroup on $E$. Define $T_t : B_b(E) \to B_b(E)$ by

$$T_t f(x) = \int_E f(S(t)x + y)\mu_t(dy).$$

$(T_t, t \geq 0)$ is a semigroup, i.e. $T_{t+s} = T_t T_s$ if and only if $(\mu_t, t \geq 0)$ is a *skew-convolution semigroup*, i.e.

$$\mu_{t+u} = \mu_u * S(u)\mu_t$$

(where $S(u)\mu_t := \mu_t \circ S(u)^{-1}$.)

Note that $T_t : C_b(E) \to C_b(E)$ but it is not (in general) strongly continuous.

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From now on, we assume that the skew-convolution semigroup $(\mu_t, t \geq 0)$ is *F-differentiable*, i.e. $a \in E^*, t \rightarrow \widehat{\mu_t}(a)$ is differentiable.

$$\text{Define } \xi(a) := \frac{d}{dt}\widehat{\mu_t}(a)\bigg|_{t=0}.$$

Then

$$\widehat{\mu_t}(a) = e^{\eta_t(a)} := \exp\left\{\int_0^t \xi(S(u)^* a)du\right\}.$$

From this it follows that $\mu_t$ is infinitely divisible for all $t \geq 0$.

Furthermore $\xi$ is the characteristic exponent of an infinitely divisible probability measure $\rho$ with characteristics $(b, R, \nu)$ (say) and the characteristics of $\mu_t$ are $(b_t, R_t, \nu_t)$ where:

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$$x_t = \int_0^t S(r)bdr + \int_0^t \int_E S(r)y(1_B(S(r)y) - 1_B(y))\nu(dy)dr,$$

$$R(t) = \int_0^t S(r)RS(r)^*)dr$$

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see Bogachev, Röckner, Schmuland, PTRF **105**, 193 (1996); Furhman,Röckner, Pot. Anal. **12**, 1 (2000)

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If $\rho$ has covariance $Q$ then $\mu_t$ has covariance

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from which it follows that

$$Q_{t+s} = Q_t + S(t)Q_sS(t)^*.$$

Let $H_t$ be RKHS of $Q_t$. Then $H_t \subseteq H_{t'}$ if $t \leq t'$. From the above $S(r)Q(t)S(r)^* = Q_{t+r} - Q_r$ and so $S(r)$ maps $\text{Im}(Q_tS(r)^*) \subseteq H_t$ to $H_{r+t}$. In fact

$$S(r)H_t \subseteq H_{t+r} \text{ and } ||S(r)||_{\mathcal{L}(H_t,H_{t+r})} \leq 1.$$

see J. van Neerven, JFA **155**, 495 (1998)

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$$S(r)H_t \subseteq H_{t+r} \text{ and } ||S(r)||_{\mathcal{L}(H_t, H_{t+r})} \leq 1.$$

see J. van Neerven, JFA **155**, 495 (1998)

If $\rho$ has covariance $Q$ then $\mu_t$ has covariance

$$\begin{aligned} Q_t &= \int_0^t S(r)QS(r)^* dr \\ &= R_t + \int_0^t \int_E \langle S(r)y, a\rangle S(r)y\nu(dy) \end{aligned}$$

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# Contraction Properties

## Theorem

*If $(T_t, t \geq 0)$ is a Mehler semigroup then $T_t$ is a contraction from $L^2(E, \mu_{t+u})$ to $L^2(E, \mu_u)$ for all $u \geq 0$.*

*Proof.* For each $f \in L^2(E, \mu_u)$,

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||T_t f||^2_{L^2(\mu_u)} &= \int_E |T_t f(x)|^2 \mu_u(dx) \\
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# Lévy Driven OU Processes

Let $A$ be the infinitesimal generator of the semigroup $(S(t), t \geq 0)$.

Let $(X(t), t \geq 0)$ be an $E$-valued Lévy process. Consider the linear SPDE with additive noise:

$$dY(t) = AY(t) + dX(t) \quad ; \quad Y(0) = Y_0$$

Unique solution is *generalised Ornstein-Uhlenbeck process*:

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Transition semigroup $T_t f(x) = \mathbb{E}(f(Y(t))|Y_0 = x)$ is a Mehler semigroup. Skew convolution semigroup $\mu_t$ is law of $\int_0^t S(t-u)dX(u) \stackrel{d}{=} \int_0^t S(u)dX(u)$ and is $F$-differentiable with $\xi$ the characteristic exponent of $X(t)$, i.e. $\mathbb{E}(e^{i\langle X(t), a\rangle}) = e^{t\xi(a)}$ for all $a \in E^*, t \geq 0$.

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Ed. Nelson

$H$ a complex Hilbert space. $\Gamma(H)$ is symmetric Fock space over $H$.

$$\Gamma(H) := \bigoplus_{n=0}^{\infty} H_s^{(n)}$$

$H^{(0)} = \mathbb{C}, H^{(1)} = H, H^{(n)}$ is $n$ fold symmetric tensor product

Exponential vectors $\{e(f), f \in H\}$ are linearly independent and total where

$$e(f) = \left(1, f, \frac{f \otimes f}{\sqrt{2!}}, \ldots, \frac{f^{\otimes^n}}{\sqrt{n!}}, \ldots\right), \quad \langle e(f), e(g)\rangle = e^{\langle f, g\rangle}.$$

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J van Neerven, JFA **155**, 495 (1998)

A. Chojnowska-Michalik and B. Goldys, J.Math. Kyoto Univ. **36**, 481 (1996)

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Assume $\mu$ infinitely divisible with $\widehat{\mu}(a) = e^{\eta(a)}$ for $a \in E^*$. We need analogues of exponential vectors. For each $a \in E^*$ define $K_a \in L^2_{\mathbb{C}}(E, \mu)$ by

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**Theorem**

*The set $\{K_a, a \in E^*\}$ is total in $L^2_{\mathbb{C}}(E, \mu)$.*

*Proof.* Let $\psi \in L^2_{\mathbb{C}}(E, \mu)$ be such that for all $a \in E^*$, $\int_E K_a(x)\overline{\psi(x)}\mu(dx) = 0$. Then $\int_E e^{i\langle x, a\rangle}\mu_\psi(dx) = 0$, where $\mu_\psi(dx) := \overline{\psi(x)}\mu(dx)$ is a complex measure. It follows by injectivity of the Fourier transform that $\mu_\psi = 0$ and hence $\psi = 0$ (a.e.) as was required. □

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*Proof.* Let $a_1, \ldots a_n \in E^*$ be distinct and $c_1, \ldots, c_n \in \mathbb{C}$ for some $n \in \mathbb{N}$ and assume that $\sum_{i=1}^n c_i K_{a_i} = 0$.

Define $\tilde{c}_i := e^{-\eta(a_i)} c_i$ for $1 \leq i \leq n$ and replace $x$ by $tx$ where $t \in \mathbb{R}$ .

Then we have $\sum_{i=1}^n \tilde{c}_i e^{it\langle x, a \rangle} = 0$ for all $t \in \mathbb{R}$. Let $t = 0$ to see that $\sum_{i=1}^n \tilde{c}_i = 0$.

Now differentiate $r$ times with respect to $t$ (where $1 \leq r \leq n - 1$) and then put $t = 0$. This yields $\sum_{i=1}^n \tilde{c}_i \langle x, a_i \rangle^r = 0$.

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We have a system of $n$ linear equations in $\tilde{c}_1, \ldots \tilde{c}_n$ and it has a non-zero solution if and only if

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \langle x, a_1 \rangle & \langle x, a_2 \rangle & \cdots & \langle x, a_n \rangle \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \langle x, a_1 \rangle^{n-1} & \langle x, a_2 \rangle^{n-1} & \cdots & \langle x, a_n \rangle^{n-1} \end{vmatrix} = 0.$$

This ia a Vandermonde determinant and so the equation simplifies to

$$\prod_{1 \leq i,j \leq n} (\langle x, a_i \rangle - \langle x, a_j \rangle) = 0.$$

Hence there exists $k, l$ with $1 \leq k, l \leq n$ such that $\langle x, a_k - a_l \rangle = 0$ for all $x \in E$. The choice of $k$ and $l$ here depends on $x$. We now prove that in fact they are independent of the choice of vector in $E$.

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To this end for each $1 \leq i, j \leq n$ define $F_{ij} := \{x \in E; \langle x, a_i - a_j \rangle = 0\}$. Then $F_{ij}$ is closed and $\bigcup_{i,j=1}^{n} F_{ij} = E$.

By the Baire category theorem, at least one pair $(i, j)$ must be such that $F_{ij}$ has non-empty interior $O_{ij}$. Let $(k, l)$ be such a pair and fix $x_0 \in O_{kl}$.

Then by linearity $\langle x - x_0, a_k - a_l \rangle = 0$ for all $x \in O_{kl}$. In other words $\langle y, a_k - a_l \rangle = 0$ for all $y \in O_{kl} - \{x_0\}$.

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Then by linearity $\langle x - x_0, a_k - a_l \rangle = 0$ for all $x \in O_{kl}$. In other words $\langle y, a_k - a_l \rangle = 0$ for all $y \in O_{kl} - \{x_0\}$.

But $O_{kl} - \{x_0\}$ contains an open neighbourhood of 0 and hence by linearity again, $\langle x, a_k - a_l \rangle = 0$ for all $x \in E$.

It follows that $a_k = a_l$ and this is a contradiction. So we must have $\tilde{c}_1 = \tilde{c}_2 = \cdots = \tilde{c}_n = 0$. Since $e^{-\eta(a)} \neq 0$ for all $a \in E^*$, we deduce that $c_1 = c_2 = \cdots = c_n = 0$, as was required. □

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# non-Gaussian Second Quantisation

Let $T \in \mathcal{L}(E^*)$. We define its *second quantisation* $\Gamma(T)$ to be the densely defined linear operator with domain $\mathcal{E} = \text{lin span}\{K_a, a \in E^*\}$ defined by linear extension of the prescription

$$\Gamma(T)K_a = K_{Ta}.$$

The following properties are straightforward to verify:

- $\Gamma(T)$ is closeable with $\mathcal{E} \subseteq \Gamma(T)^*$ and $\Gamma(T)^* = \Gamma(T^*)$,
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# The Main Result

## Theorem

*Let* $(\mu_t, t \geq 0)$ *be an F-differentiable skew convolution semigroup. For all* $t, u > 0$

$$T_t = \Gamma(S(t)^*_{t+u\to u}).$$

*Proof.* For all $a \in E^*, x \in E$

$$\begin{aligned} T_t K_{t+u,a}(x) &= \int_E K_{t+u,a}(S(t)x + y)\mu_t(dy) \\ &= e^{-\eta_{t+u}(a)} e^{i\langle S(t)x, a\rangle} \int_E e^{i\langle y, a\rangle} \mu_t(dy) \\ &= e^{-\eta_{t+u}(a)} e^{\eta_t(a)} e^{i\langle S(t)x, a\rangle}. \end{aligned}$$

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From this we see that

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Some comments:

We did not assume that $\mu_t$ has second moments and made no use of a RKHS. So our second quantisation

$$\Gamma : \mathcal{L}(E^*) \to \text{closeable lin.ops on } L^2_{\mathbb{C}}(E, \mu) \text{ preserving } \mathcal{E}.$$

If we assume that $\mu_t$ has second moments for all $t$ then $S(t)^*$ is a contraction from $H_{t+u}$ to $H_u$.

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$\lambda$ is an invariant measure for the Mehler semigroup $(T_t, t \geq 0)$ if and only if for all $t \geq 0$

$$\rho = \mu_t * S(t)\rho.$$

If $\rho$ exists it is infinitely divisible (operator self-decomposable.)

e.g. if $\mu_\infty = \text{weak-}\lim_{n \to \infty} \mu_t$ exists it is an invariant measure.

If $E$ is a Hilbert space and we are in the Ornstein-Uhlenbeck case:

A.Chojnowska-Michalik Stochastics, **21** 251 (1987)

e.g. Assume $(S_t, t \geq 0)$ is *exponentially stable* i.e. $||S(t)|| \leq Me^{-\lambda t}$ for $M \geq 1, \lambda > 0$. Then necessary and sufficient conditions for unique invariant measure are

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We get RKHS $H_\infty$ with for all $t \geq 0$

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# The Chaos Approach in the non-Gaussian case.

Based on work by

S.Peszat, JFA **260**, 3457 (2011)

$(\Omega, \mathcal{F}, P)$ is a probability space. Let $\Pi$ be a Poisson random measure defined on a measurable space $(E, \mathcal{B})$ with intensity measure $\lambda$. Let $\mathbb{Z}_+(E)$ be the non-negative integer valued measures on $(E, \mathcal{B})$. Regard $\Pi$ as a random variable on $\Omega$ taking values in $\mathbb{Z}_+(E)$ by

$$\Pi(\omega)(E) = \Pi(E, \omega)$$

Let $P_\pi$ be the law of $\Pi$ and for $F \in L^2(P_\pi), \xi \in \mathbb{Z}_+(E)$ define the "Malliavin derivative":

$$D_y F(\xi) = F(\xi + \delta_y) - F(\xi)$$

Define $T^n : L^2(P_\pi) \to L^2_{\text{symm}}(E^n, \lambda^n)$ by

$$(T^n F)(y_1, \ldots, y_n) = \mathbb{E}(D^n_{y_1,\ldots,y_n} F(\Pi)).$$

## Chaos expansion

$$\mathbb{E}(F(\Pi)G(\Pi)) = \mathbb{E}(F(\Pi))\mathbb{E}(G(\Pi)) + \sum_{n=1}^{\infty} \frac{1}{n!}\langle T^n F, T^n G\rangle_{L^2(E^n,\lambda^n)}$$

from which it follows that

$$F(\Pi) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(T^n F),$$

where $I_n$ is usual multiple Itô integral w.r.t. compensator $\tilde{\Pi} := \Pi - \lambda$.

So here $L^2(P_\pi) = \Gamma(L^2(E^n, \lambda^n))$.

see G.Last, M.Penrose, PTRF **150**, 663 (2011)

Peszat: If $E$ is a Hilbert space, $R \in \mathcal{L}(E)$, define $\rho_R^{(n)} \in \mathcal{L}(L^2(E^n, \lambda^n))$ by

$$\rho_R^{(n)} f(y_1, \ldots, y_n) = f(Ry_1, \ldots, Ry_n).$$

Second quantisation: $\Gamma_0(R) : L^2(P_\pi) \to L^2(P_\pi)$,

$$\Gamma_0(R)F = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\rho_R^{(n)}(T^n F)).$$

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# Connecting The Two Approaches in the non-Gaussian Case

For all $t \geq 0$ let $S_t := [0, t) \times E$.

Let $\Pi$ be a Poisson random measure defined on $[0, \infty) \times E$ so that $\Pi_t$ has intensity measure $\lambda_t$.

The natural filtration of $\Pi_t(\cdot) := \Pi(t, \cdot)$ is denoted $(\mathcal{F}_t, t \geq 0)$.

For $t \geq 0, f \in L^2(S_t, \lambda_t)$ define the process $(X_f(t), t \geq 0)$ by

$$X_f(t) = \int_0^t \int_E f(s, x)\tilde{\Pi}(ds, dx).$$

$\mathbb{E}(|X_f(t)|^2) = ||f||^2_{L^2(S_t, \lambda_t)} < \infty.$

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$$\eta_f(t) = \int_{S_t} (e^{if(s,x)} - 1 - if(s, x))\lambda(ds, dx).$$

Define the process $(M_f(t), t \geq 0)$ by

$$M_f(t) = \exp\{iX_f(t) - \eta_f(t)\}.$$

Then $(M_f(t), t \geq 0)$ is a square-integrable martingale with

$$dM_f(t) = \int_{S_t} (e^{if(s,x)} - 1)M_f(s-)\tilde{\Pi}(ds, dx),$$

and for all $t \geq 0$,

$$\mathbb{E}(|M_f(t)|^2) = \exp\left\{\int_{S_t} |e^{if(s,x)} - 1|^2 \lambda(ds, dx)\right\} \quad (1.1)$$

## Lemma

*For all $t \geq 0$,*

$$\mathbb{E}(|M_f(t)|^2) \leq e^{||f||^2_{L^2(S_t,\lambda_t)}}.$$

*Proof.* Using the well known inequality $1 - \cos(y) \leq \frac{y^2}{2}$ for $y \in \mathbb{R}$

$$\begin{aligned}
\mathbb{E}(|M_f(t)|^2) &= \exp\left\{2\int_{S_t}(1-\cos(f(s,x)))\lambda(ds,dx)\right\} \\
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Now let $(Y_f(t), t \geq 0)$ be the Doléans-Dade exponential which is the unique solution of the stochastic differential equation

$$dY_f(t) = Y_f(t-)dX_f(t),$$

with initial condition $Y_f(0) = 1$ (a.s.)

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Let $\mathcal{K}(t)$ be the linear span of $\{M_f(t), f \in L^2(S_t, \lambda_t)$.

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Both sets are total in $L^2(\Omega, \mathcal{F}_t, P)$.
The map $C : \mathcal{K}(t) \rightarrow \mathcal{L}(t)$ which takes each $M_f(t)$ to $Y_f(t)$ extends to an invertible linear operator on $L^2(\Omega, \mathcal{F}_t, P)$ which we continue to denote by $C$.
Note that $C$ is a contraction by above lemma.

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Now assume that $\mu_t$ has uniformly finite weak second order moments and is and for each $a \in E^*, t \geq 0$ define

$$f_a \in L^2(S_t, \lambda) \text{ by } f_a(s, x) = \langle x, a \rangle 1_{[0,t)}(s) \text{ for each } 0 \leq s \leq t, x \in E.$$

Then we have $M_f(t) = M_{t,a}$ where

$$M_{t,a}(x) = \exp\left\{ i \int_E \langle x, a \rangle \tilde{\Pi}(t, dx) - \eta_t(x) \right\},$$

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Then $M_{t,a}$ is precisely the image of $K_{t,a}$ in $L^2(\Omega, \mathcal{F}_t, P)$ under the natural embedding of $L^2(E, \mu_t)$ into that space. From now on we will identify these vectors.

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For each $t \geq 0$, we write the Doléans-Dade exponential $Y_a(t)$ when $f = f_a$ as above.

## Theorem

*For each $S \in \mathcal{L}(E^*)$*

$$\Gamma(S) = C^{-1}\Gamma_0(S^*)C,$$

*Proof.* For each $a \in E^*, t \geq 0$,

$$\begin{aligned} \Gamma(S)C^{-1}Y_a(t) &= \Gamma(S)K_{t,a} \\ &= K_{t,Sa} \\ &= C^{-1}Y_{Sa}(t) \\ &= C^{-1}\Gamma_0(S^*)Y_a(t), \end{aligned}$$

and the result follows. □

**Gracias por su atención.**