Chern degree functions

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1. Cohomological rank functions on abelian varieties
2. Stability conditions
3. Chern degree functions
4. Applications to abelian surfaces

1. Cohomological rank functions on abelian varieties

(A, L) polarized abelian variety, $g = \dim A$

Jiang-Pareschi: Given $F \in \mathrm{D}^{\mathrm{b}}(A)$ and $i \in \mathbb{Z}$, define $h^i_{F,L} : \mathbb{Q} \to \mathbb{Q}_{\geq 0}$ such that

$$h^i_{F,L}(x) \text{``} = \text{''}\, h^i(A, F \otimes L^x \otimes \alpha), \text{ for general } \alpha \in \mathrm{Pic}^0(A)$$

Concretely: if $x = \frac{a}{b} \in \mathbb{Q}$, then

$$h^i_{F,L}(x) := \frac{1}{b^{2g}} \cdot h^i(A, \mu_b^* F \otimes L^{ab} \otimes \alpha), \text{ for general } \alpha \in \mathrm{Pic}^0(A)$$

where $\mu_b : A \to A$ is the multiplication-by-b isogeny.

The number $h^i_{F,L}(x)$ is well defined.

Theorem (Jiang-Pareschi)

1. *The functions are given by polynomials in the neighborhood of every rational number: For every* $x_0 \in \mathbb{Q}$*, there exists* $\epsilon > 0$ *and polynomials* $P^+_{x_0}, P^-_{x_0}$ *such that*

$$h^i_{F,L}(x) = P^-_{x_0}(x) \quad \text{if } x \in (x_0 - \epsilon, x_0] \cap \mathbb{Q}$$

$$h^i_{F,L}(x) = P^+_{x_0}(x) \quad \text{if } x \in [x_0, x_0 + \epsilon) \cap \mathbb{Q}$$

2. $h^i_{F,L}$ *extends to a continuous function* $h^i_{F,L} : \mathbb{R} \to \mathbb{R}_{\geq 0}$

These results do not imply that cohomological rank functions are piecewise polynomial!

Example 1: Elliptic curves

Let E be an elliptic curve, $F \in \mathrm{Coh}(E)$.

- If F is μ-semistable, then

$$h^0_{F,L}(x) = \begin{cases} 0 & \chi_{F,L}(x) \leq 0 \\ \chi_{F,L}(x) = \operatorname{rk} F \cdot x + \deg F & \chi_{F,L}(x) \geq 0 \end{cases}$$

- More generally, the Harder-Narasimhan filtration

$$0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_r = F$$

determines $h^0_{F,L}$ and $h^1_{F,L}$.

Example 2: Ideal sheaf of one point

$g = \dim A \geq 2$, $\mathcal{I}_0$=ideal sheaf of $0 \in A$, $d = h^0(L)$

$$h^0_{F,L}(x) = \begin{cases} 0 & x \leq 0 \\ ?? & 0 \leq x \leq 1 \\ \chi_{\mathcal{I}_0,L}(x) = dx^g - 1 & x \geq 1 \end{cases}, \quad h^1_{F,L}(x) = \begin{cases} 1 & x \leq 0 \\ ?? & 0 \leq x \leq 1 \\ 0 & x \geq 1 \end{cases}$$

The number $\eta(L) := \inf\{x \in \mathbb{Q} \mid h^1_{\mathcal{I}_0,L}(x) = 0\}$ is an interesting invariant:

- **Jiang-Pareschi:** $\eta(L) = 1 \iff L$ has base points

  $\eta(L) < \frac{1}{2} \implies L$ is projectively normal

- **Caucci:** $\eta(L) < \frac{1}{p+2} \implies L$ satisfies the property (N_p)

  (**Ito:** Also for $\eta(L) = \frac{1}{p+2}$, if $h^1_{\mathcal{I}_0,L}$ is of class $\mathcal{C}^1$)

2. Stability conditions

Classically: Mumford (for curves), Takemoto, Gieseker,...

Bridgeland: Define stability in the derived category $\mathrm{D}^{\mathrm{b}}(X)$ (X smooth projective variety)

Definition

A *Bridgeland stability condition* on $\mathrm{D}^{\mathrm{b}}(X)$ is a pair $\sigma = (\mathcal{A}, \mu)$, where:

1. $\mathcal{A}$ is the heart of a bounded t-structure on $\mathrm{D}^{\mathrm{b}}(X)$
2. $\mu = \frac{D}{R}$ is a slope for objects of $\mathcal{A}$, such that:
   - D, R are additive for short exact sequences in $\mathcal{A}$
   - $R(E) \geq 0$ for every $E \in \mathcal{A} \setminus \{0\}$, and $R(E) = 0 \implies D(E) > 0$
   - Every $E \in \mathcal{A} \setminus \{0\}$ admits a Harder-Narasimhan filtration
3. Support property (technical condition)

2. Stability conditions

Classically: Mumford (for curves), Takemoto, Gieseker,...

Bridgeland: Define stability in the derived category $\mathrm{D}^{\mathrm{b}}(X)$ (X smooth projective variety)

Definition

A *Bridgeland* (resp. *weak*) *stability condition* on $\mathrm{D}^{\mathrm{b}}(X)$ is a pair $\sigma = (\mathcal{A}, \mu)$, where:

1. $\mathcal{A}$ is the heart of a bounded t-structure on $\mathrm{D}^{\mathrm{b}}(X)$
2. $\mu = \frac{D}{R}$ is a slope for objects of $\mathcal{A}$, such that:
    - D, R are additive for short exact sequences in $\mathcal{A}$
    - $R(E) \geq 0$ for every $E \in \mathcal{A} \setminus \{0\}$, and $R(E) = 0 \Longrightarrow D(E) > 0$ (resp. $D(E) \geq 0$)
    - Every $E \in \mathcal{A} \setminus \{0\}$ admits a Harder-Narasimhan filtration
3. Support property (technical condition)

Example: Classical slope-stability is a weak stability condition.

The (α, β)-plane

(X, L) smooth polarized surface

(α, β)-*plane* of stability conditions: $\sigma_{\alpha,\beta} = (\mathrm{Coh}^\beta X, \nu_{\alpha,\beta})$ for $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$

- If $\beta \in \mathbb{R}$,

  $$\mathrm{Coh}^\beta X := \{E \in \mathrm{D}^{\mathrm{b}}(X) \mid \mathcal{H}^i(E) = 0 \text{ for } i \neq 0, -1, \;\; \mu_+(\mathcal{H}^{-1}(E)) \leq \beta, \;\; \mu_-(\mathcal{H}^0(E)) > \beta\}$$

- $\nu_{\alpha,\beta}(E) = \frac{\mathrm{ch}_2(E \otimes L^{-\beta}) - \frac{\alpha^2}{2} L^2 \cdot \mathrm{ch}_0(E)}{L \cdot \mathrm{ch}_1(E \otimes L^{-\beta})}$

Bridgeland, Arcara-Bertram: $\sigma_{\alpha,\beta} = (\mathrm{Coh}^\beta X, \nu_{\alpha,\beta})$ is a Bridgeland stability condition for every $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$

Important: When $\alpha = 0$ and $\beta \in \mathbb{Q}$, $\sigma_{0,\beta}$ is a WEAK stability condition.

3. Chern degree functions

(X, L) smooth polarized surface

$F \in \mathrm{D}^{\mathrm{b}}(X)$, $k \in \mathbb{Z} \rightsquigarrow$ Define $\mathrm{chd}^k_{F,L} : \mathbb{Q} \to \mathbb{Q}_{\geq 0}$

Let $x \in \mathbb{Q}$, and set $\beta = -x$.

- If $F \in \mathrm{Coh}^\beta(X)$ has HN filtration $0 = F_0 \hookrightarrow F_1 \hookrightarrow \dots \hookrightarrow F_s \hookrightarrow \dots \hookrightarrow F_r = F$ with respect to $\sigma_{0,\beta}$,

$$\mathrm{chd}^0_{F,L}(-\beta) := \mathrm{ch}_2(F_s \otimes L^{-\beta}), \quad \mathrm{chd}^1_{F,L}(-\beta) := \mathrm{ch}_2(F/F_s \otimes L^{-\beta}),$$

$$\mathrm{chd}^k_{F,L}(-\beta) = 0 \text{ for } k \neq 0, 1$$

- For arbitrary $F \in \mathrm{D}^{\mathrm{b}}(X)$, use the "partition" of F into pieces of $\mathrm{Coh}^\beta(X)$:

$$\mathrm{chd}^k_{F,L}(-\beta) := \mathrm{chd}^0_{\mathcal{H}^k_\beta(F),L}(-\beta) + \mathrm{chd}^1_{\mathcal{H}^{k-1}_\beta(F),L}(-\beta)$$

"Cohomological" properties:

- $\sum_{k\in\mathbb{Z}}(-1)^k \cdot \mathrm{chd}^k_{F,L}(x) = \mathrm{ch}_2(F \otimes L^x)$ for every $x \in \mathbb{Q}$
- Serre vanishing
- Serre duality
- Under certain assumptions, chd^0 is non-decreasing and chd^1 is non-increasing

Theorem

Given an object $F \in \mathrm{D}^{\mathrm{b}}(X)$ *and* $k \in \mathbb{Z}$:

1. *Every rational number* $x_0 \in \mathbb{Q}$ *admits a left (resp. right) neighborhood where the function* $\mathrm{chd}^k_{E,L}$ *is given by a polynomial* P^- *(resp.* P^+*) depending on* x_0.
2. $\mathrm{chd}^k_{E,L}$ *extends to a continuous function* $\mathrm{chd}^k_{E,L} : \mathbb{R} \to \mathbb{R}_{\geq 0}$

Again, these results do not imply that Chern degree functions are piecewise polynomial!

Theorem

Let $\beta_0 \in \mathbb{Q}$ *and* $F \in \mathrm{Coh}^{\beta_0} X$. *Then:*

$$\mathrm{chd}^0_{F,L},\ \mathrm{chd}^1_{F,L} \text{ are not of class } \mathcal{C}^1 \text{ at } -\beta_0 \iff F \text{ has a HN factor with } \nu_{0,\beta_0} = 0$$

4. Applications to abelian surfaces

Theorem

If (X, L) is a polarized abelian surface, then for every $F \in \mathrm{D}^{\mathrm{b}}(X)$ and $k \in \mathbb{Z}$

$$\mathrm{chd}^k_{F,L} = h^k_{F,L}$$

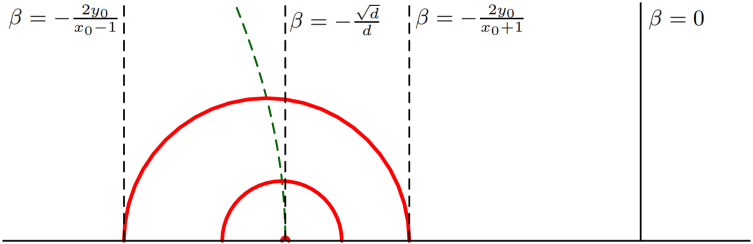
Analogy with the case of elliptic curves:

- Heart whose objects only have functions h^0 and h^1.
- For stable objects of the heart only one of the functions survives.
- The HN filtration breaks the function into smaller pieces (LOCALLY)

(X, L) polarized abelian surface of type $(1, d)$ ($\implies d = h^0(L)$), $NS(X) \cong \mathbb{Z} \cdot [L]$

We want to compute the cohomological rank functions of $\mathcal{I}_0$

- $\mathcal{I}_0 \in \mathrm{Coh}^{\beta}(X)$ for every $\beta < 0$
- $\mathcal{I}_0 \in \mathrm{Coh}^{\beta}(X)$ is $\sigma_{\alpha,\beta}$-semistable for all $\beta < 0, \alpha \gg 0$
- Possible walls for $\mathcal{I}_0 \longleftrightarrow$ Positive solutions to $x^2 - 4dy^2 = 1$



Theorem

1. *If d is a perfect square, then*

$$h^0_{\mathcal{I}_0,L}(x) = \begin{cases} 0 & x \leq \frac{\sqrt{d}}{d} \\ dx^2 - 1 & x \geq \frac{\sqrt{d}}{d} \end{cases}$$

2. *If d is not a perfect square, then $\eta(L) \in \{\frac{2y_0}{x_0-1}, \frac{2y_1}{x_1-1}\}$ where (x_0, y_0) and (x_1, y_1) are the two smallest positive solutions to $x^2 - 4dy^2 = 1$.*

   Furthermore, $h^0_{\mathcal{I}_0,L}$ is of class $\mathcal{C}^1$ everywhere.

Corollary

Let (X, L) *be a polarized abelian surface of type* $(1, d)$, *such that* $NS(X) \cong \mathbb{Z} \cdot [L]$.

1. *If* $d \geq 7$, *then* L *is projectively normal.*
2. *If* $p \geq 1$ *and* $d > (p+2)^2$, *then* L *satisfies the property* (N_p).

Gracias!