Leaps of the integrability (in the sense of Hasse-Schmidt)

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1 Motivation

Proposition (Seidenberg '66)

Let A *be a noetherian domain containing the rational numbers. Let us denote* Σ *its fraction field and we consider* $\delta \in \mathrm{Der}(\Sigma)$*. If* $\delta \in \mathrm{Der}(A)$*, then* $\delta \in \mathrm{Der}\left(\overline{A}\right)$*.*

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If A is a local complete domain of characteristic 0 and k is a coefficient field of A, then $\mathrm{rank}(\mathrm{Der}_k(A)) \leq \dim A$.

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Let A be a noetherian domain of characteristic $p > 0$. Let us denote Σ its fraction field and we consider $\delta \in \mathrm{IDer}_k(\Sigma)$. If $\delta \in \mathrm{IDer}(A)$, then $\delta \in \mathrm{IDer}\left(\overline{A}\right)$.

Proposition (Molinelli '77)

If A is a local complete domain of characteristic $p > 0$ and k is a coefficient field of A, then $\mathrm{rank}(\mathrm{IDer}_k(A)) \leq \dim A$.

2 Introduction to the Hasse-Schmidt derivations

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A Hasse-Schmidt derivation (HS-derivation for short) of A (over k) of length $m \geq 1$ (resp. of length ∞) is a sequence $D := (D_0, D_1, \ldots, D_m)$ (resp. $D = (D_0, D_1, \ldots)$) of k-linear maps $D_r : A \to A$, satisfying the conditions:

$$D_0 = \mathrm{Id}_A, \quad D_r(xy) = \sum_{i+j=r} D_i(x) D_j(y)$$

for all $x, y \in A$ and for all r. We write $\mathrm{HS}_k(A; m)$ (resp. $\mathrm{HS}_k(A)$) for the set of HS-derivations of A (over k) of length m (resp. ∞).

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$$\mathrm{Der}_k(A) \equiv \mathrm{HS}_k(A; 1)$$

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- Let us consider $A = k[x_1, \ldots, x_d]$. The Taylor differential operators $\Delta^{(\alpha)} : A \to A$, $\alpha \in \mathbb{N}^d$, are defined by

$$f(x_1 + T_1, \ldots, x_d + T_d) = \sum_{\alpha \in \mathbb{N}^d} \Delta^{(\alpha)}(f) T^{\alpha} \quad \forall f \in A$$

Then, $\Delta_j = (\Delta^{(0,\ldots,\overbrace{i}^{j},\ldots,0)})_{i\geq 0} \in \mathrm{HS}_k(A)$ for all $j = 1, \ldots, d$.

Group structure

If $D \in \mathrm{HS}_k(A;m)$ then $D = \sum \mu^i D_i \in \mathrm{End}_k(A)[\![\mu]\!]_m$.

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If $D, D' \in \mathrm{HS}_k(A;m)$, $D'' = D \circ D' \in \mathrm{HS}_k(A;m)$ such that

$$D''_r = \sum_{i+j=r} D_i \circ D'_j$$

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The map $\delta \in \mathrm{Der}_k(A) \mapsto (\mathrm{Id}, \delta) \in \mathrm{HS}_k(A;1)$ is a group isomorphism.

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Let us denote

$$\mathrm{Hom}^{\circ}_{k-\mathsf{alg}}(A, A[\![\mu]\!]_m) := \{f \in \mathrm{Hom}_{k-\mathsf{alg}}(A, A[\![\mu]\!]_m) \mid f(x) \equiv x \mod \mu \ \forall x \in A\}$$

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The map

$$\begin{array}{ccc} \mathrm{HS}_k(A; m) & \to & \mathrm{Hom}^{\circ}_{k-\mathsf{alg}}(A, A[\![\mu]\!]_m) \\ D & \mapsto & \left[\varphi_D : x \in A \mapsto \sum_{i=0}^{m} D_i(x)\mu^i \in A[\![\mu]\!]_m\right] \end{array}$$

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If $\psi : A[\![\mu]\!]_m \to A[\![\mu]\!]_n$ is a substitution map and $D \in \mathrm{HS}_k(A; m)$, then $\psi \circ \varphi_D \in \mathrm{Hom}^{\circ}_{k-\mathsf{alg}}(A, A[\![\mu]\!]_n)$. We denote $\psi \bullet D \in \mathrm{HS}_k(A; n)$ the HS-derivation determined by $\psi \circ \varphi_D$.

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Let $D \in \mathrm{HS}_k(A;m)$ with $m \in \mathbb{N} \cup \{\infty\}$.

- For each $a \in A$, we define $a : \mu \in A[\![\mu]\!]_m \mapsto a\mu \in A[\![\mu]\!]_m$. Then, $a \bullet D = (a^r D_r)_{r \geq 0} \in \mathrm{HS}_k(A;m)$.

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- For $1 \leq n \leq m$, $\tau_{mn} : \mu \in A[\![\mu]\!]_m \mapsto \mu \in A[\![\mu]\!]_n$. Then, $\tau_{mn}(D) = (\mathrm{Id}, D_1, \ldots, D_n)$.

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- For each integer $n \geq 1$, $\psi : \mu \in A[\![\mu]\!]_m \mapsto \mu^n \in A[\![\mu]\!]_{mn}$. Then, $D[n] = \psi \bullet D \in \mathrm{HS}_k(A;mn)$:

$$D[n]_r = \begin{cases} D_{r/n} & \text{if } r = 0 \mod n \\ 0 & \text{otherwise} \end{cases}$$

3 Integrability in the sense of Hasse-Schmidt

Integrable derivations

Definition

Let $D \in \mathrm{HS}_k(A;m)$ where $m \in \mathbb{N} \cup \{\infty\}$ and $n \geq m$. We say that D is n-integrable if there exists $E \in \mathrm{HS}_k(A;n)$ such that $\tau_{nm}(E) = D$. Any such E will be called an n-integral of D. If D is ∞-integrable we simply say that D is integrable.

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Let $\delta \in \mathrm{Der}_k(A)$ *and* $n \in \mathbb{N} \cup \infty$. *We say that* δ *is* n*-integrable if there exists* $E \in \mathrm{HS}_k(A; n)$ *such that* $E_1 = \delta$.

We write $\mathrm{IDer}_k(A; n)$ *for the module of* n*-integrable derivations and* $\mathrm{IDer}_k(A) := \mathrm{IDer}_k(A; \infty)$.

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Theorem (H. Matsumura (1986))

If A is 0-smooth over k, then any Hasse-Schmidt derivation of length $m < \infty$ is ∞-integrable.

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Proposition (L.Narváez Macarro, MPTH., (2021))

$\mathrm{IDer}_k(A;n)$ *is a Lie-Rinehart algebra (anchor map=inclusion)* $\forall n \in \mathbb{N} \cup \{\infty\}$.

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Proposition (L. Narváez Macarro (2012))

If A *is a finitely presented* k*-algebra,* $n \in \mathbb{N}$ *and* $\delta \in \mathrm{Der}_k(A)$. *Then* $\delta \in \mathrm{IDer}_k(A;n)$ *iff* $\delta_{\mathfrak{p}} \in \mathrm{IDer}_k(A_{\mathfrak{p}};n)$ $\forall\ \mathfrak{p} \in \mathrm{Spec}(A)$.

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We have the chain

$$\mathrm{Der}_k(A) = \mathrm{IDer}_k(A;1) \supseteq \mathrm{IDer}_k(A;2) \supseteq \mathrm{IDer}_k(A;3) \supseteq \cdots$$

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Let $s > 1$ *be an integer. We say that the* k*-algebra* A *has a leap at* $s > 1$ *if the inclusion* $\mathrm{IDer}_k(A; s-1) \supsetneq \mathrm{IDer}_k(A; s)$ *is proper. The set of leaps of* A *over* k *is denoted by* $\mathrm{Leaps}_k(A)$*.*

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$$\mathrm{IDer}_k(A; n) = \begin{cases} \langle \partial_x \rangle & \text{if } n < p \\ \langle x\partial_x \rangle & \text{if } n \geq p \end{cases}$$

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Leaps are not determined by the semigroup of the curve.

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Theorem (MPTH)

Let k be a ring of characteristic $p > 0$ and A a k-algebra. Then, $\mathrm{Leaps}_k(A) \subseteq \{p^\tau \mid \tau \geq 1\}$.

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Definition

Let $I \subseteq A$ *be an ideal and* $m, n \in \mathbb{N} \cup \{\infty\}$

- $D \in \mathrm{HS}_k(A; m)$ *is* I*-logarithmic if* $D_r(I) \subseteq I$ *for all* r. *The set of* I*-logarithmic HS-derivations is denoted by* $\mathrm{HS}_k(\log I; m)$.

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- *Let* $\delta \in \mathrm{Der}_k(\log I) \equiv \mathrm{HS}(\log I; 1)$. *We say that* δ *is* I*-logarithmically* n*-integrable if there exists* $E \in \mathrm{HS}_k(\log I; n)$ *such that* E *is an* n*-integral of* δ. *We denote* $\mathrm{IDer}_k(\log I; n)$ *the set of* I*-logarithmically* n*-integrable derivations*

HS-derivations of polynomial rings

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Proposition (L. Narváez Macarro (2012))

Under the above conditions, the map $\Pi_m^I : \mathrm{IDer}_k(\log I; n) \to \mathrm{IDer}_k(A; n)$ *is a surjective homomorphism of* R*-modules for all* $m \in \mathbb{N} \cup \{\infty\}$.

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Corollary

A has a leap at $s > 1$ if and only if the inclusion $\mathrm{IDer}_k(\log I; s-1) \supsetneq \mathrm{IDer}_k(\log I; s)$ is proper.

Idea of the proof

Theorem

If $\mathrm{char}(k) = p > 0$, R *a polynomial ring and* $I \subseteq R$ *an ideal. Then,* $\forall n > 1$ *not a power of* p, $\mathrm{IDer}_k(\log I; n-1) = \mathrm{IDer}_k(\log I; n)$.

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- Let $\delta \in \operatorname{IDer}_k(\log I; n-1)$
- Then there is $D \in \operatorname{HS}_k(R; n)$ an n-integral of δ such that $\tau_{n,n-1}(D) \in \operatorname{HS}_k(\log I; n-1)$.

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  - $E_1 = 0$
  - $\tau_{n,n-1}(E) \in \mathrm{HS}_k(\log I; n-1)$.
  - $E_n = -D_n + H$ where $H(I) \subseteq I$.

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- $D \circ E \in \mathrm{HS}_k(\log I; n)$ is an n-integral of δ.

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For example: $\mathrm{char}(k) = p > 0$, $S = \{p^i \mid i \in \mathcal{I}\}$ where $\mathcal{I} \subseteq \mathbb{N}$ and $A = k[x_i \mid i \in \mathcal{I}]/\langle x_i^{p^i} \mid i \in \mathcal{I}\rangle$.

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If $\mathcal{I} = \mathbb{N}$, then $\mathrm{Leaps}_k(A) = \{p^\tau \mid \tau \geq 1\}$.

Thanks for your attention