The moduli space of principal bundles with formal trivializations over algebraic curves

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Previous work

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4. A. Langer, T. L. Gómez, A. H. W. Schmitt, I. Sols, *Moduli spaces for principal bundles in large characteristic*

Contents

Goals

- Construction of a fine moduli space for principal bundles over algebraic curves.

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- Relations with the stack of principal bundles.

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- Immersion of the moduli space in the scheme of sections of a projective bundle.

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3. Principal bundles have automorphisms, so we add extra data (rigidification of the problem).
4. We translate the problem of principal bundles with formal trivializations to a problem concerning vector bundles with a formal trivialization of the vector bundle.

Definition

A G-**bundle** over X with respect to the Zariski topology (resp. étale, fppf, fpqc) is a G-system (P, π) over X such that π is quasi-compact, quasi-separated and such that there exists a covering $\{f : U_i \to X\}_{i \in I}$ of X with respect to the Zariski topology (resp. étale, fppf, fpqc) such that for each index $i \in I$ there exists an isomorphism of G-systems over U_i

$$\begin{array}{ccc} f_i^* P & \xrightarrow{\sim} & U_i \times G \\ & \searrow \quad \swarrow & \\ & U_i & \end{array}$$

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2. *π is smooth, surjective, and the G-action on P is free and transitive, that is, the following natural map*

$$P \times_{\Bbbk} G \to P \times_X P$$
$$(p, g) \mapsto (p, p \cdot g)$$

is an isomorphism of X-schemes.

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- If F is a quasi-projective scheme equipped with a G-action, then there exists the associated fiber space $P \times^G F := (P \times F)/G \to X$, which is the fiber space over X with typical fibre F.

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- Extension and reduction of the structure group.

Serre Theorem

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Theorem

Let X be an algebraic variety X over $\mathbb{k}$, and let $H \subset G$ be an algebraic closed subgroup of G. To give the data of an H-bundle is equivalent to give a G-bundle together with a section s of the associated fiber space of typical fiber G/H

$$P \times^{G} (G/H) = P/H \longrightarrow X, \qquad s \colon X \to P/H$$

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$$\begin{array}{ccc} P & \longrightarrow & \mathbb{I} = \underline{\mathrm{Isom}}(V_X, E_P) \\ \downarrow & & \downarrow \\ X & \xrightarrow[s_P]{} & \mathbb{I}/G = \underline{\mathrm{Isom}}(V_X, E_P)/G \end{array}$$

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- Conversely, given a G-reduction (E, s), we have a G-bundle

$$\begin{array}{ccc} P_E := s^*(\underline{\mathrm{Isom}}(V_X, E)) & \longrightarrow & \underline{\mathrm{Isom}}(V_X, E) \\ \downarrow & & \downarrow \\ X & \xrightarrow[s]{} & \underline{\mathrm{Isom}}(V_X, E)/G \end{array}$$

- In $\mathbb{H} := \underline{\mathrm{Hom}}(V_X, E)$ we have the universal map $\mathfrak{h} : V_{\mathbb{H}} \to \mathcal{E}_{\mathbb{H}}$ which induces $\wedge V_{\mathbb{H}} \to \wedge \mathcal{E}_{\mathbb{H}}$ This latter one can be expressed as

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- Since G is reductive, $\mathbb{H} \to \mathbb{H}//G := \mathrm{Spec}(S^{\bullet}_{\mathcal{O}_X}(V^{\vee} \otimes \mathcal{E})^G)$ is a universal good quotient. The determinant line sheaf descends to the quotient, as well as the determinant section. The descent of the determinant section is denoted by $\mathfrak{det}$.

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- Since ρ lands in $\mathrm{SL}(V)$, $\mathbb{I}/G \hookrightarrow \mathbb{H}//G$ is an open subscheme and it is characterized as the complement of the vanishing locus of $\mathfrak{det}$.

Serre Theorem

Theorem

Let X be a $\Bbbk$-scheme and let G be a linear algebraic semisimple group equipped with a faithful representation $\rho : G \hookrightarrow \mathrm{SL}(V)$, with V an n-dimensional $\Bbbk$-vector space. There exists an equivalence of groupoids

$$\Big\{ G\text{-bundles over } X \Big\} \simeq \Big\{ G\text{-reductions over } X \Big\}$$

we send each G-bundle to (E_P, s_P), and, conversely, each G-reduction (E, s) is mapped to the G-bundle $P_E := s^*(\underline{\mathrm{Isom}}(V_X, E))$.
The above correspondence is functorial in X: given a morphism of schemes $f : S \to X$, it holds that $(E_{f^*P}, s_{f^*P}) = (f^*E_P, f^*s_P)$ y $P_{f^*E} = f^*P_E$.

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2. We have proved that the correspondence is well behaved under base change.
3. The extra data we want to introduce must be compatible with the Serre Correspondence

Formal trivializations

Definition

Let $\mathfrak{X}$ a formal scheme and let $G = \mathrm{Spec}(\mathbb{k}[G])$ a linear algebraic group. A G-bundle over $\mathfrak{X}$ is a formal scheme $\mathfrak{P}$ over $\mathfrak{X}$ equipped with a G-action together with a morphism of formal schemes $\pi : \mathfrak{P} \to \mathfrak{X}$ which is G-invariant, qc, faithfully flat, and there exists an isomorphism of formal schemes

$$\mathfrak{P} \times_{\mathsf{Spf}(\mathbb{k})} \times G \simeq \mathfrak{P} \times_{\mathfrak{X}} \mathfrak{P}$$

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In this setting, the trivial G-bundle over $\mathfrak{X}$ is defined as the formal scheme $\mathfrak{X} \times_{\mathrm{Spf}(\Bbbk)} G$ equipped with the G-action given by the multiplication of G. The map $\pi : \mathfrak{X} \times_{\mathrm{Spf}(\Bbbk)} G \to \mathfrak{X}$ is the projection on the first factor.

Formal trivializations

Definition

Let X be a $\Bbbk$-scheme and let $Z \subset X$ be a closed subscheme defined by an ideal $\mathcal{J}$. Let us use the notation $Z_n := \mathrm{Spec}(\mathcal{O}_X/\mathcal{J}^{n+1})$. The **completion** of X along Z is defined as the formal scheme $\mathfrak{Z}$ whose underlying topological space is $|Z|$, and the sheaf of rings is $\mathcal{O}_{\mathfrak{Z}} := \varprojlim \mathcal{O}_{Z_n}$.

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Definition

Let $P \to X$ be a G-bundle over X and let Z be a closed subscheme. We define G-**formal bundle associated** to P over the completion of X along Z as the pair $(\mathfrak{P}, \mathcal{O}_{\mathfrak{P}})$ being $\mathfrak{P}$ the underlying topological space of the pullback of P to Z, and $\mathcal{O}_{\mathfrak{P}}$ the sheaf of rings obtained as the projective limit of the sheaves $\mathcal{O}_{P_n}$, being P_n the pullback of P to Z_n.

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    1. *The associated formal G-bundle $\mathfrak{P} \to \mathfrak{Z}$ is isomorphic to $\mathfrak{Z} \times G$.*

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    1. *The associated formal G-bundle $\mathfrak{P} \to \mathfrak{Z}$ is isomorphic to $\mathfrak{Z} \times G$.*
    2. *For each $n \geq 1$, we have G-equivariant isomorphisms $P_n \simeq Z_n \times G$ compatibles with the inductive systems defined by (Z_n) and (P_n).*

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    2. *For each $n \geq 1$, we have G-equivariant isomorphisms $P_n \simeq Z_n \times G$ compatibles with the inductive systems defined by (Z_n) and (P_n).*
    3. *For some n, P_n is trivializable.*

Formal trivializations

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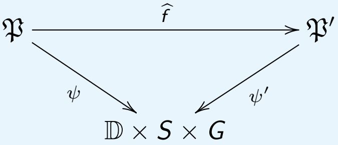
3. There exists a bijection between the set of formal trivializations of P and the set of formal trivializations of the associated vector bundle E_P.Let us point out that a formal trivialization of $\mathcal{E}$ is an isomorphism $\widehat{\mathcal{E}}_{p\times S} \simeq \mathcal{O}_S[[t]]^{\oplus n}$.

Definition

Let (E, s) be a G-reduction over $C \times S$. A formal trivialization of (E, s) is an isomorphism of sheaves $\widehat{\mathcal{E}}_{p\times S} \simeq \mathcal{O}_S[[t]]^{\oplus n}$.

Definition

Let $(P,\psi)(P',\psi')$ be two G-bundles over $C \times S$ equipped with formal trivializations. A morphism of pairs $(P,\psi) \to (P',\psi')$ is a morphism of G-bundles $f : P \to P'$, such that the induced map between the associated formal bundles $\widehat{f} : \mathfrak{P} \to \mathfrak{P}'$, is compatible with the given formal trivializations



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$$\begin{array}{ccc} \mathfrak{P} & \xrightarrow{\widehat{f}} & \mathfrak{P}' \\ & \searrow^{\psi} \quad \swarrow^{\psi'} & \\ & \mathbb{D} \times S \times G & \end{array}$$

Definition

Let (E, s, ψ), and (E', s', ψ') be two G-reductions equipped with a formal trivialization of the vector bundles. A morphism of those objects is a morphism of G-reductions $f : (E, s) \to (E', s')$ such that the induced map between the compeltions of E and E' along $p \times S$ is compatible with ψ and ψ'.

Formal trivializations

Theorem

There exists an equivalence of groupoids which is functorial on S

$$\left\{ \begin{array}{c} G\text{-bundles over } C \times S \\ \text{with formal trivializations} \\ \text{along } \{p\} \times S \end{array} \right\} \simeq \left\{ \begin{array}{c} G\text{-reductions over } C \times S \text{ with} \\ \text{formal trivializations} \\ \text{along } \{p\} \times S \end{array} \right\}$$

The moduli space $\mathrm{Bun}^{\infty}_{G,C}$

- Let us consider the moduli space $\mathcal{U}^{\infty}_{C}$. A point of this moduli space is a pair (E, ψ) being E a vector bundle of rank n over $C \times S$, and ψ a formal trivialization of E along $\{p\} \times S$. The moduli space is a closed subscheme of the infinite Grassmannian $\mathrm{Gr}(\Bbbk((t))^{\oplus n})$.

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Definition

We define the determinant morphism $\det^{\infty}$ as the map between functors

$$\begin{aligned} \det^{\infty} : \mathcal{U}^{\infty}_C(S) &\to \text{Pic}^{\bullet}_C(S) \\ (\mathcal{E}, \psi) &\mapsto \det \mathcal{E} := \wedge^n \mathcal{E} \end{aligned}$$

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- We define the moduli space of vector bundles with formal trivialization and trivial determinant as $\mathcal{U}_C^{\infty, triv} := \mathcal{U}_C^{\infty} \times_{\mathrm{Pic}_C} \{[\mathcal{O}_C]\}$

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The universal pair of $\mathcal{U}^{\infty, triv}_C$ is denoted by $(\mathcal{E}_{\mathcal{U}}, \psi_{\mathcal{U}})$. We use the notation $\mathcal{A}_{\mathcal{U}} := \left(S^{\bullet}_{\mathcal{O}_{C \times \mathcal{U}^{\infty, triv}_C}}(V_{\mathcal{U}} \otimes \mathcal{E}^{\vee}_{\mathcal{U}})\right)^G$ where $V_{\mathcal{U}} := V \otimes_{\Bbbk} \mathcal{O}_{C \times \mathcal{U}^{\infty, triv}_C}$.

Theorem

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Proof.

Let us consider the scheme $\big(\mathrm{Spec}(\mathcal{A}_{\mathcal{U}})\big)_{\mathfrak{det}_{\mathcal{U}}}$, where $\mathfrak{det}_{\mathcal{U}}$ is the descent of the determinant section attached to $\mathcal{E}_{\mathcal{U}}$ y $V_{\mathcal{U}}$. $\big(\mathrm{Spec}(\mathcal{A}_{\mathcal{U}})\big)_{\mathfrak{det}_{\mathcal{U}}}$ denotes the open subscheme of $\mathrm{Spec}(\mathcal{A}_{\mathcal{U}})$ where the section $\mathfrak{det}_{\mathcal{U}}$ does not vanish.

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Since the representation $\rho : G \hookrightarrow \mathrm{SL}(V)$ is fixed, through the Serre correspondence, we have defined a forgetful map from $\mathrm{Bun}^{\infty}_{G,C}$ to the functor of points of $\mathcal{U}_C^{\infty,triv}$:

$$\mathrm{Bun}^{\infty}_{G,C} \to (\mathcal{U}_C^{\infty,triv})^{\bullet}$$

□

Proof.

To prove the representability of $\text{Bun}^{\infty}_{G,C}$ is equivalent to prove that the forgetful map is representable, which is equivalent to prove the representability of the scheme of sections of the map

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The scheme of sections is representable due to the good properties of $\mathcal{A}_{\mathcal{U}} = \left(S^{\bullet}_{\mathcal{O}_{C\times\mathcal{U}_C^{\infty,triv}}}(V_{\mathcal{U}} \otimes \mathcal{E}_{\mathcal{U}}^{\vee})\right)^G$ □

Contents

The forgetful mapπ^{∞}

- ¿How are relate $\mathrm{Bun}^{\infty}_{G,C}$ and $\mathrm{Bun}_{G,C}$?

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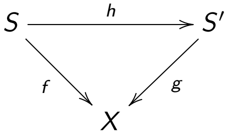

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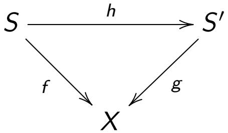

Definition

We define the forgetful map $\pi^{\infty} : \mathcal{S}\,\mathrm{Bun}^{\infty}_{G,C} \to \mathrm{Bun}_{G,C}$ by sending each triple (S, P, ψ), to the pair (S, P).

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If $\mathbf{Sch}_{\Bbbk}$ is equipped with the Zariski topology, then π^{∞} is surjective when G is special.

If we replace the Zariski topology by the fppf topology or the étale topology, then π^{∞} is surjective for every group G.

Loop Group

Definition

Given a group G, we define the positive Loop group of G, L^+G, as the representative of the functor that assign to each affine $\mathbb{k}$-scheme $\mathrm{Spec}(R)$, the group $\mathrm{Hom}(\mathbb{k}[G], R[[z]])$.

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Let us give a geometric interpretation. Let us consider the functor

$$\mathrm{Aut}_G : \mathbf{Sch}_{\mathbb{k}} \to \mathbf{Groups}$$

assigning to each $\mathbb{k}$-scheme S, the automorphism group of the trivial G-bundle over $\mathbb{D} \times S$, this is

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Theorem

The sheafification of Aut_G *is canonically isomorphic to* L^+G.

The uniformization Theorem

- The group Aut_G acts on the moduli space $\mathrm{Bun}^{\infty}_{G,C}$.

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Theorem

Let G be a linear semisimple special algebraic group. There exists an isomorphism of stacks over $\mathbf{Sch}_{\Bbbk}$ (equipped with the Zariski topology).

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The theorem is true for any linear semisimple group G if we consider the étale topology on $\mathbf{Sch}_{\mathbb{k}}$.

Contents

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- We prove that there exists a canonical immersion of $\mathrm{Spec}(\mathcal{A}_U)$ in a projective bundle.
- To obtain the above results we need to study the algebra of invariants $\mathcal{A}_U = S^{\bullet}(V_{\mathcal{U}} \otimes \mathcal{E}_{\mathcal{U}})^G$.

Pgg-algebras and properties

Definition

Let R be a ring. A **partial algebra** over R with support I is to give $\{A_i, m_{ij}\}_{i,j\in I}$ where each A_i is an R-module, and each m_{ij} is a bilinear map of R-modules, which is known as the multiplication map

$$m_{ij} : A_i \times A_j \to A_{i+j} \quad \text{para todo } i,j \in I \text{ con } i+j \in I$$

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The multiplication maps must satisfy the commutative property $m_{ij}(a_i, a_j) = m_{ji}(a_j, a_i)$ for every $a_i \in A_i$, $a_j \in A_j$ and every pair of index i,j with $i, j, i+j \in I$; The multiplication maps must verify the associative property (when it makes sense)

$$m_{i+j,k} \circ (m_{ij}, 1) = m_{i,j+k} \circ (1, m_{jk})$$

Let $\mathbf{Alg}_R$ be the category of R-algebras. Given a natural number $t > 0$ and a graded R-algebra $A = \oplus_{n\geq 0} A_n$, there exists a natural transformation of functors on $\mathbf{Alg}_R$

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Definition

Let $A = \oplus_{n \geq 0} A_n$ be a graded R-algebra. We say that A is a graded partially generated R-algebra, or for the sake of simplicity, a pgg-R-algebra, if $A_0 = R$ and there exists a natural number t such that $\Phi_{\leq t}$ is an isomorphism of functors on $\mathbf{Alg}_R$. In this case we say that A is a t-pgg-R-algebra.

- Any graded algebra of finitely presented is a pgg-algebra.

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- Let V be $\mathbb{k}$-vector space and let G be a linear algebraic semisimple group acting on V through a faithful representation $\rho : G \hookrightarrow \mathrm{SL}(V)$. The ring of invariants $\mathbb{k}[V]^G := (S^{\bullet}_{\mathbb{k}} V^*)^G$ is a pgg-$\mathbb{k}$-algebra.

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- If A is a t-pgg-R-algebra and $R \to R'$ is a ring homomorphism, then, $A_{R'} := A \otimes_R R'$ is a t-pgg-R'-algebra.

Canonical immersion

If $A = \oplus_{i \geq 0} A_i$ is a graded R-algebra generated by its elements of degree 1, there exists an immersion locally closed $\mathrm{Spec}(A) \to \mathbb{P}(R \oplus A_1)$.

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Theorem

Let A be a t_A-pgg-R-algebra and let $t := t_A + 1$. Then, the map $\operatorname{Spec} A \to \operatorname{Spec} R$ *factorizes through a locally closed immersion of* $\operatorname{Spec} R$*-schemes*

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The closure of the image is given by $\mathrm{Proj}(A \otimes_R R[T])$. Besides, there exists a natural action of $\mathcal{G} := \mathrm{Aut}_{\Bbbk-alg-grad}(A)$ over $\mathbb{P}\Big(\bigoplus_{1\leq |\underline{d}| \leq t} S^{d_1} A_1 \otimes_R \cdots \otimes_R S^{d_t} A_t \Big)$ such that the above immersion is $\mathcal{G}$-equivariant.

Generalization of Nagata Theorem

- Nagata proves that for any finitely generated $\mathbb{k}$-algebra A and every faithful representation $\rho : G \to \mathrm{Aut}_{\mathbb{k}-alg}(A)$, the ring of invariants $A^G \subset A$ is a finitely generated $\mathbb{k}$-algebra when G is reductive.

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- Seshadri proves a generalization of the above Theorem. If R is a finitely generated $\mathbb{k}$-algebra, G a reductive algebraic group, M finitely generated free R-module and $\rho : G^{\bullet}_R \to \underline{\mathrm{Aut}}_{R-mod}(M)$ a representation, then the algebra of invariants $S^{\bullet}_R(M^{\vee})^{G_R}$ is finitely generated over R.

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- We study the algebra $(S_T^\bullet(M^\vee \otimes_R N))^{G_T}$, with R an arbitrary $\mathbb{k}$-algebra, M being a free finitely generated R-module, $\rho : G_R^\bullet \to \underline{\mathrm{Aut}}_{R-mod}(M)$ a faithful representation, T an arbitrary R-algebra and N a T-module.

Theorem

Let R be a $\Bbbk$-algebra, T a R-algebra, M a free R-module of finite rank, and N a T-module. Let G be a linear semisimple algebraic group over $\Bbbk$ together with a faithful representation $\rho : G^{\bullet}_R \hookrightarrow \mathrm{Aut}_{R-mod}(M)$. Let A denotes the graded T-algebra $(S^{\bullet}_T(M^{\vee} \otimes_R N)^{G_T}$. Then

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4. *If the representation is $\rho : G^{\bullet} \to \underline{\mathrm{Aut}}_{\mathbb{k}-mod}(V)$, being V a $\mathbb{k}$-vector space, and we assume that $M := V \otimes_{\mathbb{k}} R$ then, if N is a flat T-module, then A is a flat T-algebra.*

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Let R be a $\Bbbk$-algebra, T a R-algebra, M a free R-module of finite rank, and N a T-module. Let G be a linear semisimple algebraic group over $\Bbbk$ together with a faithful representation $\rho : G^\bullet_R \hookrightarrow \mathrm{Aut}_{R-mod}(M)$. Let A denotes the graded T-algebra $(S^\bullet_T(M^\vee \otimes_R N))^{G_T}$. Then

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Besides, if $R \to T$ is a finitely presented map (resp. flat), 1), 2) (resp. 3)) are still valid if we replace T by R.

Projective immersion

Theorem

Let R be a $\Bbbk$-algebra, T a R-algebra, M a free R-module of finite rank and N a T-module. Let G be a linear semisimple algebraic group over $\Bbbk$ together with a faithful representation $\rho : G_R^\bullet \hookrightarrow \underline{\mathrm{Aut}}_{R-mod}(M)$. If N is finitely presented, or G is a classical group (Sl_n, SO_n, Sp_{2n}), then there exists a natural locally closed immersion

$$\Phi_{M,N,G} : \mathrm{Spec}\left(\left(S_T^\bullet(M^\vee \otimes_R N)\right)^{G_T}\right) \hookrightarrow \mathbb{P}\Big(\bigoplus_{1\leq|\underline{d}|\leq d} S^{d_1}A_1 \otimes_R \cdots \otimes_R S^{d_t}A_t\Big)$$

with $A_i = (S^i(M^\vee \otimes_R N))^{G_T}$, for some natural number d.

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Let R be a $\Bbbk$-algebra, T a R-algebra, M a free R-module of finite rank and N a T-module. Let G be a linear semisimple algebraic group over $\Bbbk$ together with a faithful representation $\rho : G^{\bullet}_{R} \hookrightarrow \underline{\mathrm{Aut}}_{R-mod}(M)$. If N is finitely presented, or G is a classical group (Sl_n, SO_n, Sp_{2n}), then there exists a natural locally closed immersion

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with $A_i = \left(S^i(M^{\vee} \otimes_R N)\right)^{G_T}$, for some natural number d. Besides, if G is one of the classical groups, d depends on G but does not depend on N. In any case, $\Phi_{M,N,G}$ is $\mathrm{Aut}_T(N)$-equivariant.

Theorem

Let G be a linear semisimple classical algebraic group $(\mathrm{SL}_n, \mathrm{SO}_n, \mathrm{Sp}_{2n})$ over $\Bbbk$ equipped with a faithful representation $\rho : G^{\bullet} \hookrightarrow \mathrm{SL}(V)$. There exists a canonical immersion of $\mathrm{Bun}^{\infty}_{G,C}$ in the scheme of sections of the projective bundle

$$\mathbb{P}\Big(\bigoplus_{1\leq|\underline{d}|\leq t} S^{d_1}A_1 \otimes_R \cdots \otimes_R S^{d_t}A_t\Big) \to \mathcal{U}_C^{\infty,triv} \times C$$

with $A_i = \Big(S^i_{\mathcal{O}_{C\times\mathcal{U}_C^{\infty,triv}}}(V_{\mathcal{U}} \otimes \mathcal{E}^{\vee}_{\mathcal{U}})\Big)^G$. Therefore, $\mathrm{Bun}^{\infty}_{G,C}$ can be described as a subscheme of the scheme of sections of $\mathrm{Proj}(\mathcal{A}_{\mathcal{U}}[T]) \to \mathcal{U}_C^{\infty,triv} \times C$.

Thank you for your attention.