Gröbner's problem and the geometry of $GT-$varieties.

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Content.

1. New contributions to Gröbner's problem.
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2. The geometry of $\overline{G}-$varieties.
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   - The homogeneous ideal.
   - The canonical module.
   - The Castelnuovo–Mumford regularity.

Notation.

- $\mathbb{K}$, algebraically closed field of characteristic zero.
- $R = \mathbb{K}[x_0, \dots, x_n]$, the polynomial ring.

$$R = \bigoplus_{t \geq 0} R_t, \quad \dim_{\mathbb{K}} R_t = \binom{n+t}{n} =: N_{n,t}$$

- $GL(n+1, \mathbb{K})$, the group of invertible $(n+1)$-square matrices over $\mathbb{K}$.
- R^{Λ}, the ring of invariants of a finite group $\Lambda \in \mathrm{GL}(n+1, \mathbb{K})$.
- *CM*, Cohen-Macaulay and
- *aCM*, arithmetically Cohen-Macaulay.

Gröbner's problem.

The *Veronese variety* $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ of $\mathbb{P}^n$ is the $n-$dimensional projective variety parameterized by the set of all monomials of degree d in R:

$$\mathcal{M}_{n,d} = \{x_0^{a_0} \cdots x_n^{a_n} \in R \mid a_0 + \cdots + a_n = d\}.$$

Definition

A *monomial projection* $Y_{n,d} \subset \mathbb{P}^{\mu_{n,d}-1}$ of $X_{n,d}$ is a projective variety parameterized by a subset $\Omega_{n,d} \subset \mathcal{M}_{n,d}$ of $\mu_{n,d} \leq N_{n,d}$ monomials.

- For $N_{n,d} - \mu_{n,d} = 1, 2, 3,$ $Y_{n,d}$ is called, respectively, a *simple, double, triple* monomial projection of the Veronese variety $X_{n,d}$.
- For $N_{n,d} - \mu_{n,d} \geq 4,$ $Y_{n,d}$ is called a *multiple* monomial projection of the Veronese variety $X_{n,d}$.

Gröbner's problem.

$$\Omega_{n,d} = \{m_{i_1}, \ldots, m_{i_{\mu_{n,d}}}\} \subset \mathcal{M}_{n,d} = \{m_1, \ldots, m_{N_{n,d}}\}$$

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\nu_{n,d}} & X_{n,d} = \nu_{n,d}(\mathbb{P}^n) \\ & \searrow^{\varphi_{\Omega_{n,d}}} & \downarrow \pi \\ & & Y_{n,d} = \overline{\varphi_{\Omega_{n,d}}(\mathbb{P}^n)} \end{array}$$

$$\nu_{n,d} : \mathbb{P}^n \longrightarrow \mathbb{P}^{N_{n,d}-1}, \quad (y_0 : \ldots : y_n) \to (m_1(p) : \cdots : m_{N_{n,d}}(p))$$

$$\varphi_{\Omega_{n,d}} : \mathbb{P}^n \longrightarrow \mathbb{P}^{\mu_{n,d}-1}, \quad (y_0 : \ldots : y_n) \to (m_{i_1}(p) : \cdots : m_{i_{\mu_{n,d}}}(p))$$

π is the projection of $X_{n,d}$ from the linear subspace $\langle p_i = (0 : \cdots : 0 : 1 : 0 : \cdots : 0), m_i \notin \Omega_{n,d} \rangle \subset \mathbb{P}^{N_{n,d}-1}$ to the linear subspace $V(w_{m_i},\ m_i \notin \Omega_{n,d}) \subset \mathbb{P}^{N_{n,d}-1}$.

Gröbner's problem.

(1967) Gröbner proved that the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ is an aCM variety.

The *simple* monomial projection $Y_{n,d} \subset \mathbb{P}^{N_{n,d}-2}$ of $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ parameterized by $\Omega_{n,d} = \mathcal{M}_{n,d} \setminus \{x_i^{d-2}x_j^2\}$ is a **non aCM** variety.

The *simple* monomial projection $Y_{n,d} \subset \mathbb{P}^{N_{n,d}-2}$ of $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ parameterized by $\Omega_{n,d} = \mathcal{M}_{n,d} \setminus \{x_i^d\}$ is an **aCM** variety.

Problem (Gröbner's problem)

To determine when a monomial projection $Y_{n,d}$ of the Veronese variety $X_{n,d}$ is an aCM variety.

Gröbner's problem.

The *rational normal curve* $X_{1,d} \subset \mathbb{P}^d$ is the Veronese curve parameterized by all monomials of degree d in $\mathbb{K}[x_0, x_1]$:

$$\mathcal{M}_{1,d} = \{x_0^d, x_0^{d-1}x_1, x_0^{d-2}x_1^2, \dots, x_0x_1^{d-1}, x_1^d\}.$$

Example

We take $d = 4$, $\mathcal{M}_{1,4} = \{x_0^4, x_0^3x_1, x_0^2x_1^2, x_0x_1^3, x_1^4\}$.

$$\Omega_{1,4} = \{x_0^4, x_0^3x_1, x_0x_1^3, x_1^4\} = \mathcal{M}_{1,4} \setminus \{x_0^2x_1^2\}.$$

$\mathbb{K}[\Omega_{1,4}]$ is the first example of a non CM domain due to Macaulay, so the simple monomial projection $Y_{1,4} \subset \mathbb{P}^3$ parameterized by $\Omega_{1,4}$ is a non aCM curve.
If instead of $x_0^2x_1^2$ we delete the monomial x_0^4, we obtain a rational twisted cubic in $\mathbb{P}^3$, which is an aCM curve.

Gröbner's problem.

Consider a monomial projection $Y_{n,d} \subset \mathbb{P}^{\mu_{n,d}-1}$ of $X_{n,d}$ parameterized by $\Omega_{n,d} = \{m_1, \dots, m_{\mu_{n,d}}\}$.

We take $w_1, \dots, w_{\mu_{n,d}}$ new variables and we set $S = \mathbb{K}[w_1, \dots, w_{\mu_{n,d}}]$.

$\mathrm{I}(Y_{n,d})$ is the kernel of the morphism

$$\rho : S \to \mathbb{K}[m_1, \dots, m_{\mu_{n,d}}] =: \mathbb{K}[\Omega_{n,d}], \quad \rho(w_i) = m_i$$

$A(Y_{n,d}) \cong \mathbb{K}[\Omega_{n,d}]$, i.e the semigroup ring of the affine semigroup

$$\mathrm{H}(\Omega_{n,d}) = \langle (a_0, \dots, a_n) \mid x_0^{a_0} \dots x_n^{a_n} \in \Omega_{n,d} \rangle \subset \mathbb{Z}_{\geq 0}^{n+1}.$$

$\mathrm{I}(Y_{n,d})$ is the binomial prime ideal generated by

$$\left\{ \prod_{i=1}^{\mu_{n,d}} w_i^{\alpha_i} - \prod_{i=1}^{\mu_{n,d}} w_i^{\beta_i} \;\middle|\; \prod_{i=1}^{\mu_{n,d}} m_i^{\alpha_i} = \prod_{i=1}^{\mu_{n,d}} m_i^{\beta_i}, \alpha_i, \beta_i \in \mathbb{Z}_{\geq 0} \right\}.$$

Gröbner's problem. Techniques

- *Normal* affine semigroups $H \subset \mathbb{Z}^{n+1}_{\geq 0}$, i.e. if $h_1, h_2, h_3 \in H$ and $zh_1 = zh_2 + h_3$ for some $z \in \mathbb{Z}_{\geq 0}$, then there exists $h \in H$ such that $h_3 = zh$.

Theorem (Hochster)

If H is normal, then $\mathbb{K}[H]$ is a CM ring.

- *Simplicial* affine semigroups $H \subset \mathbb{Z}^{n+1}_{\geq 0}$, i.e. there are $\mathbb{Q}-$linearly independent elements $e_0, \ldots, e_n \in H$ and $z \in \mathbb{Z}_{\geq 0}$ such that zH is contained in the affine semigroup $\langle e_0, \ldots, e_n \rangle$.

Theorem (Goto, Suzuki, Watanabe, Hoa, Trung)

Set $H_1 := \{h \in \overline{H} \mid h + e_i \in H \text{ and } h + e_j \in H \text{ for some } 0 \leq i \neq j \leq n\}$. Then, $\mathbb{K}[H]$ is a CM ring if and only if $H = H_1$.

Results on Gröbner's problem

Gröbner's problem in terms of the deleted monomials.

- (Schenzel) Simple monomial projections.
- (Trung) Double monomial projections.
- (Hoa) Triple monomial projections.

Monomial projections of the rational normal curve $X_{1,d} \subset \mathbb{P}^d$

- (Cavaliere, Niesi) Simplicial case.
- (Trung) In general.

Gröbner's problem for multiple monomial projections of the Veronese varieties remains barely known.

Invariants of finite abelian groups

Let $G \subset \mathrm{GL}(n+1, \mathbb{K})$ be a finite abelian group of order $d = d_1 \cdots d_s$.

$$G = \Gamma_1 \oplus \cdots \oplus \Gamma_s \subset \mathrm{GL}(n+1, \mathbb{K})$$

where each Γ_i is a cyclic group of order d_i with $d_i | d_i + 1$ and it is generated by a diagonal matrix of the form
$M_{d_i;\gamma_0,\dots,\gamma_n} := \mathrm{diag}(e_i^{\gamma_0}, \dots, e_i^{\gamma_n})$, i.e.

$$\begin{pmatrix} e_i^{\gamma_0} & 0 & \cdots & 0 \\ 0 & e_i^{\gamma_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_i^{\gamma_n} \end{pmatrix}$$

with e_i a d_ith primitive root of $1 \in \mathbb{K}$ and $\mathrm{GCD}(d_i, \gamma_0, \dots, \gamma_n) = 1$.

The ring of invariants of G

$$R^G = \{p \in R \mid g(p) = p, \forall g \in G\} = \oplus_{t\geq 0} R_t^G, \; R_t^G = R_t \cap R^G.$$

- R^G has a minimal set $\{m_1, \ldots, m_N\}$ of monomial generators of degree at most $\mathrm{ord}(G) = d$, i.e. $R^G = \mathbb{K}[m_1, \ldots, m_N]$. $\{m_1, \ldots, m_N\}$ is called a *minimal set of fundamental invariants of G.*

Theorem (Eagon, Hochster)

The ring R^Λ of invariants of a finite group $\Lambda \subset \mathrm{GL}(n+1, \mathbb{K})$ is a CM ring.

Invariants of finite abelian groups

Example

We take $G = \langle M_{3;0,1,2} = \mathrm{diag}(1, e, e^2)\rangle \subset \mathrm{GL}(3, \mathbb{K})$. A minimal set of generators of R^G consists of monomials of degree 1, 2 and 3:

$$\{x_0, x_1x_2, x_1^3, x_2^3\}.$$

We consider the dth Veronese subalgebra $R^{\overline{G}}$ of R^G:

$$R^{\overline{G}} = \bigoplus_{t\geq 0} R_t^{\overline{G}} = \bigoplus_{t\geq 0} R_{td}^G \subset \bigoplus_{t\geq 0} R_t^G.$$

$R^{\overline{G}}$ is the ring of invariants of the abelian group $\overline{G} \subset \mathrm{GL}(n+1, \mathbb{K})$ generated by the generators of G and the diagonal matrix $\mathrm{diag}(e, \ldots, e)$, where e is a dth primitive root of $1 \in \mathbb{K}$. $\overline{G}$ is called the *cyclic extension of G*.

Example (Last example)

We take $G = \langle M_{3;0,1,2} = \mathrm{diag}(1, e, e^2)\rangle \subset \mathrm{GL}(3, \mathbb{K})$. A minimal set of generators of R^G consists of monomials of degree 1, 2 and 3:

$$\{x_0, x_1x_2, x_1^3, x_2^3\}.$$

Example (Taking the cyclic extension)

We take the cyclic extension $\overline{G} = \langle M_{3;0,1,2}, M_{3;1,1,1}\rangle \subset \mathrm{GL}(3, \mathbb{K})$, an abelian group of order 9. A minimal set of generators of $R^{\overline{G}}$ consist of monomials of the same degree 3:

$$\{x_0^3, x_1^3, x_2^3, x_1x_2x_3\}.$$

They parameterize an aCM monomial projection $Y_{2,3} \subset \mathbb{P}^3$ of the Veronese surface $X_{2,3} \subset \mathbb{P}^9$.

Theorem

The set $\mathcal{B}_1$ of all monomial invariants of G of degree d is a minimal set of generators of $R^{\overline{G}}$, i.e $R^{\overline{G}} = \mathbb{K}[\mathcal{B}_1]$.

$\mathcal{B}_1 = \{m_1, \dots, m_{\mu_d}\} \subset R$ with $|\mathcal{B}_1| = \mu_d$.

Theorem

*$\mathcal{B}_1$ parameterizes an aCM monomial projection $X_d \subset \mathbb{P}^{\mu_d - 1}$ of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$. We call X_d a $\overline{G}-$*variety with group $G \subset \mathrm{GL}(n+1, \mathbb{K})$.

Invariants of finite abelian groups from Combinatorics.

The set of monomial invariants of G of degree td is a $\mathbb{K}-$basis of $R_{td}^{\overline{G}}$, we denote it by $\mathcal{B}_t$. They are univocally determined by the set of $\mathbb{Z}_{\geq 0}^{n+1}-$solutions of the system of congruences $(*)_{\mathcal{A};t,r_1,\ldots,r_s}$:

$$\begin{cases} y_0 + y_1 + \cdots + y_n = td \\ \alpha^1_{\sigma_1(0)}y_0 + \alpha^1_{\sigma_1(1)}y_1 + \cdots + \alpha^1_{\sigma_1(n)}y_n = r_1 d_1 \\ \vdots \\ \alpha^s_{\sigma_s(0)}y_0 + \alpha^s_{\sigma_s(1)}y_1 + \cdots + \alpha^s_{\sigma_s(n)}y_n = r_s d_s \end{cases}$$

$0 \leq r_i \leq \frac{\alpha^i_n td}{d_i}$, $i = 1, \ldots, s$.

Remark

$(td, 0, \ldots, 0), \ldots, (0, \ldots, 0, td)$ are solutions of $(*)_{\mathcal{A};t;r_1,\ldots,r_s}$.

The set of all these points forms a *normal simplicial affine semigroup* $H_{\mathcal{A}} \subset \mathbb{Z}_{\geq 0}^{n+1}$. By Hochster's result, we obtain that $\mathbb{K}[H_{\mathcal{A}}]$ is a CM ring.

We denote by $I_d \subset R$ the monomial artinian ideal generated by the set $\mathcal{B}_1 = \{m_1, \ldots, m_{\mu_d}\}$ of fundamental invariants of $\overline{G}$.

Definition

An artinian ideal $J \subset R$ fails the *weak Lefschetz property, shortly WLP*, in degree i_0 if for any linear form $L \in (R/L)_1$ the multiplication map

$$\times L : (R/J)_{i_0} \to (R/J)_{i_0+1}$$

does not have maximal rank, i.e. it is neither injective nor surjective.

Example

We take $J = \langle x_0^3, x_1^3, x_2^3, x_0x_1x_2 \rangle$. The ideal J fails the WLP in degree 2 since the multiplication map $\times L : (R/J)_2 \to (R/J)_3$ is not injective for any $L \in R_1$.

Togliatti systems

A connection between artinian ideals failing the WLP and the existence of projective varieties satisfying at least one Laplace equation:

Theorem (Mezzetti, Miró-Roig, Ottaviani (Tea Theorem))

Let $J \subset R$ be an artinian ideal generated by r forms $F_1, \ldots, F_r$ of degree d and let J^{-1} be its Macaulay's inverse system. If $r \leq N_{n-1,d} = \binom{n-1+d}{n-1}$, then the following conditions are equivalent.

(i) *J fails the WLP in degree $d-1$.*

(ii) *$F_1, \ldots, F_r$ become $\mathbb{K}-$linearly dependent on a general hyperplane $H \subset \mathbb{P}^n$.*

(iii) *The $n-$dimensional variety $Y := \overline{\varphi(\mathbb{P}^n)}$ satisfies at least one Laplace equation of order $d-1$, where $\varphi = \mathbb{P}^n \dashrightarrow \mathbb{P}^{N_{n,d}-r-1}$ is the rational map associated to J_d^{-1}.*

They called a *Togliatti system* to any ideal J satisfying the equivalent conditions (i),(ii) and (iii).

$GT-$systems and $GT-$varieties.

Definition ($GT-$systems and $GT-$varieties)

Let $\Lambda \subset \mathrm{GL}(n+1, \mathbb{K})$ be a finite group of order d with $2 \leq n < d$. A Togliatti system $J \subset R$ generated by $r \leq N_{n-1,d}$ forms $F_1, \dots, F_r$ of degree d is said to be a *$GT-$system with group* Λ if the associated morphism $\varphi_J : \mathbb{P}^n \longrightarrow \mathbb{P}^{r-1}$ is a Galois covering with group Λ. In this case, $X = \varphi_J(\mathbb{P}^n)$ is called a *$GT-$variety with group* Λ.

Proposition (The Galois covering condition)

Let $\Lambda \subset \mathrm{GL}(n+1, \mathbb{K})$ *be a finite group of order* $|\Lambda|$ *and* $\mathcal{B} = \{g_1, \dots, g_r\}$ *a minimal set of homogeneous fundamental invariants of* Λ *with* $\deg(g_1) = \cdots = \deg(g_r) =: d$. *Let* $\varphi_{\mathcal{B}} : \mathbb{P}^n \longrightarrow \mathbb{P}^{r-1}$ *be the morphism defined by* $(g_1 : \cdots : g_r)$. *It holds:*

(i) R^Λ *is the homogeneous coordinate ring of the projective variety* $X := \varphi_{\mathcal{B}}(\mathbb{P}^n) \subset \mathbb{P}^{r-1}$. *Thus* X *is the quotient variety* $\mathbb{P}^n/\Lambda$ *and it is an aCM variety.*

(ii) $\varphi_{\mathcal{B}} : \mathbb{P}^n \longrightarrow \mathbb{P}^{r-1}$ *is a Galois covering of* $\mathbb{P}^n$ *with group* Λ.

$\overline{G}-$varieties and $GT-$varieties.

Let $G \subset \mathrm{GL}(n+1,\mathbb{K})$ an abelian group of order d, $\mathcal{B}_1 = \{m_1, \ldots, m_{\mu_d}\}$ the set of fundamental invariants of $\overline{G}$ and $I_d = (\mathcal{B}_1) \subset R$.

Proposition

If $\mu_d \leq N_{n-1,d}$, then I_d is a $GT-$system with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$.

- Let $L \in (R/I_d)_1$ be a linear form.
- We set $F := \prod_{g \in G \setminus \{Id\}} g(L)$, which is a form of degree $d-1$.
- $L \cdot F = \prod_{g \in G} g(L)$ is a form of degree d. By construction, $L \cdot F$ is an invariant of G of degree d, thus $L \cdot F \in I_d$.
- The multiplication map $\times L : (R/I_d)_i \to (R/I_d)_{i+1}$ has maximal rank for any $0 \leq i \leq d-2$.
- Since $\mu_d \leq N_{n-1,d}$, $\times L : (R/I_d)_{d-1} \to (R/I_d)_d$ is neither injective nor surjective. So I_d fails the WLP in degree $d-1$.

Goals

- Hilbert function, Hilbert polynomial and Hilbert series.
- Degree.
- A minimal set of generators of the homogeneous ideal.
- Canonical module, Cohen-Macaulay type and Castelnuovo–Mumford regularity.

A minimal free resolution

We set $c := \operatorname{codim}(X_d) = \mu_d - n - 1$. A minimal graded free $S-$resolution $F_\bullet$ of $A(X_d)$:

$$F_\bullet : \quad 0 \longrightarrow F_c \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow S \longrightarrow A(X_d) \longrightarrow 0,$$

where

$$F_i \cong \bigoplus_{j\geq 1}^{f_i} S(-j-i)^{\beta_{i,j}}$$

and $\beta_{i,f_i} > 0$, $1 \leq i \leq c$.

Hilbert function and Hilbert series

Hilbert function

$$\mathrm{HF}(A(X_d), t) = \dim_{\mathbb{K}}(S/I(X_d))_t = \dim_{\mathbb{K}} R_t^{\overline{G}}.$$

Proposition

For any $t \in \mathbb{Z}_{\geq 0}$, $\mathrm{HF}(A(X_d), t)$ *equals to the number of monomial invariants of* $\overline{G}$ *of degree* td. *Equivalently, it is the number of* $\mathbb{Z}_{\geq 0}^{n+1}$*−solutions of the linear system of congruences:*

$$\begin{cases} y_0 + y_1 + \cdots + y_n = td \\ \alpha^1_{\sigma_1(0)} y_0 + \alpha^1_{\sigma_1(1)} y_1 + \cdots + \alpha^1_{\sigma_1(n)} y_n \equiv 0 \bmod d_1 \\ \vdots \\ \alpha^s_{\sigma_s(0)} y_0 + \alpha^s_{\sigma_s(1)} y_1 + \cdots + \alpha^s_{\sigma_s(n)} y_n \equiv 0 \bmod d_s \end{cases}$$

It provides a method to effectively compute $\mathrm{HF}(A(X_d), t)$ in low dimension.

Hilbert function of $GT-$surfaces

Let $G = \langle M_{d;0,\alpha_1,\alpha_2} \rangle \subset \mathrm{GL}(3, \mathbb{K})$ be a cyclic group of order $d \geq 3$ and X_d a $GT-$surface with group G. We consider integers:

$$\alpha_1' = \frac{\alpha_1}{\mathrm{GCD}(\alpha_1, d)}, \; d' = \frac{d}{\mathrm{GCD}(\alpha_1, d)}$$

μ and $0 < \lambda \leq d'$ such that $\alpha_2 = \lambda \alpha_1' + \mu d'$.

Theorem (–, Mezzetti, Miró-Roig)

Set $\theta(\alpha_1, \alpha_2, d) := (\alpha_1, d) + (\lambda, d') + (\lambda - (\alpha_1, d), d')$. *Then,*

(i) $\mathrm{HF}(X_d, t) = \frac{d}{2} t^2 + \frac{1}{2} \theta(\alpha_1, \alpha_2, d) t + 1.$

(ii) $\mathrm{HS}(X_d, z) = \dfrac{\frac{d - \theta(\alpha_1, \alpha_2, d) + 2}{2} z^2 + \frac{d + \theta(\alpha_1, \alpha_2, d) - 4}{2} z + 1}{(1 - z)^3}.$

From Invariant Theory: Molien series

Hilbert series

$$\mathrm{HS}(A(X_d), z) = \sum_{t \geq 0} \mathrm{HF}(A(X_d), t) z^t = \frac{h_s z^s + \cdots + h_1 z + 1}{(1-z)^{n+1}}.$$

Proposition (An expression only in terms of the group $\overline{G}$)

$\mathrm{HS}(A(X_d), z^d)$ *is the Molien series of* $\overline{G}$*:*

$$\frac{1}{|\overline{G}|} \sum_{g \in \overline{G}} \frac{1}{\det(Id - zg)}.$$

Example

We take $2 \leq n < d$ with d prime and a cyclic group of order d $G = \langle M_{d;\alpha_0,\dots,\alpha_n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ such that $\alpha_0 < \alpha_1 < \cdots < \alpha_n < d$. Expanding the Molien series of $\overline{G}$, we obtain:

$$\mathrm{HF}(A(X_d), t) = \frac{1}{d} \binom{td+n}{n} + \frac{d-1}{d}.$$

Hilbert series and secondary invariants

Proposition (Cohen-Macaulayness of $R^{\overline{G}}$)

$x_0^d, \dots, x_n^d$ is a h.s.o.p of $R^{\overline{G}}$ and $R^{\overline{G}}$ is a free $\mathbb{K}[x_0^d, \dots, x_n^d]-$module with a Hironaka decomposition

$$R^{\overline{G}} \cong \bigoplus_{i=0}^{D} \theta_i \mathbb{K}[x_0^d, \dots, x_n^d],$$

where $\theta_0, \dots, \theta_D$ are all the monomial invariants $x_0^{a_0} \cdots x_n^{a_n}$ of G such that $a_0 < d, \dots, a_n < d$. $\{\theta_0, \dots, \theta_D\}$ is called a **set of secondary invariants of $\overline{G}$.**

Corollary (Hilbert series and secondary invariants)

Let $\delta_0, \dots, \delta_n$ the sequence of multiplicities of degrees of $\theta_0, \theta_1, \dots, \theta_D$, then

$$\mathrm{HS}(A(X_d), z) = \frac{\delta_n z^n + \cdots + \delta_1 z + 1}{(1-z)^{n+1}}.$$

Hilbert series and secondary invariants

Corollary

(i) *Complete description of the h−vector of* $A(X_d)$ *in terms of invariants of* $\overline{G}$.

(ii) *The degree of* X_d *is* $\deg(X_d) = \frac{d^{n+1}}{|\overline{G}|}$.

Example

Take $G = \langle M_{3;0,1,2} \rangle \subset \mathrm{GL}(3, \mathbb{K})$ a cyclic group of order 3.

$$\begin{aligned} \mathcal{B}_1 &= \{x_0^3, x_1^3, x_2^3, x_0x_1x_2\} \\ \mathcal{B}_2 &= \{x_0^6, x_0^3x_1^3, x_0^4x_1x_2, x_1^6, x_0x_1^4x_2, x_0^2x_1^2x_2^2, x_0^3x_2^3, x_1^3x_2^3, x_0x_1x_2^4, x_2^6\}. \end{aligned}$$

x_0^3, x_1^3, x_2^3 is a h.s.o.p of $R^{\overline{G}}$ and $\{x_0x_1x_2, x_0^2x_1^2x_2^2\}$ is a set of secondary invariants of $\overline{G}$. We obtain:

$$\mathrm{HS}(A(X_3), z) = \frac{z^2 + z + 1}{(1 - z)^3}.$$

Homogeneous ideal of $\overline{G}$−varieties.

$\mathcal{B}_1 = \{m_1, \ldots, m_{\mu_d}\}$, $S = \mathbb{K}[w_1, \ldots, w_{\mu_d}]$.

The homogeneous ideal of X_d

$\mathrm{I}(X_d)$ is the binomial prime ideal kernel of the morphism

$$\rho : S \to \mathbb{K}[m_1, \ldots, m_{\mu_d}], \quad \rho(w_i) = m_i.$$

It is generated by binomials of degree at least 2:

$$\left\{ \prod_{i=1}^{\mu_{n,d}} w_i^{\alpha_i} - \prod_{i=1}^{\mu_{n,d}} w_i^{\beta_i} \;\middle|\; \prod_{i=1}^{\mu_{n,d}} m_i^{\alpha_i} = \prod_{i=1}^{\mu_{n,d}} m_i^{\beta_i}, \alpha_i, \beta_i \in \mathbb{Z}_{\geq 0} \right\}.$$

Main objectives

(i) To determine a minimal set of binomial generators of $\mathrm{I}(X_d)$.

(ii) To characterize such generators.

(iii) Which is the maximum degree $k \geq 2$ needed to minimally generate $\mathrm{I}(X_d)$?

The homogeneous ideal of $\overline{G}-$varieties

For each integer $k \geq 2$, $I(X_d)_k$ denotes the set of binomials in $\mathrm{I}(X_d)$ of degree k, which we call *$k-$binomials.*

$$I(X_d) = \sum_{k \geq 2} (I(X_d)_k).$$

Theorem

$\mathrm{I}(X_d)$ *is generated by binomials of degree at most* 3, *i.e.*

$$\mathrm{I}(X_d) = (\mathrm{I}(X_d)_2, \mathrm{I}(X_d)_3).$$

Sharpness: there are $\overline{G}-$varieties in any dimension with $I(X_d)$ minimally generated by 2 and $3-$binomials.

The canonical module

The normal simplicial affine semigroup of X_d

$$A(X_d) \cong R^{\overline{G}} = \mathbb{K}[\mathcal{B}_1] = \mathbb{K}[H_{\mathcal{A}}],$$

where $H_{\mathcal{A}}$ is the normal simplicial affine semigroup of the $\mathbb{Z}_{\geq 0}^{n+1}$−solutions of the system of congruences associated to $\overline{G}$.

Relative interior ideal

We denote by $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ the ideal of $R^{\overline{G}}$ generated by all monomial invariants $x_0^{a_0} \cdots x_n^{a_n}$ of $\overline{G}$ such that $a_0 > 0, \ldots, a_n > 0$, i.e. $(a_0, \ldots, a_n)$ belongs to the relative interior of the affine semigroup $H_{\mathcal{A}} \subset \mathbb{Z}_{\geq 0}^{n+1}$.

Theorem (Danilov-Stanley)

$\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ *is the canonical module of* $\mathbb{K}[H_{\mathcal{A}}]$.

The canonical module

For each integer $k \geq 1$, $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_k$ denotes the set of monomials of degree kd in $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = (x_0^{a_0} \cdots x_n^{a_n} \in R^{\overline{G}} \mid a_0 \cdots a_n \neq 0)$.

Theorem

$\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ *is generated by monomial invariants of* G *of degree* d *and* $2d$*, i.e.*

$$\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = (\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1, \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_2).$$

- To determine when a monomial $m \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})_2$ belongs to $(\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$ depends strongly on the group G.

The canonical module

- $A(X_d)$ is a *Gorenstein* ring if $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ is principal.
- $A(X_d)$ is a *level* ring if $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = (\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1)$ or $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1 = \emptyset$.

Example

(i) Take $n \geq 2$ an even integer and $G = \langle M_{n+1;0,1,2,\ldots,n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ a cyclic group of order $n+1$. Then $R^{\overline{G}}$ is a Gorenstein ring with

$$\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = (x_0 \cdots x_n).$$

(ii) Now we take $k \geq 2$ an integer and $G_k = \langle M_{k(n+1);0,1,2,\ldots,n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ a cyclic group of order $n+1$. Then, $R^{\overline{G_k}}$ is a level ring with

$$\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = (\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1).$$

The Castelnuovo–Mumford regularity

(i) $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_2 \neq \emptyset$. This assures the existence of secondary invariants of degree $(n-1)d$, i.e. $\delta_{n-1} > 0$.

(ii) The monomials in $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$ uniquely determine the secondary invariants of $\overline{G}$ of degree nd and vice versa.

$$\mathrm{HS}(A(X_d), z) = \frac{\delta_n z^n + \delta_{n-1} z^{n-1} + \cdots + \delta_1 z + 1}{(1-z)^{n+1}}.$$

Theorem

For the Castelnuovo–Mumford regularity $\mathrm{reg}(A(X_d))$ *of* $A(X_d)$ *it holds:*

$$n \leq \mathrm{reg}(A(X_d)) \leq n+1$$

with equality $\mathrm{reg}(A(X_d)) = n+1$ *if and only if* $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1 \neq \emptyset$.

The Betti table

	0	1	2	3	$\cdots$	$c-2$	$c-1$	c
0	1	–	–	–	$\cdots$	–	–	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{c-2,1}$	$\beta_{c-1,1}$	–
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{c-2,2}$	$\beta_{c-1,2}$	–
3	–	–	$\beta_{2,3}$	$\beta_{3,3}$	$\cdots$	$\beta_{c-2,3}$	$\beta_{c-1,3}$	–
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	–	–	$\beta_{2,n-1}$	$\beta_{3,n-1}$	$\cdots$	$\beta_{c-2,n-1}$	$\beta_{c-1,n-1}$	$\beta_{c,n-1}$
n	–	–	$\beta_{2,n}$	$\beta_{3,n}$	$\cdots$	$\beta_{c-2,n}$	$\beta_{c-1,n}$	$\beta_{c,n}$

Thanks for your attention!