Yano's conjecture

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Bernstein-Sato polynomial

Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ a non-constant polynomial. There exists a differential operator $P(s) \in D_{\mathbb{C}^n} \otimes \mathbb{C}[s]$ and $b_{f,P}(s) \in \mathbb{C}[s]$ s.t.

$$P(s) \cdot f^{s+1} = b_{f,P}(s) f^s \qquad (*)$$

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The monic generator of the ideal in $\mathbb{C}[s]$ generated by $b_{f,P}(s)$ fulfilling $(*)$ is the Bernstein-Sato polynomial $b_f(s)$ of f.

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Theorem (Kashiwara '76, Lichtin '89)

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Similarly for the local case: $f \in \mathbb{C}\{x_0, \dots, x_n\}$, then $\exists\, b_{f,\mathbf{0}}(s)$.

Brieskorn lattice

Let $f : (\mathbb{C}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ defining an isolated singularity.

The Brieskorn lattice: $''H^n := \Omega^{n+1}_{X,\mathbf{0}} / (\mathrm{d}f \wedge \mathrm{d}\Omega^{n-1}_{X,\mathbf{0}}) \circlearrowleft \partial_t$ free $\mathbb{C}\{t\}$-module of rank μ (Milnor number).

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Theorem (Malgrange '75)

The reduced Bernstein-Sato poly. $\tilde{b}_{f,\mathbf{0}}(s) := b_{f,\mathbf{0}}(s)/(s+1)$ of f is the minimal polynomial of the endomorphism

$$-\overline{\partial_t t} : {''\widetilde{H}^n} / t\, {''\widetilde{H}^n} \longrightarrow {''\widetilde{H}^n} / t\, {''\widetilde{H}^n} \quad (**),$$

where ${''\widetilde{H}^n} := \sum_{k \geq 0} (\partial_t t)^k ({''H^n})$ is the saturation of ${''H^n}$.

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The b-exponents: roots of the characteristic polynomial of the endomorphism $\overline{\partial_t t}$ $(**)$.

Motivating example

Let $f = y^4 - x^9, \mu = 24$. The roots of $b_{f,\mathbf{0}}(s)$ are $\left\{ -\frac{4\alpha+9\beta}{36} \mid {}^{0<\alpha<9,}_{0<\beta<4} \right\}$.

$$-\frac{13}{36}, -\frac{17}{36}, \ldots, -\frac{50}{36}, -\frac{54}{36}, -\frac{55}{36}, -\frac{59}{36}$$

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Consider $f_{\mathbf{t}}$, μ-constant deformations of f:

- $f_{\mathbf{t}} = f + t_1 x^7 y^2 : -\frac{59}{36} \longrightarrow -\frac{59}{36} - 1 = -\frac{23}{36}, \quad \text{if} \quad t_1 \neq 0.$

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- $f_{\mathbf{t}} = f + t_2 x^6 y^2 + t_1 x^7 y^2 : -\frac{23}{36}, -\frac{55}{36} \longrightarrow -\frac{19}{36}, \quad \text{if} \quad t_2 \neq 0.$

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- $f_{\mathbf{t}} = f + t_4 x^7 y + t_3 x^5 y^2 : -\frac{23}{36}, -\frac{19}{36}, -\frac{50}{36} \longrightarrow -\frac{14}{36}, \quad \text{if} \quad t_3 + 3t_4^2 \neq 0.$

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Finally, if $f_{\mathbf{t}} = f + t_4 x^7 y + t_3 x^5 y^2 + t_2 x^6 y^2 + t_1 x^7 y^2$, in a Zariski open set of the base of the deformation:

- The roots of $b_{f_t}(s)$ are between $-\mathrm{lct}(f) = -\frac{13}{36}$ and $-\frac{13}{36} - 1$.

Yano's conjecture

Let $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a plane branch with characteristic sequence $(n; \beta_0, \ldots, \beta_1)$. Define $e_i := \gcd(n, \beta_1, \ldots, \beta_i)$,

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$$r_i := \frac{\beta_i + n}{e_i}, \quad R_i := \frac{\beta_i e_{i-1} + \beta_{i-1}(e_{i-2} - e_{i-1}) + \cdots + \beta_1(e_0 - e_1)}{e_i},$$

$$r'_0 := 2, \quad r'_i := r_{i-1} + \left\lfloor \frac{\beta_i - \beta_{i-1}}{e_{i-1}} \right\rfloor + 1 = \left\lfloor \frac{r_i e_i}{e_{i-1}} \right\rfloor + 1,$$

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Consider:

$$R\big((n, \beta_1, \ldots, \beta_g), t\big) := \sum_{i=1}^{g} t^{\frac{r_i}{R_i}} \frac{1 - t}{1 - t^{\frac{1}{R_i}}} - \sum_{i=0}^{g} t^{\frac{r'_i}{R'_i}} \frac{1 - t}{1 - t^{\frac{1}{R'_i}}} + t,$$

Yano's conjecture (II)

Conjecture (Yano ('82))

For generic curves in some μ-constant deformation of f, the b-exponents $\{\alpha_1, \alpha_2, \ldots, \alpha_\mu\}$ are

$$\sum_{i=1}^{\mu} t^{\alpha_i} = R\big((n, \beta_1, \ldots, \beta_g), t\big).$$

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- $K_\pi = \mathrm{Div}(\pi^* \mathrm{d}x \wedge \mathrm{d}y) = \sum_{i=1}^{m} k_i E_i$ is the canonical divisor.

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Alternatively, if $E_i, i = 1, \ldots, g$ rupture divisors of f:

$$\{\alpha_1, \ldots, \alpha_\mu\} = \bigcup_{i=1}^{g} \left\{ \sigma_{i,\nu} = \frac{k_i + 1 + \nu}{N_i} \,\middle|\, 0 < \nu < N_i, N_i^{(j)} \sigma_{i,\nu} \notin \mathbb{Z}, j = 1, 2, 3 \right\}$$

where $E_i^{(j)} \in \mathrm{Supp}(F_\pi)$ crossing E_i (i.e., $E_i^{(j)} \cap E_i \neq \varnothing$).

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- Using analytic continuation of the complex zeta function: We give a prove for any $g > 0$ & assuming that the monodromy eigenvalues are pair-wise different.
- We give a prove for the general case using periods of integrals in the Milnor fiber (solutions of the Gauss-Manin connection).

Asymptotic expansion of periods of integrals

Let $f : (\mathbb{C}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function defining an isolated singularity. For $0 < \delta \ll \epsilon \ll 1$,

$$X := B_\epsilon \cap f^{-1}(T), \quad X' := X \smallsetminus f^{-1}(0), \quad X_t := B_\epsilon \cap f^{-1}(t), \quad t \in T_\delta.$$

where X_t is the Milnor fiber and $\widetilde{H}^n(X_t, \mathbb{C}) = \mathbb{C}^\mu$ and zero otherwise.

Milnor fiber

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(Co)homological Milnor fibration:

$$f^* : H^n := \bigcup_{t \in T^*} H^n(X_t, \mathbb{C}) \longrightarrow T'$$

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Gauss-Manin connection: $\nabla^n : \mathcal{H}^n \longrightarrow \Omega^1_{T'} \otimes_{T'} \mathcal{H}^n.$

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Since the Gauss-Manin connection has regular singularities (Brieskorn '70), for any $\omega \in \Gamma(X, \Omega_X^{n+1})$ (Malgrange '74)

$$\int_{\gamma(t)} \frac{\omega}{\mathrm{d}f} = \sum_{\lambda \in \Lambda} \sum_{\alpha \in L(\lambda)} \sum_{0 \le k \le n} a_{\alpha-1,k} t^{\alpha-1} (\ln t)^k,$$

Λ eigenvalues monodromy, and $L(\lambda) := \mathbb{Q}_{\ge 0} \cap (2\pi i)^{-1} \log \Lambda$.

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  for $\omega \in \Gamma(X, \Omega^{n+1}_X)$.
- After a result of Varchenko ('80), having $A^{\omega}_{\alpha,k} \neq 0$ and α in $(0,1)$ implies that $\alpha+1$ is a b-exponent.

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$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{\rho} & \overline{X} & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle \pi^* f} & & \downarrow{\scriptstyle f} \\ \widetilde{T} & \xrightarrow{\sigma} & T & = & T. \end{array}$$

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- Hence, $\widetilde{\omega}/\mathrm{d}\tilde{f}$ is well-defined on $\widetilde{E}_i^\circ$, where $\widetilde{\omega} := (\pi\rho)^*\omega$.
- However, $\widetilde{X}$ is an orbifold (mild singularities).

Asymptotic expansion

Locally around E_i, $\overline{\omega} = \pi^*\omega = \overline{\omega}_0 + \overline{\omega}_1 + \cdots + \overline{\omega}_\nu + \cdots$

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Then, one shows that

$$\int_{\gamma(t)} \frac{\omega}{\mathrm{d}f} = \sum_{\nu \geq 0} t^{\sigma_{i,\nu}(\omega)-1} \int_{\tilde{\gamma}(\tilde{t})} R_{i,\nu}(\omega),$$

where $R_{i,\nu}(\omega)$ extends to a multivalued form on E_i° and

$$\sigma_{i,\nu}(\omega) = \frac{v_i(\omega_\nu) + 1}{N_i} = \frac{k_i + 1 + \nu}{N_i}.$$

Asymptotic expansion (II)

Then, as long as the integral is finite, $A^{\omega}_{\sigma_{i,\nu}-1,0}(t)$ is

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On the other hand, the form $R_{i,\nu}(\omega)$ defines a $\mathbb{Q}$-divisors on E_i°

$$\mathrm{Div}(R_{i,\nu}(\omega)) = \sum_j \epsilon_{j,\nu}(\omega) D_{i,j} + \sum_k \delta_{k,\nu} x_k$$

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Lemma

For any plane curve singularity,

$$\sum_j \epsilon_{i,\nu}(\omega) + \sum_k \delta_{k,\nu}(\omega) + \nu E_i^2 = -2.$$

- Since for plane curves $E_i \cong \mathbb{P}^1_{\mathbb{C}}$, $R_{i,\nu}(\omega)$ defines a multivalued form on $E_i^{\circ} \cong \mathbb{P}^1_{\mathbb{C}} \setminus \{s_1, \ldots, s_r\}$.

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Proposition (Deligne-Mostow '86)

Let $\omega \in \Gamma(\mathbb{P}^1_{\mathbb{C}}, \Omega^1(-\sum \epsilon_{j,\nu}(\omega)s_j - \sum \delta_{k,\nu}(\omega)x_k)(L))$. Assume that $\sum_{s\in S} \epsilon_{j,\nu}(\omega) \geq r-1$ and that $\epsilon_{j,\nu}(\omega) \notin \mathbb{Z}$ for all $s \in S$. Then, ω defines a non-zero cohomology class in $H^1(\mathbb{P}^1_{\mathbb{C}} \setminus S, L)$.

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Proposition (B.)

Let E_i be a rupture divisor of the minimal resolution of f with divisorial valuation v_i. Then, for any $v > N_i$ there exists a one-parameter μ-constant deformation of f of the form $f + tg_t$ such that $v_i(g_t) = v$, for all values of the parameter t.

Overview of the proof

Recall, Yano's candidates:

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- Since $0 \leq \nu < N_i$, by Varchenko's theorem, $\sigma_{i,\nu}$ is a b-exponent of the generic fibers.
- Generalizing a result of Varchenko: the b-exponents are semicontinuous under μ-constant deformations. Hence, we can apply this argument to all the candidates.

Yano’s conjecture

Theorem (B.)

For any irreducible plane curve singularity, Yano’s conjecture holds true.

References

- B., *Yano's conjecture*, Invent. Math., **226** (2021), 421-465.

Thank you!