On the difference between Milnor number and Tjurina number of isolated singularities

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- The Milnor number is defined as

$$\mu := \dim_{\mathbb{C}} M_f.$$

It is a topological invariant of the singularity

Toy example

Let us consider the curve $f(x,y) = y^7 - x^9 = 0$. In this case $M_f = T_f$, by using SINGULAR we calculate $\mu = \tau = 48$ and a basis for this algebra:

$$\{x^7y^5, x^6y^5, x^5y5, x^4y^5, x^3y^5, x^2y^5, xy^5, y^5, x^7y^4, x^6y^4, x^5y^4, x^4y^4, x^3y^4, x^2y^4, xy^4, y^4, x^7y^3, x^6y^3, x^5y^3, x^4y^3, x^3y^3, x^2y^3, xy^3, y^3, x^7y^2, x^6y^2, x^5y^2, x^4y^2, x^3y^2, x^2y^2, xy^2, y^2, x^7y, x^6y, x^5y, x^4y, x^3y, x^2y, xy, y, x^7, x^6, x^5, x^4, x^3, x^2, x, 1\}$$

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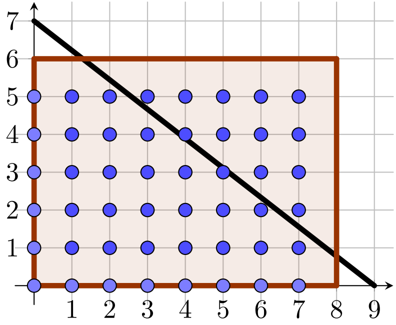

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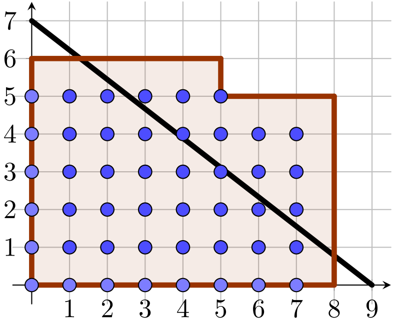
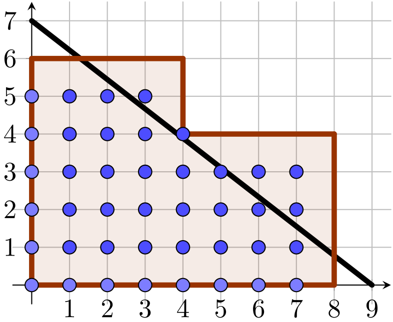


$$f(x,y) = y^7 - x^9 + x^5y^5$$
$$\mu = 48$$
$$\tau = 45$$



$$f(x,y) = y^7 - x^9 + x^4y^4$$
$$\mu = 48$$
$$\tau = 40$$

Topology vs Analysis

Definition

Two isolated hypersurface singularities defined by f and g have the same topological type if there is a homeomorphism $\varphi : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ such that $\varphi(V_f) = V_g$.

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Theorem (Lê 1973)

If two isolated hypersurface singularities defined by f and g have the same topological type then $\mu(f) = \mu(g)$.

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Theorem (Mather-Yau 1982)

The hypersurface isolated singularities defined by f and g are analytically equivalent if and only if their Tjurina algebras are isomorphic as $\mathbb{C}$–algebras.

In particular, same analytic type $\Rightarrow \ \tau(f) = \tau(g)$.

(a) John Milnor 1931-

(b) Galina N. Tjurina 1938-1970

Comparing μ and τ

Theorem (K. Saito 1971)

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a germ of analytic function defining an isolated hypersurface singularity at the origin. Then

$$\mu = \tau \Leftrightarrow \text{ f is quasihomogeneous}$$

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Theorem (Liu 2017)

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a germ of analytic function defining an isolated hypersurface singularity at the origin. Then

$$\frac{\mu}{\tau} \leq n$$

Dimca and Greuel Question

Example (Dimca-Greuel)

Consider the families of curves

$$X_a : x^{2a+1} + x^a y^{a+1} + y^{2a} = 0, \quad X_b : x^{2b+1} + x^{b+1} y^{b+1} + y^{2b+1} = 0.$$

For those families $\tau(X_a) = 3a^2$, $\mu(X_a) = 2a(2a-1)$, $\mu(X_b) = 4b^2$, $\tau(X_b) = 4b^2 - (b-1)^2$. Therefore, it follows that

$$\mu/\tau \xrightarrow[a\to\infty]{} 4/3. \quad \mu/\tau \xrightarrow[b\to\infty]{} 4/3$$

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Conjecture (Dimca-Greuel 2017)

Is for any plane curve singularity $\frac{\mu}{\tau} < \frac{4}{3}$?

Dimca and Greuel Question

Proposition (A.-Blanco, 2018)

f is said to be semi-quasi-homogeneous with weight $w = (n, m)$ such that $\gcd(n, m) \geq 1$, $n, m \geq 2$ and $f = y^n - x^m + h.d.t.$ Then for any semi-quasi-homogeneous plane curve singularity $\mu/\tau < 4/3$.

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Theorem (Alberich-A.-Blanco-Melle; Genzmer-Hernandes 2019)

For any equisingular class of germs of irreducible plane curve singularity,

$$\tau_{min} = \sigma(n) + \frac{n^2 + 3n - 6}{2} + \sum_{p \text{ free}} \frac{(e_p - 1)(e_p + 2) + 2\sigma(e_p + 1)}{2} + \sum_{p \text{ sat.}} \frac{e_p(e_p - 1) + 2\sigma(e_p + 2)}{2},$$

where the summation runs on all points p equal or infinitely near to the origin and $\sigma(k) = \frac{(k-2)(k-4)}{4}$ if k is even and $\sigma(k) = \frac{(k-3)^2}{4}$ if k is odd.

Dimca and Greuel question for plane branches

Corollary

For any plane branch singularity,

$$\frac{\mu}{\tau} < \frac{4}{3}.$$

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Problems of the τ_{min} approach:

- Too restrictive: The results to prove the formula do not work for non irreducible curves.
- Too hard: We may find a formula for τ_{min} and not to be able to estimate it.

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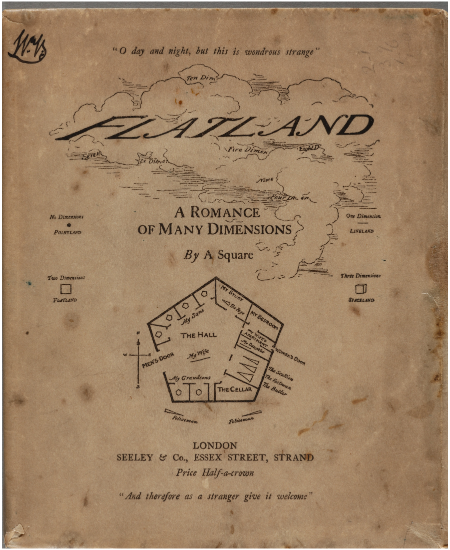
- Too hard: We may find a formula for τ_{min} and not to be able to estimate it.
- Why $4/3$?



...The little Hexagon meditated on this a while and then said to me; "But you have been teaching me to raise numbers to the third power: I suppose three-to-the-third must mean something in Geometry; what does it mean?" "Nothing at all", replied I, "not at least in Geometry; for Geometry has only Two Dimensions"....

"Flatland, A Romance of Many Dimensions" by Edwin Abbott.

Milnor and Tjurina numbers for surface singularities

Let $(X, 0) \in (\mathbb{C}^3, 0)$ be an isolated surface singularity defined by an equation $f \in \mathcal{O}_{\mathbb{C}^3,0}$. Let $\widetilde{X} \to X$ be a resolution of singularities of X.

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Definition

A holomorphic 2–form in $\mathbb{C}\{x, y, z\}$ is defined as

$$\omega = a(x, y, z)dx \wedge dy + b(x, y, z)dy \wedge dz + c(x, y, z)dx \wedge dz$$

Definition

A holomorphic form ω on $U' = X \setminus \{0\}$ is called of *first kind* if there exists a resolution $\pi : \widetilde{X} \to X$ of the singularity X such that $\pi^*(\omega)$ extends holomorphically to $\widetilde{X}$

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The geometric genus

$$p_g := \dim \frac{\{\text{Holomorphic } 2\text{–forms on } U\}}{\{2\text{–forms of first kind}\}}.$$

DG Question from other perspective

Consider the germ of isolated surface singularity

$$\Sigma : z^2 + f(x, y) = 0.$$

Remark: $C : f(x, y) = 0$ then $\tau(C) = \tau(\Sigma)$ and $\mu(C) = \mu(\Sigma)$.

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Theorem (Tomari 1991)

Let p_g be the geometric genus of Σ and μ its Milnor number. Then

$$8p_g + 1 \leq \mu.$$

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Theorem (Wahl 1985)

Let $(X,0) \in (\mathbb{C}^3,0)$ be an isolated surface singularity defined by an equation $f \in \mathcal{O}_{\mathbb{C}^3,0}$. Then

$$\mu - \tau \leq 2p_g$$

DG Question from other perspective

Theorem (A. 2019)

For any germ of plane curve singularity

$$\frac{\mu}{\tau} < \frac{4}{3}.$$

Proof: Let $f(x,y) = 0$ be an equation of a germ of a plane curve singularity. Consider the surface singularity $f(x,y) + z^2 = 0$. Then, Wahl $+$ Tomari give

$$\mu - \tau \leq 2p_g < \mu/4 \quad \square$$

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Consequence: The bound $4/3$ can be inferred from the geometry of the singularity.

Durfee conjecture and the bound for surface singularities

Is $\mu/\tau < 4/3$ for any surface singularity?

Durfee conjecture and the bound for surface singularities

Is $\mu/\tau < 4/3$ for any surface singularity? NO

$$f = x^{14} + y^6 z^8 + z^{14} + x^9 z^5 + (x + y + z)^{15}.$$

We can compute with SINGULAR that the Milnor number is $\mu = 2288$ and the Tjurina number is $\tau = 1660$. Therefore, $\mu/\tau > 4/3$.

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Conjecture (Durfee 1978)

For any isolated surface singularity $(X, 0) \subset (\mathbb{C}^3, 0)$

$$6p_g \leq \mu.$$

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Some partial results by: Tomari (91), Ashikaga (93), Némethi (98), Melle-Hernández (2000), Kóllar and Némethi (2017), Enokizono (2018). Still open

Bound for surface singularities

Proposition

Let $(X, 0) \subset (\mathbb{C}^3, 0)$ *be an isolated surface singularity of one of the following types:*

(1) *Quasi-homogeneous singularities,*

(2) $(X, 0)$ *of multiplicity* 3,

(3) *absolutely isolated singularity,*

(4) *suspension of the type* $\{f(x, y) + z^N = 0\}$,

(5) *the link of the singularity is an integral homology sphere,*

(6) *the topological Euler characteristic of the exceptional divisor of the minimal resolution is positive.*

Then

$$\frac{\mu}{\tau} < \frac{3}{2}$$

Remark: All these cases are the cases for which Durfee conjecture is known to be true.

Bound for surface singularities

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$$\tau_{min} = (2d-3)(d+1)(d-1)/3.$$

Also, $\mu = (d-1)^3$. Then

$$\frac{\mu}{\tau_{\min}} \xrightarrow[d\to\infty]{} \frac{3}{2}.$$

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Conjecture (A. 2019)

For any $(X,0) \subset (\mathbb{C}^3,0)$ *isolated surface singularity:*

$$\frac{\mu}{\tau} < \frac{3}{2}.$$

Exponents

$f : \mathbb{C}^{n+1} \to \mathbb{C}$ germ of isolated hypersurface singularity, $t \in \mathbb{C}$.
The *Brieskorn lattice* is defined as

$$H_0'' := \frac{\Omega^{n+1}_{\mathbb{C}^{n+1},0}}{df \wedge d\Omega^{n-1}_{\mathbb{C}^{n+1},0}}$$

In the Brieskorn lattice there is an action of the differential operator ∂_t^{-1} defined as

$$\partial_t^{-1}[\omega] := [df \wedge \alpha],$$

where $\omega \in \Omega^{n+1}_{\mathbb{C}^{n+1},0}$ and $\alpha \in \Omega^{n}_{\mathbb{C}^{n+1},0}$ such that $d\alpha = \omega$. Also, $t\omega := f\omega$.

Proposition (Pham 70's)

H_0'' is a $\mathbb{C}\{\{\partial_t^{-1}\}\}$–module

Exponents

Theorem (M. Saito 1989)

There exists a basis $\{v_i\}$ *of* H_0'' *as* $\mathbb{C}\{\{\partial_t^{-1}\}\}$*–module and matrices with complex coefficient* A_0, A_1 *such that*

$$tv = A_0 v + A_1 \partial_t^{-1} v$$

where $v = (v_1, \ldots, v_\mu)^t$*. Moreover,* A_0 *is nilpotent and* A_1 *is semisimple.*

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Definition

The *exponents* of f are defined as the set of eigenvalues of the matrix A_1.

Quasi-homogeneous singularities

$f(x_0, \ldots, x_n)$ quasi-homogeneous of degree 1 with respect to the weigths $(w_0, \ldots, w_n)$, i.e. $f(\eta^{w_0} x_0, \ldots, \eta^{w_n} x_n) = \eta f(x_0, \ldots, x_n)$.

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Consider the generating function

$$\mathrm{Sp}_f(t) := \sum_{i=1}^{\mu} t^{\alpha_i} \in \mathbb{Z}[\mathbb{Q}]$$

with $\alpha_1, \ldots, \alpha_\mu$ the exponents of f.
In this case we can compute $\mathrm{Sp}_f(t)$ as

$$\mathrm{Sp}_f(t) = \frac{\prod_{i=0}^{n}(t - t^{w_i})}{\prod_{i=0}^{n}(t^{w_i} - 1)}.$$

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If we do $t=\exp(\pi i\tau)$ and $\chi_f=(1/\mu)\,\mathrm{Sp}_f(t)$,

$$\lim_{w_0,\dots,w_n\to 0}\chi_f(t) = \left(\exp(\pi i\tau)\frac{\sin(\pi\tau)}{\pi\tau}\right)^{n+1}$$

A continuous distribution

$$N_{n+1}(s)ds := \int_{x_0+\cdots x_n=s} \varphi(x_0)\cdots\varphi(x_n)dx_0\cdots dx_n,$$

where φ is the indicator function of the unit interval $[0,1]$,

$$\varphi(x) := \begin{cases} 1 & \text{if } x \in [0,1], \\ 0 & \text{if } x \notin [0,1]. \end{cases}$$

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The Fourier transform $\mathcal{F}(N_{n+1}(\tau))$ of $N_{n+1}(s)$ is:

$$\int \exp(2\pi i\tau s)N_{n+1}(s)ds = \mathcal{F}(\varphi)(\tau)^{n+1} = \left(\exp(\pi i\tau)\frac{\sin(\pi\tau)}{\pi\tau}\right)^{n+1}.$$

K. Saito's problem

The normalized spectrum of f,

$$\chi_f(t) := \frac{\mathrm{Sp}_f(T)}{\mu} = \frac{1}{\mu}\sum_{j=1}^{\mu} T^{\alpha_j},$$

Making $T = \exp(2\pi i t)$, one gets the Fourier transform representation:

$$\chi_f(t) := \frac{1}{\mu}\int \exp(2\pi i s\tau)\sum_{i=1}^{\mu}\delta(s-\alpha_i)ds,$$

where $\delta(s)$ is Dirac's delta function.

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Question (K. Saito 1983)

Let $\alpha_1, \dots, \alpha_\mu$ be the spectral values of an isolated hypersurface singularity, is

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$$\lim \chi_f = \mathcal{F}(N_{n+1}), \text{ or equivalently, } \lim \sum_{i=1}^{\mu} \delta(s-\alpha_i)ds = N_{n+1}ds \quad ?$$

Theorem (K. Saito 1983)

1. *For quasi-homogeneous singularities of degree with weights* $(w_0, \dots, w_n)$*:* $\lim_{w_0,\dots,w_n \to 0} \chi_f = \mathcal{F}(N_{n+1})$.
2. *For an irreducible plane curve singularity with Puiseux pairs* $(n_1, l_1), \dots, (n_g, l_g)$, $\lim_{n_g \to \infty} \chi_f = \mathcal{F}(N_2)$.

Theorem (A.-Schulze 2020)

For a fixed Newton diagram Γ*, consider the Newton diagrams* $\varpi\Gamma$ *obtained from* Γ *by scaling with the factor* ϖ*. Then we have* $\lim_{\varpi\to\infty} \chi_{f_\varpi} = \mathcal{F}(N_{n+1})$*, where the limit runs over all Newton non-degenerate* f_ϖ *of* $n+1$ *variables with Newton diagram* $\varpi\Gamma$*.*

Dominating values

Consider the function

$$\Phi_f : [0,1] \to \mathbb{R}, \quad r \mapsto \int_0^r \Big(N_{n+1}(s) - \frac{1}{\mu} \sum_{i=1}^{\mu} \delta(s - \alpha_i) \Big) ds$$

By definition $0 < r < \frac{n+1}{2}$ is a *dominating value* if $\Phi_f(r) > 0$ for all f in $n+1$ variables. A *weakly dominating value* is defined by replacing $<$ by $\leq$ and $\int_0^r$ by $\int_0^{r-\epsilon}$ for all $\epsilon > 0$.

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Problem (K. Saito 1983)

1. *Determine the set of all dominating values and weakly dominating values for each* n.
2. *Is* $1/2$ *a dominating value for all* $n \geq 1$*?*
3. *Is* 1 *a dominating value for all* $n \geq 2$*?*

Related problems

The geometric genus

$$p_g := \dim \frac{\{\text{Holomorphic } n\text{–forms on } U\}}{\{n\text{–forms of first kind}\}}.$$

Theorem (M. Saito 1983)

Let $\{\alpha_1, \dots, \alpha_\mu\}$ *be the exponents of* f*. Then,* $p_g = |\{i \mid \alpha_i \leq 1\}|$.

Question (K. Saito 1983)

Is 1 a dominating value for all $n \geq 2$? In other words, for is f in $n+1$ variables, is the geometric genus bounded by

$$p_g < \frac{\mu}{(n+1)!}?$$

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$$\mu = \dim \frac{H_0''}{\partial_t^{-1} H_0''} \qquad \tau = \dim \frac{H_0''}{tH_0'' + \partial_t^{-1} H_0''} \qquad \mu - \tau = \dim \frac{tH_0'' + \partial_t^{-1} H_0''}{\partial_t^{-1} H_0''}$$

Open problem

Definition

For isolated complete intersection singularities defined by an ideal $\mathcal{I} = (f_1, \ldots, f_k)$ of dimension $n = N - k$:

$$\mu := \operatorname{rk} H_n(F), \quad \tau := \dim_{\mathbb{C}}(\operatorname{Ext}^1_{\mathcal{O}_{(X,0)}}(\Omega^1_{(X,0)}, \mathcal{O}_{(X,0)})).$$

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Theorem (($\Leftarrow$) Greuel 1980– ($\Rightarrow$) Vosegaard 2002)

If (X, x) is an isolated complete intersection singularity of dimension $n \geq 1$,

$$\mu = \tau \Leftrightarrow \ (X, x) \ \textit{is quasihomogeneous.}$$

Problem

Problem

Let $(X,0) \subset (\mathbb{C}^N, 0)$ be an isolated complete intersection singularity of dimension $n \geq 1$ and codimension $r = N - n$. Is there exist an optimal $\frac{b}{a} \in \mathbb{Q}$ with $b < a$ such that

$$\mu - \tau < \frac{b}{a}\mu \text{ ?}$$

Where optimal means that there exist a family of singularities such that μ/τ tends to $\frac{a}{a-b}$ when the multiplicity at the origin tends to infinity.

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- Case $N = 3$ and $r = 1$ Durfee $\Rightarrow\ a = 3,\ b = 1$

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Thanks for the attention!!