Generalized Geometry, an introduction
Assignment 3

Universitat Autònoma de Barcelona
Summer course, 8-19 July 2019

Problem 1. Let T be the reversing operator on $\wedge^\bullet V^*$: for $\alpha_j \in V^*$,

$$(\alpha_1 \wedge \ldots \wedge \alpha_r)^T = \alpha_r \wedge \ldots \wedge \alpha_1.$$

We define a pairing $(\cdot, \cdot)$ on $\wedge^\bullet V^*$ with values on $\det V^* = \wedge^{top} V^*$ by

$$(\varphi, \psi) = (\varphi^T \wedge \psi)_{top},$$

where $\varphi, \psi \in \wedge^\bullet V^*$ and *top* denotes the top degree component.

- For $v \in V + V^*$, prove $(v \cdot \varphi, \psi) = (\varphi, v \cdot \psi)$.
- For $x \in \mathrm{Cl}(V + V^*)$, prove $(x \cdot \varphi, \psi) = (\varphi, x^T \cdot \psi)$.
- For $g \in \mathrm{Spin}(V + V^*)$, $(g \cdot \varphi, g \cdot \psi) = \pm(\varphi, \psi)$.

Problem 2. Let $L = \mathrm{Ann}(\varphi)$ a maximally isotropic subspace

- Prove that $L \cap V = \{0\}$ if and only if $\varphi_{top} \neq 0$.
- Prove that $L \cap L(E', 0) = \{0\}$ if and only if $(\varphi, vol_{\mathrm{Ann}\, E'}) \neq 0$.
- For $L' = \mathrm{Ann}(\psi)$ be another maximally isotropic subspace, prove that $L \cap L' = \{0\}$ if and only if $(\varphi, \psi) \neq 0$.
- Do we need L and L' to be maximally isotropic subspace for the previous statement to be true?

Problem 3. We have defined $\mathrm{GL}(n, \mathbb{C})$ as a subgroup of $\mathrm{GL}(2n, \mathbb{R})$, but there is another possible, and more intuitive, interpretation, as invertible matrices with complex entries. For instance, the elements of $\mathrm{GL}(1, \mathbb{C})$ are just non-zero complex numbers.

- What is the matrix in $\mathrm{GL}(2, \mathbb{R})$ corresponding to $z = a + ib \in \mathrm{GL}(1, \mathbb{C})$?
- What can you say in general for $\mathrm{GL}(n, \mathbb{C})$?

On the other hand, we saw, in term of matrices, that

$$\mathrm{O}(2n) \cap \mathrm{Sp}(2n) = \mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C}). \tag{1}$$

We shall show that this is actually the unitary group $\mathrm{U}(n)$, whose definition we recall. First, a hermitian metric on a complex vector space V is a map $h : V \times V \to \mathbb{C}$ that is $\mathbb{C}$-linear on the first component and satisfies $h(v, u) = \overline{h(u, v)}$, which implies that is anti-linear on the second component. The usual example is $h(u, v) = u^T \overline{v}$. Define

$$\mathrm{U}(n) := \{M \in \mathrm{GL}(n, \mathbb{C}) \mid h(Mu, Mv) = h(u, v), \text{ for } u, v \in \mathbb{C}^n\}.$$

- Prove that this group equals the intersections in (1).
- Try to make statement (1) valid for any linear riemannian metric, complex stucture, etc., not just the ones given by Id, J, etc.

Problem 4.

A couple of questions about the group $\mathrm{O}(V + V^*)$:

- ** Is the action of $\mathrm{O}(V + V^*)$ on maximally isotropic subspaces transitive?
- ** Describe the Lie algebra of $\mathrm{O}(V + V^*)$.