POLYTOPALITY AND CARTESIAN PRODUCTS OF GRAPHS

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Abstract. We study the question of polytopality of graphs: when is a given graph the graph of a polytope? We first review the known necessary conditions for a graph to be polytopal, and we present three families of graphs which satisfy all these conditions, but which nonetheless are not graphs of polytopes.

Our main contribution concerns the polytopality of Cartesian products of non-polytopal graphs. On the one hand, we show that products of simple polytopes are the only simple polytopes whose graph is a product. On the other hand, we provide a general method to construct (non-simple) polytopal products whose factors are not polytopal.

Even though graphs are perhaps the most prominent feature of polytopes, we are still far from being able to answer several basic questions regarding them. For applications, one of the most important ones is to bound the diameter of the graph in terms of the number of variables and inequalities defining the polytope [San10]. From a theoretical point of view, it is striking that we cannot even efficiently decide whether a given graph occurs as the graph of a polytope or not [RG96].

In this paper, we study how polytopality behaves with respect to some common operations on graphs and polytopes. We start by reviewing in Section 1 some necessary conditions for a graph to be polytopal: Balinski's Theorem [Bal61], the d-Principal Subdivision Property [Bar67] and the Separation Property [Kle64]. Our guideline is to construct graphs satisfying these properties, but which nonetheless are not graphs of polytopes: we say that these graphs are non-polytopal for "non-trivial reasons". We present three infinite families of such graphs, to illustrate different methods to prove non-polytopality and to introduce general notions and results useful in the rest of the paper.

The second part of this paper is dedicated to the study of the polytopality of Cartesian products of graphs. Cartesian products of polytopal graphs are automatically polytopal, and their polytopality range (*i.e.* the set of possible dimensions of their realizations) has been the subject of recent research [JZ00, Zie04, SZ10, MPP09]. The main contribution of this paper concerns the polytopality of Cartesian products of non-polytopal graphs. On the one hand, we show in Section 2.1 that products of simple polytopes are the only simple polytopes whose graph is a product. On the other hand, we provide in Section 2.2 a general method to construct (non-simple) polytopal products whose factors are not polytopal.

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1. Non-polytopal graphs for non-trivial reasons

Definition 1.1. *A graph G is* polytopal *if it is isomorphic to the graph of some polytope P. If P is d-dimensional, we say that G is d-polytopal.*

In small dimension, polytopality is easy to deal with. The first interesting question is 3-polytopality, which is characterized by Steinitz' "Fundamental Theorem of convex types" (see [Grü03, Zie95] for a discussion on different proofs):

Theorem 1.2 (Steinitz [Ste22])**.** *A graph G is the graph of a 3-polytope P if and only if G is planar and 3-connected. Moreover, the combinatorial type of P is uniquely determined by G.* □

The first step to realizing a graph G is to understand the possible face lattice of a polytope whose graph is G. For example, it is often difficult to decide which cycles of G can define 2-faces of a d-polytope realizing G. In dimension 3, graphs of 2-faces are characterized by the following separation condition:

Theorem 1.3 (Whitney [Whi32])**.** *Let G be the graph of a 3-polytope P. The graphs of the 2-faces of P are precisely the induced cycles in G that do not separate G.* □

In contrast to the easy 2- and 3-dimensional worlds, d-polytopality becomes much more involved as soon as $d \geq 4$. For example, neighborly 4-polytopes (whose graph is complete) illustrate the difference between the behavior of 3- and 4-dimensional polytopes:

(i) Starting from a neighborly 4-polytope, and stacking vertices on undesired edges, Perles observed that every graph is an induced subgraph of the graph of a 4-polytope (while only planar graphs are induced subgraphs of graphs of 3-polytopes).
(ii) The existence of combinatorially different neighborly polytopes proves that the 2-faces of a 4-polytope cannot be derived from its graph (compare with Whitney's Theorem).

As a consequence of his work on realization spaces of 4-polytopes, Richter-Gebert underlined several deeper negative results: among others, 4-polytopality is NP-hard and cannot be characterized by a finite set of "forbidden minors" (see [RG96, Chapter 9]).

1.1. Necessary conditions for polytopality.

The first part of this paper focusses on the following necessary conditions for a graph to be polytopal:

Proposition 1.4. *A d-polytopal graph G satisfies the following properties:*

(1) Balinski's Theorem*: G is d-connected* [Bal61].
(2) Principal Subdivision Property *(d-PSP): Every vertex of G is the principal vertex of a principal subdivision of K_{d+1}. Here, a* subdivision *of K_{d+1} is obtained by replacing edges by paths, and a* principal subdivision *of K_{d+1} is a subdivision in which all edges incident to a distinguished* principal vertex *are not subdivided* [Bar67].
(3) Separation Property*: The maximal number of components into which G may be separated by removing $n > d$ vertices equals $f_{d-1}\big(C_d(n)\big)$, the maximum number of facets of a d-polytope with n vertices* [Kle64]. □

Remark 1.5. The principal subdivision property together with Steinitz' Theorem ensure that no graph of a 3-polytope is d-polytopal for $d \neq 3$. In other words, any 3-polytope is the unique polytopal realization of its graph.

The necessary conditions of Proposition 1.4 are sometimes sufficient to determine the polytopality of small graphs. We refer for example the interested reader to the discussion in [Pil10, Example 6.8] on the polytopal realizations of all circulant graphs with at most 8 vertices.

However, there are graphs which are not polytopal although they satisfy these conditions. We say that such a graph is *non-polytopal for non-trivial reasons*. The remainder of this section is devoted to the presentation of three families of non-polytopal graphs for non-trivial reasons, which illustrate three different ways to prove non-polytopality. The first family is that of the complete bipartite graphs:

Example 1.6 ([Bar67])**.** The complete bipartite graph $K_{m,n}$ is not polytopal for any integers $m, n \geq 3$. However $K_{n,n}$ satisfies all properties of Proposition 1.4 to be 4-polytopal as soon as $n \geq 7$.

Proof. Assume that $K_{n,m}$ is the graph of a d-polytope P. Then $d \geq 4$ because $K_{n,m}$ is non-planar. Consider the induced subgraph H of $K_{n,m}$ corresponding to some 3-face F of P. Because H is induced and has minimum degree at least 3, it contains a $K_{3,3}$ minor, so F was not a 3-face after all. □

Remark 1.7. The *polytopality range* of a graph is the set of possible dimensions of its realizations. What subsets of $\mathbb{N}$ can be polytopality ranges of graphs? We know that if a polytopality range contains 1, 2 or 3, then it is a singleton (Remark 1.5) and that every singleton is a polytopality range (*e.g.* stacking a vertex in every facet of a simplex [Kle64]), as well as any interval $\{4, \dots, n\}$ (complete graph). We suspect that any interval $\{m, \dots, n\}$ with $4 \leq m \leq n$ is a polytopality range. One way of getting non-singleton polytopality ranges is to project polytopes preserving their graph. For example, [MPP09] obtain that for any sequence of integers $n_1, \dots, n_r$ (with $n_i \geq 2$), the product $\triangle_{n_1} \times \cdots \times \triangle_{n_r}$ can be projected from dimension $\sum n_i$ until dimension $r+3$ preserving its graph (a particular example of that is the projection of the simplex until dimension 4). This raises the question of whether there exist graphs whose polytopality range is not an interval of $\mathbb{N}$.

1.2. Simple polytopes.

A d-polytope is *simple* if its vertex figures are simplices. In other words, its facet-defining hyperplanes are in general position, so that a vertex is contained in exactly d facets, and also in exactly d edges (and thus the graph of a simple d-polytope is d-regular). Surprisingly, a d-regular graph can be realized by at most one simple polytope:

Theorem 1.8 ([BML87, Kal88])**.** *Two simple polytopes are combinatorially equivalent if and only if they have the same graph.* □

This property, conjectured by Perles, was first proved by Blind and Mani [BML87]. Kalai [Kal88] then gave a very simple (but exponential) algorithm for reconstructing the face lattice from the graph, and Friedman [Fri09] showed that this can even be done in polynomial time.

As mentioned previously, the first step to find a polytopal realization of a graph is often to understand what the face lattice of this realization can look like. Theorem 1.8 ensures that if the realization is simple, there is only one choice. This motivates us to temporarily restrict the study of realization of regular graphs to simple polytopes:

Definition 1.9. *A graph is* simply d-polytopal *if it is the graph of a simple d-polytope.*

We can exploit properties of simple polytopes to obtain results on the simple polytopality of graphs. For us, the key property turns out to be that any k-tuple of edges incident to a vertex of a simple polytope is contained in a k-face. For example, this implies the following result:

Proposition 1.10. *All induced cycles of length* 3*,* 4 *and* 5 *in the graph of a simple d-polytope P are graphs of 2-faces of P.*

Proof. For 3-cycles, the result is immediate: any two adjacent edges of a 3-cycle induce a 2-face, which must be a triangle because the graph is induced.

Next, let $\{a, b, c, d\}$ be consecutive vertices of a 4-cycle in the graph of a simple polytope P. Any pair of edges emanating from a vertex lies in a 2-face of P. Let C_a be the 2-face of P that contains $\text{conv}\{d, a, b\}$. Similarly, let C_c be the 2-face of P that contains $\{b, c, d\}$. If C_a and C_c were distinct, they would intersect improperly, at least in the two vertices b and d. Thus, $C_a = C_c = \text{conv}\{a, b, c, d\}$ is a 2-face of P.

The case of 5-cycles is a little more involved. We first show it for 3-polytopes. If a 5-cycle C in the graph G of a simple 3-polytope does not define a 2-face, it separates G into two nonempty subgraphs A and B (Theorem 1.3). Since G is 3-connected, both A and B are connected to C by at least three edges. But the endpoints of these six edges must be distributed among the five vertices of C, so one vertex of C receives two additional edges, and this contradicts simplicity.

For the general case, we show that any 5-cycle C in a simple polytope is contained in some 3-face, and apply the previous argument (a face of a simple polytope is simple). First observe that any three consecutive edges in the graph of a simple polytope lie in a common 3-face. This is true because any two adjacent edges define a 2-face, and a 2-face together with another adjacent edge defines a 3-face. Thus, four of the vertices of C are already contained in a 3-face F. If the fifth vertex w of C lies outside F, then the 2-face defined by the two edges of C incident to w intersects improperly with F. □

Remark 1.11. Observe that Proposition 1.10 cannot be extended neither for 6-cycles (the 3-dimensional cube has a missing 6-cycle) nor for non-simple polytopes (for $p \geq 3$, the double pyramid over a p-cycle has a missing p-cycle).

Corollary 1.12. *A simply polytopal graph cannot:*

(i) be separated by an induced cycle of length 3*,* 4 *or* 5*.*
(ii) contain two induced cycles of length 4 *or* 5 *which share* 3 *vertices.*
(iii) contain an induced $K_{2,3}$ *or an induced Petersen graph.*

Proof. Parts (i) and (ii) are immediate consequences of Proposition 1.10 since the 2-faces of a polytope are non-separating cycles and pairwise intersect in at most one edge. Part (iii) arises from Part (ii) since $K_{2,3}$ (resp. the Petersen graph) contains two induced 4-cycles (resp. two 5-cycles) which share 3 vertices. □

Proposition 1.10 provides a simple proof of the non-polytopality of the following family of graphs[1].

Example 1.13. Consider the family of graphs suggested in Figure 1. The nth graph of this family is the graph G_n whose vertex set is $\mathbb{Z}_{2n+3} \times \mathbb{Z}_2$ and where the vertex (x, y) is related with the vertices $(x+y+1, y)$, $(x+y, y+1)$, $(x-y-1, y)$ and $(x+y-1, y+1)$.

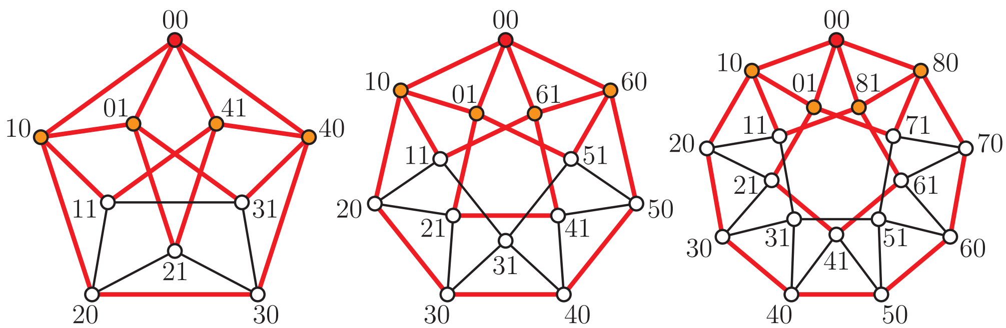

FIGURE 1. An infinite family of non-polytopal graphs (for non-trivial reasons). The vertex 00 is the principal vertex of a principal subdivision of K_5, whose edges are colored in red.

Observe first that the graphs of this family satisfy all necessary conditions of Proposition 1.4:

(1) They are 4-connected: when we remove 3 vertices, either the external cycle $\{i0 \mid i \in \mathbb{Z}_{2n+3}\}$ or the internal cycle $\{i1 \mid i \in \mathbb{Z}_{2n+3}\}$ remains a path, to which all the vertices are connected.
(2) They satisfy the principal subdivision property for dimension 4: the edges of a principal subdivision of K_5 with principal vertex 00 are colored in Figure 1.
(3) They satisfy the separation property: the cyclic 4-polytope on m vertices has $\frac{m(m-3)}{2}$ facets, while removing m vertices from G_n cannot create more than m connected components.

Consider the first graph G_1 of this family (on the left in Figure 1). Since the 5-cycles induced by $\{00, 10, 20, 30, 40\}$ and $\{00, 10, 20, 21, 41\}$ share two edges, G_1 is not polytopal (because of Theorem 1.3 in dimension 3 and of Proposition 1.10 in dimension 4). In fact, Proposition 1.10 even excludes all graphs of the family:

Lemma 1.14. *None of the graphs of the infinite family described above is polytopal.*

Proof. Since they contain a subdivision of K_5, they are not 3-polytopal.

Assume the graph to be 4-polytopal, *i.e.* it is the graph of a simple 4-polytope P. Denote by e_i the edge of the external cycle from vertex $i0$ to vertex $(i+1)0$. Since P is simple, there are three 2-faces incident to each edge e_i. Two of them are easy to find: e_i together with the vertex $i1$ forms a 3-cycle A_i, and e_i together with the vertices $(i-1)1$ and $(i+1)1$ forms an induced 4-cycle B_i. By Proposition 1.10 these

[1] This family was communicated to us by Guedes de Oliveira and Noy [dON09], with a different proof for non-polytopality, based on acyclic orientations.

cycles A_i and B_i come from 2-faces. The third 2-face of P containing e_i must use the unique edges incident to $i0$ and $(i+1)0$ not used so far, namely e_{i-1} and e_{i+1}. We conclude that the entire external cycle of the figure forms the third 2-face C incident to every e_i.

We look now at 3-faces. At each e_i there is one, call it F_i, using C and A_i and one, call it E_i, using C and B_i. But then B_{i+1} must be in F_i and A_{i+1} must be in E_i (otherwise, they would intersect improperly). That is, $F_0 = E_1 = F_2 = E_3 = \dots$, which eventually gives $F_i = E_i$ since $2n+3$ is odd. This is impossible. ☐

1.3. **Truncation and star-clique operation.** We consider the polytope $\tau_v(P)$ obtained by cutting off a single vertex v in a polytope P. The set of inequalities defining $\tau_v(P)$ is that of P together with a new inequality satisfied strictly by all the vertices of P but not satisfied by v. The faces of $\tau_v(P)$ are:

(i) all the faces of P which do not contain v;
(ii) the truncations $\tau_v(F)$ of all faces F of P containing v; and
(iii) the vertex figure of v in P together with all its faces.

In particular, if v is a simple vertex in P, then the truncation of v in P replaces v by a simplex. On the graph of P, this translates into the following transformation:

Definition 1.15. *Let G be a graph and v be a vertex of degree d of G. The star-clique operation (at v) replaces vertex v by a d-clique K, and assigns one edge incident to v to each vertex of K. The resulting graph $\sigma_v(G)$ has $d-1$ more vertices and $\binom{d}{2}$ more edges.*

Proposition 1.16. *Let v be a vertex of degree d in a graph G. Then $\sigma_v(G)$ is d-polytopal if and only if G is d-polytopal.*

Proof. If a d-polytope P realizes G, then the truncation $\tau_v(P)$ realizes $\sigma_v(G)$.

For the other direction, consider a d-polytope Q which realizes $\sigma_v(G)$. We first show that the d-clique replacing v forms a facet of Q. Let its vertices be denoted $v_1, \dots, v_d$. Observe that all these vertices have degree d in $\sigma_v(G)$. That is, Q is "simple at those vertices". This implies that for every subset S of neighbors of, say, v_1, there is a face of dimension $|S|$ containing S and v_1. In particular, there is a facet F of Q containing $v_1, \dots, v_d$. By simplicity of all these vertices, F cannot contain any other vertex.

Up to a projective transformation, we can assume that the d facets of Q adjacent to F intersect beyond F. Then, removing the inequality defining F from the facet description of Q creates a polytope which realizes G. ☐

We can exploit Proposition 1.16 to construct families of non-polytopal graphs:

Lemma 1.17. *Any graph obtained from a 4-regular 3-polytopal graph by a finite nonempty sequence of star-clique operations is non-polytopal.*

Proof. No such graph can be 3-polytopal since it is not planar (it contracts easily to K_5). If the resulting graph were 4-polytopal, Proposition 1.16 would assert that the original graph was also 4-polytopal, which would contradict Remark 1.5. ☐

This observation allows us to construct our third infinite family of non-polytopal graphs for non-trivial reasons:

Example 1.18. For $n \geq 3$, consider the family of graphs suggested by Figure 2. They are constructed as follows: place a regular $2n$-gon C_{2n} into the plane, centered at the origin. Draw a copy C'_{2n} of C_{2n} scaled by $\frac{1}{2}$ and rotated by $\frac{\pi}{2n}$, and lift the vertices of C'_{2n} alternately to heights 1 and -1 into the third dimension. The graph $✡_n$ is the graph of the convex hull of the result.

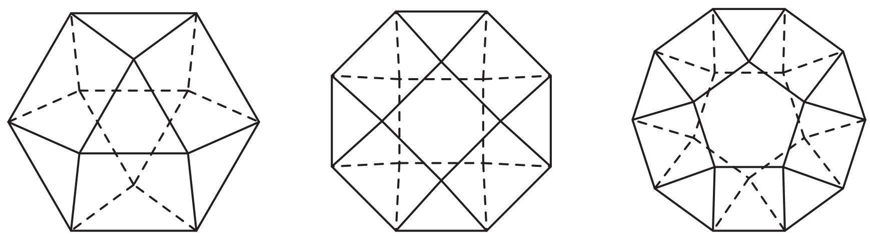

FIGURE 2. The graphs $✡_n$ for $n \in \{3, 4, 5\}$.

In other words, the graph $✡_n$ is the graph of the Minkowski sum of two pyramids over an n-gon (the first pyramid obtained as the convex hull of the even vertices of C'_{2n} together with the point $(0, 0, 1)$, and the second pyramid obtained as the convex hull of the odd vertices of C'_{2n} together with the point $(0, 0, -1)$).

Let $✡^\star_n$ be the result of successively applying the star-clique operation to all vertices on the intermediate cycle C_{2n}. Corollary 1.17 ensures that $✡^\star_n$ is not polytopal, although it satisfies all necessary conditions to be 4-polytopal (we skip this discussion which is similar to that in Example 1.13).

Remark 1.19. Corollary 1.17 fails in higher dimension: the graph obtained from a d-polytopal graph by a star-clique operation on a vertex of degree $\delta > d$ may still be polytopal. For example, the complete graph K_n is a $(n-1)$-regular 4-polytopal graph, and the graph $K_{n-1} \times K_2$ obtained by a star-clique operation on a vertex of K_n is still 4-polytopal [MPP09].

2. Polytopality of products of graphs

Define the *Cartesian product* $G \times H$ of two graphs G and H to be the graph whose vertex set is the product $V(G \times H) := V(G) \times V(H)$, and whose edge set is $E(G \times H) := \big(V(G) \times E(H)\big) \cup \big(E(G) \times V(H)\big)$. In other words, for $a, c \in V(G)$ and $b, d \in V(H)$, the vertices (a, b) and (c, d) of $G \times H$ are adjacent if either $a = c$ and $\{b, d\} \in E(H)$, or $b = d$ and $\{a, c\} \in E(G)$. Notice that this product is usually denoted by $G \square H$ in graph theory. We choose to use the notation $G \times H$ to be consistent with the Cartesian product of polytopes: if G and H are the graphs of the polytopes P and Q respectively, then the product $G \times H$ is the graph of the product $P \times Q$. In this section, we focus on the polytopality of products of non-polytopal graphs.

The factors of a polytopal product are not necessarily polytopal: consider for example the product of a triangle by a path, or the product of a segment by two glued triangles (see Figure 3 and more generally Proposition 2.7). We neutralize these elementary examples by further requiring the product $G \times H$, or equivalently the factors G and H, to be regular (the degree of a vertex (v, w) of $G \times H$ is the sum of the degrees of the vertices v of G and w of H). In this case, it is natural to investigate when such regular products can be simply polytopal. The answer is given by Theorem 2.2.

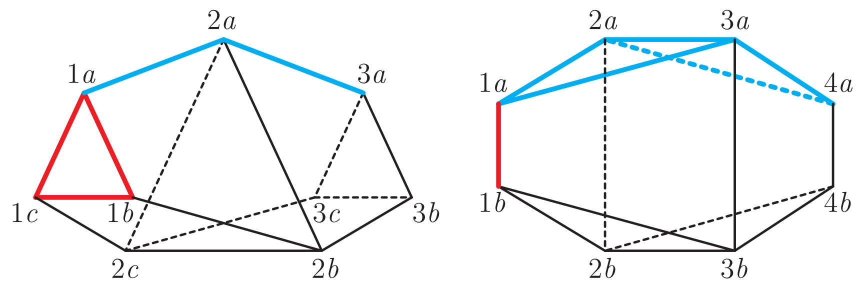

FIGURE 3. Polytopal products of non-polytopal graphs: the product of a triangle abc by a path 123 (left) and the product of a segment ab by two glued triangles 123 and 234 (right).

Before starting, let us observe that the necessary conditions of Proposition 1.4 are preserved under Cartesian products in the following sense:

Proposition 2.1. *If two graphs G and H are respectively d- and e-connected, and respectively satisfy d- and e-PSP, then their product $G \times H$ is $(d+e)$-connected and satisfies $(d+e)$-PSP.*

Proof. The connectivity of a Cartesian product of graphs was studied in [CS99]. In fact, it is even proved in [Špa08] that

$$\kappa(G \times H) = \min(\kappa(G)|H|, \kappa(H)|G|, \delta(G) + \delta(H)) \geq \kappa(G) + \kappa(H),$$

where $\kappa(G)$ and $\delta(G)$ respectively denote the connectivity and the minimum degree of a graph G.

For the principal subdivision property, consider a vertex (v, w) of $G \times H$. Choose a principal subdivision of K_{d+1} in G with principal vertex v and neighbors N_v, and a principal subdivision of K_{e+1} in H with principal vertex w and neighbors N_w. This gives rise to a principal subdivision of K_{d+e+1} in $G \times H$ with principal vertex (v, w) and neighbors $(N_v \times \{w\}) \cup (\{v\} \times N_w)$. Indeed, for $x, x' \in N_v$, the vertices (x, w) and (x', w) are connected by a path in $G \times w$ by construction; similarly, for $y, y' \in N_w$, the vertices (v, y) and (v, y') are connected by a path in $v \times H$. Finally, for each $x \in N_v$ and $y \in N_w$, connect (x, w) to (v, y) via the path of length 2 that passes through (x, y). All these paths are disjoint by construction. □

2.1. **Simply polytopal products.** A product of simply polytopal graphs is automatically simply polytopal. We prove that the reciprocal statement is also true:

Theorem 2.2. *A product of graphs is simply polytopal if and only if its factors are.*

Applying Theorem 1.8, we obtain a strong characterization of the simply polytopal products:

Corollary 2.3. *The simple polytope realizing the above product of graphs is unique. Therefore, combinatorial products of simple polytopes are the only simple polytopes whose graph is a product.* □

Let G and H be two connected regular graphs of degree d and e respectively, and assume that the graph $G \times H$ is the graph of a simple $(d+e)$-polytope P. By

Proposition 1.10, for all edges a of G and b of H, the 4-cycle $a \times b$ is the graph of a 2-face of P.

Lemma 2.4. *Let F be any facet of P, let v be a vertex of G, and let $\{x, y\}$ be an edge of H such that $(v, x) \in F$ and $(v, y) \notin F$. Then $G \times \{x\} \subset F$ and $G \times \{y\} \cap F = \emptyset$.*

Proof. Since the polytope is simple, all neighbors of (v, x) except (v, y) are connected to (v, x) by an edge of F. Let v' be a neighbor of v in G, and let C be the 2-face $\mathrm{conv}\{v, v'\} \times \mathrm{conv}\{x, y\}$ of P. If (v', y) were a vertex of F, the intersection $C \cap F$ would consist of exactly three vertices (because $(v, y) \notin F$), a contradiction. In summary, $(v', x) \in F$ and $(v', y) \notin F$, for all neighbors v' of v. Repeating this argument and using the fact that G is connected yields $G \times \{x\} \subset F$ and $G \times \{y\} \cap F = \emptyset$. □

Lemma 2.5. *The graph of any facet of P is either of the form $G' \times H$ for a $(d-1)$-regular induced subgraph G' of G, or of the form $G \times H'$ for an $(e-1)$-regular induced subgraph H' of H.*

Proof. Assume that the graph of a facet F is not of the form $G' \times H$. Then there exists a vertex v of G and an edge $\{x, y\}$ of H such that $(v, x) \in F$ and $(v, y) \notin F$. By Lemma 2.4, the subgraph H' of H induced by the vertices $y \in H$ such that $G \times \{y\} \subset F$ is nonempty. We now prove that the graph $\mathrm{gr}(F)$ of F is exactly $G \times H'$.

The inclusion $G \times H' \subset \mathrm{gr}(F)$ is clear: by definition, $G \times \{y\}$ is a subgraph of $\mathrm{gr}(F)$ for any vertex $y \in H'$. For any edge $\{x, y\}$ of H' and any vertex $v \in G$, the two vertices (v, x) and (v, y) are contained in F, so the edge between them is an edge of F; if not, we would have an improper intersection between F and this edge. For the other inclusion, define $H'' := \{y \in H \mid G \times \{y\} \cap F = \emptyset\}$ and let $H''' := H \setminus (H' \cup H'')$. If $H''' \neq \emptyset$, the fact that H is connected ensures that there is an edge between some vertex of H''' and either a vertex of H' or H''. This contradicts Lemma 2.4.

We have proved that $G \times H' = \mathrm{gr}(F)$. The fact that F is a simple $(d+e-1)$-polytope and the d-regularity of G together ensure that H' is $(e-1)$-regular. □

Proof of Theorem 2.2. One direction is clear. For the other direction, proceed by induction on $d+e$, the cases $d = 0$ and $e = 0$ being trivial. Now assume that $d, e \geq 1$, that $G \times H = \mathrm{gr}(P)$, and that G is not the graph of a d-polytope. By Lemma 2.5, all facets of P are of the form $G' \times H$ or $G \times H'$, where G' (resp. H') is an induced $(d-1)$-regular (resp. $(e-1)$-regular) subgraph of G (resp. H). By induction, the second case does not arise. We fix a vertex w of H. Then induction tell us that $F_w := G' \times \{w\}$ is a face of P, and $G' \times H$ is the only facet of P that contains F_w by Lemma 2.5. This cannot occur unless F_w is a facet, but this only happens in the base case $H = \{w\}$. □

Example 2.6. Consider a graph G that is d-regular, d-connected, and satisfies d-PSP, but is not simply d-polytopal. Then, any product of G by a simply e-polytopal graph is $(d+e)$-regular, $(d+e)$-connected, satisfies $(d+e)$-PSP, but is not simply $(d+e)$-polytopal.

From this observation, we can construct other families of non-polytopal graphs for non-trivial reasons, *e.g.* the product of the circulant graph $C_{2m}(1, m)$ by a segment.

2.2. **Polytopal products of non-polytopal graphs.** Is the product of two Petersen graphs polytopal? This specific question (posed by Ziegler [Zie10]) inspired our study of polytopality of Cartesian products of non-polytopal graphs. Corollary 1.12 and Theorem 2.2 give two reasons why this product cannot be simply polytopal. However, there is no apparent reason why it could not be 4- or 5-polytopal. In fact, this section provides examples of polytopal products of non-polytopal graphs.

We first show a general construction that produces polytopal products starting from a polytopal graph G and a non-polytopal one H. We need the graph H to be the graph of a *regular subdivision* of a polytope Q, that is, the graph of the upper[2] envelope (the set of all upper facets with respect to the last coordinate) of the convex hull of the point set $\{(q, \omega(q)) \,|\, q \in V(Q)\} \subset \mathbb{R}^{e+1}$ obtained by lifting the vertices of $Q \subset \mathbb{R}^e$ according to a *lifting function* $\omega : V(Q) \to \mathbb{R}$.

Proposition 2.7. *If G is the graph of a d-polytope P, and H is the graph of a regular subdivision of an e-polytope Q, then $G \times H$ is $(d+e)$-polytopal. In the case $d > 1$, the regular subdivision of Q can even have internal vertices.*

Proof. Let $\omega : V(Q) \to \mathbb{R}_{>0}$ be a lifting function that induces a regular subdivision of Q with graph H. Assume without loss of generality that the origin of $\mathbb{R}^d$ lies in the interior of P. For each $p \in V(P)$ and $q \in V(Q)$, we define the point $\rho(p, q) := (\omega(q)p, q) \in \mathbb{R}^{d+e}$. Consider

$$R := \operatorname{conv} \{\rho(p, q) \,|\, p \in V(P), q \in V(Q)\}.$$

Let g be a facet of Q defined by the linear inequality $\langle \psi \,|\, y\rangle \le 1$. Then the inequality $\langle (0, \psi) \,|\, (x, y)\rangle \le 1$ defines a facet of R, with vertex set $\{\rho(p, q) \,|\, p \in P, q \in g\}$, and isomorphic to $P \times g$.

Let f be a facet of P defined by the linear inequality $\langle \phi \,|\, x\rangle \le 1$. Let c be a cell of the subdivision of Q, and let $\psi_0 h + \langle \psi \,|\, y\rangle \le 1$ be the linear inequality that defines the upper facet corresponding to c in the lifting. Then we claim that the linear inequality

$$\chi(x, y) = \psi_0 \langle \phi \,|\, x\rangle + \langle \psi \,|\, y\rangle \le 1$$

selects a facet of R with vertex set $\{\rho(p, q) \,|\, p \in f, q \in c\}$ that is isomorphic to $f \times c$. Indeed,

$$\chi\big(\rho(p, q)\big) = \chi(\omega(q)p, q) = \psi_0 \omega(q) \langle \phi \,|\, p\rangle + \langle \psi \,|\, q\rangle \le 1,$$

where equality holds if and only if $\langle \phi \,|\, p\rangle = 1$ and $\psi_0 \omega(q) + \langle \psi \,|\, q\rangle = 1$, so that $p \in f$ and $q \in c$.

The above set $\mathcal{F}$ of facets of R contains all facets: indeed, any $(d+e-2)$-face of a facet in $\mathcal{F}$ is contained in precisely two facets in $\mathcal{F}$. Since the union of the edge sets of the facets in $\mathcal{F}$ is precisely $G \times H$, it follows that the graph of R equals $G \times H$.

A similar argument proves the same statement in the case when $d > 1$ and H is a regular subdivision of Q with internal vertices (meaning that not only the vertices of Q are lifted, but also a finite number of interior points). □

We already mentioned two examples obtained by such a construction in the beginning of this section (see Figure 3): the product of a polytopal graph by a path and the product of a segment by a subdivision of an n-gon with no internal vertex. Proposition 2.7 even produces examples of regular polytopal products which are not simply polytopal:

[2]The usual convention is to take the lower envelope instead, but taking the upper one simplifies our presentation.

Example 2.8. Let H be the graph obtained by a star-clique operation from the graph of an octahedron. It is non-polytopal (Corollary 1.17), but it is the graph of a regular subdivision of a 3-polytope (see Figure 4). Consequently, the product of H by any regular polytopal graph is polytopal. Thus, there exist regular polytopal products which are not simply polytopal.

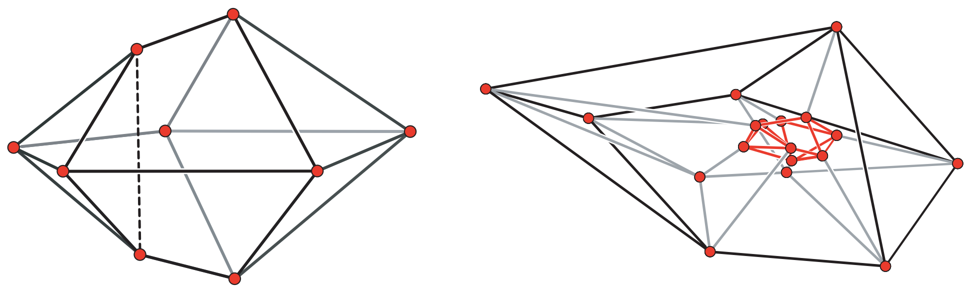

FIGURE 4. A non-polytopal 4-regular graph H which is the graph of a regular subdivision of a 3-polytope (left) and the Schlegel diagram of a 4-polytope whose graph is the product of H by a segment (right).

Finally, Proposition 2.7 also produces polytopal products of two non-polytopal graphs:

Example 2.9. Define the *p-domino graph* D_p to be the product of a path P_p of length p by a segment. Let $p, q \geq 2$. Observe that D_p and D_q are not polytopal and that $D_p \times P_q$ is the graph of a regular subdivision of a 3-polytope. Consequently, the product of dominos $D_p \times D_q$ is a 4-polytopal product of two non-polytopal graphs (see Figure 5).

Finally, let us observe that the product $D_p \times D_q = P_p \times P_q \times (K_2)^2$ can be decomposed in different ways into a product of two graphs. However, in any such decomposition, at least one of the factors is non-polytopal.

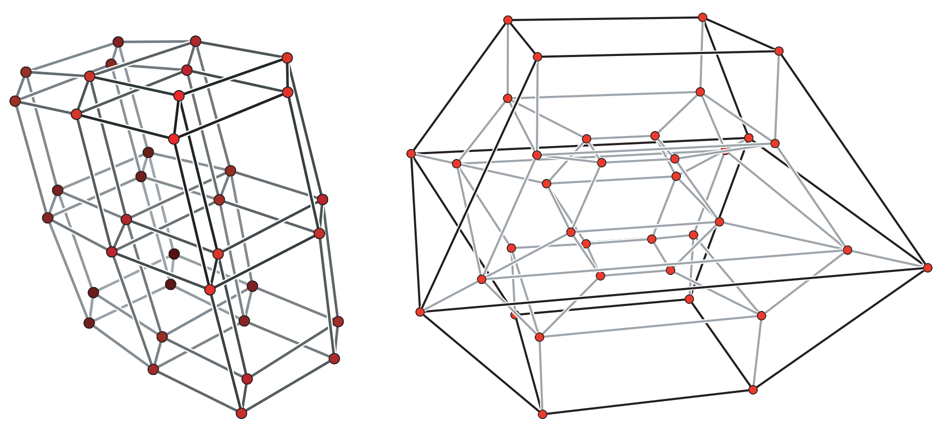

FIGURE 5. The graph of the product of two 2-dominos (left) and the Schlegel diagram of a realizing 4-polytope (right).

2.3. **Product with a segment.** In this section, we illustrate all possible behaviors, regarding polytopality, of the product of a segment by a regular graph H:

(1) If H is polytopal, then $K_2 \times H$ is polytopal. However, some ambiguities can appear:
  (a) The dimension can be ambiguous. For example, $K_2 \times K_n$ is realized by the product of a segment by any neighborly polytope. See [MPP09] for a discussion on dimensional ambiguity of products of complete graphs.
  (b) The dimension can be unambiguous, but the combinatorics of the polytope can be ambiguous. In this case, H is not simply polytopal (Theorem 1.8). In Proposition 2.11, we determine all possible realizations of the graph $K_2 \times K_{2,2,2}$ of a prism over an octahedron.
  (c) There can be no ambiguity at all. This happens for example if H is a cycle.
(2) If H is not polytopal, then $K_2 \times H$ is not simply polytopal (Theorem 2.2). However:
  (a) $K_2 \times H$ can be polytopal in smaller dimension (Example 2.8).
  (b) $K_2 \times H$ can be non-polytopal. This happens for example when H is the complete graph $K_{n,n}$ (Proposition 2.10).

Proposition 2.10. *For $n \geq 3$, the graph $K_2 \times K_{n,n}$ is not polytopal.*

Proof. Observe that $K_2 \times K_{n,n}$ is not d-polytopal for $d \leq 3$ because it contains a $K_{3,3}$-minor. Suppose, to seek a contradiction, that $K_2 \times K_{n,n}$ is the graph of a d-polytope P, for some $d \geq 4$. Denote by A and B the two maximal independent sets in $K_{n,n}$, and by A_0, B_0, A_1, B_1 their corresponding copies in the Cartesian product $K_2 \times K_{n,n}$. We look at how many vertices can a 3-face of P have in each of them.

- Suppose that a 3-face F has a single vertex x in one of them, say A_0. Then this vertex must have (at least two) neighbors in B_0 since it cannot have more than one in A_1. But then those vertices of F in B_0 have degree at most two in F, since they can only be joined to x and to their corresponding vertices in B_1.
- Suppose that a 3-face F has at least three vertices x, y, z in one of them, say A_0. Then it cannot have more than two vertices in B_0, because otherwise its graph would contain a $K_{3,3}$. In fact, there must be exactly two vertices u, v in B_0: since any vertex of F has degree at least 3, and each vertex in A_0 can only be connected to vertices in B_0 or to its corresponding neighbor in A_1, each vertex of F in A_0 must have at least, and thus exactly, two neighbors in B_0 and one in A_1. Thus, F also has at least three vertices in A_1 and, by the same reasoning, exactly two vertices in B_1; call one of them w. But now $\{x, y, z\}$ and $\{u, v, w\}$ are the two maximal independent sets of a subdivision of $K_{3,3}$ included in F, a contradiction.

So, every 3-face F has exactly two vertices in each of the sets A_0, B_0, A_1, B_1. In order for them to have degree at least (and then equal to) three, they must consist of two corresponding copies of a $K_{2,2}$. But it is impossible for every 3-face of P to be of this form, since for every vertex x of P in, say, A_0 there must be a 3-face containing x and not using the edge to its corresponding vertex in A_1. □

Observe that the arguments in the proof show a bit more than what we state: no induced subgraph of $K_2 \times K_{n,n}$ is d-polytopal for $d \geq 4$, and its only 3-polytopal induced subgraph is $K_2 \times K_{2,2}$ (the graph of a 3-dimensional cube).

Proposition 2.11. *The graph $K_2 \times K_{2,2,2}$ (of the prism over the octahedron) is realized by four combinatorially different polytopes.*

Proof. In order to exhibit four different realizations, we recall the situation and the proof of Proposition 2.7. Given the graph G of a d-polytope P and the graph H of a regular subdivision of an e-polytope Q defined by a lifting function $\omega : V(Q) \to \mathbb{R}$, we construct a $(d+e)$-polytope with graph $G \times H$ as follows: we start from the product $P \times Q$ and we lift each face $\{p\} \times Q$ using ω. This subdivides $\{p\} \times Q$, creating the subgraph $\{p\} \times H$ of the product $G \times H$. Observe now that the deformation can be different at each vertex of P: we can use a different lifting function at each vertex of P, and produce combinatorially different polytopes.

To come back to our example, observe that the octahedron has four regular subdivisions with no additional edges: the octahedron itself (for a constant lifting function), and the three subdivisions into two Egyptian pyramids glued along their square face (for a lifting function that vanishes in the common square face and is negative at the other two vertices). This leads to four combinatorially different realizations of $K_2 \times K_{2,2,2}$: in our previous construction, we can choose either the octahedron at both ends of the segment (thus obtaining the prism over the octahedron), or the octahedron at one end and the glued Egyptian pyramids at the other, or the glued Egyptian pyramids at both ends of the segment (and this leads to two possibilities according to whether we choose the same square or two orthogonal squares to subdivide the two octahedra). □

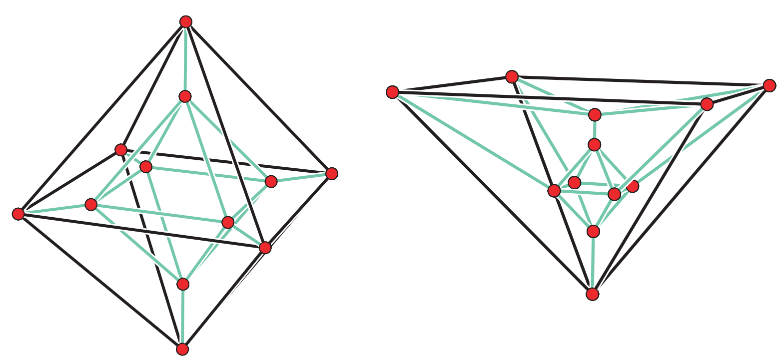

FIGURE 6. The prism over the octahedron (left) and a combinatorially different polytope with the same graph (right).

It can also be proved (see [Pil10, Proposition 6.36]) that any realization of the graph $K_2 \times K_{2,2,2}$ is combinatorially equivalent to one of the four described above, but we omit this part.

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