SIGNALETIC OPERADS

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Abstract. We introduce k-signaletic operads and their Koszul duals, generalizing the dendriform, diassociative and duplicial operads (which correspond to the $k = 1$ case). We show that the Koszul duals of the k-signaletic operads act on multipermutations and that the resulting algebras are free, thus providing combinatorial models for these operads. Finally, motivated by these actions on multipermutations, we introduce similar operations on multiposets which yield yet another relevant operad obtained as Manin powers of the L-operad.

Contents

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1. Introduction

Diassociative and dendriform algebras and operads. In his study of Leibniz algebras, a kind of non commutative Lie algebras, J.-L. Loday introduced two more families of algebras named diassociative and dendriform [Lod01]. They are vector spaces endowed with two binary operations $\prec$ and $\succ$ satisfying certain axioms regarding the compositions of two operations (see Section 2.4.1 for precise definitions). In particular, both operations are associative for diassociative algebras, and the sum of the two operations is associative for dendriform algebras.

The close connections between these two families of algebras are better understood via the theory of (non-symmetric) operads. An operad is an algebraic structure abstracting a type of algebras. Their theory allows, via a suitable notion of morphism, to understand the link between types. More precisely they provide a formalization of operations transforming several inputs into one output, of the ways to compose these operations, and of the relations between the different possible compositions (see Section 2.1). It turns out that the diassociative operad Diass and the dendriform operad Dend are both binary quadratic Koszul operads and that they are Koszul duals to each other (see Section 2.3). An enumerative consequence of this duality is that the Hilbert series $\mathcal{H}_{\mathsf{Diass}}$ and $\mathcal{H}_{\mathsf{Dend}}$ of these two operads are Lagrange inverse to each other:

$$\mathcal{H}_{\mathsf{Diass}}\big(-\mathcal{H}_{\mathsf{Dend}}(-t)\big) = t,$$

$$\text{where} \qquad \mathcal{H}_{\mathsf{Diass}}(t) = \sum_{p\geq 1} p\, t^p = \frac{t}{1-t^2} \qquad \text{and} \qquad \mathcal{H}_{\mathsf{Dend}}(t) = \sum_{p\geq 1} C_p\, t^p = \frac{1-\sqrt{1-4t}}{2t} - 1$$

where $C_p := \frac{1}{p+1}\binom{2p}{p}$ denotes the p-th Catalan number.

Combinatorial models for the free algebras on the diassociative and dendriform operads were described in [Lod01, LR98]. The free diassociative algebra is described in [Lod01] in terms of middle entries in monomials. The free dendriform algebra was described by J.-L. Loday and M. Ronco in [LR98] as an algebra on planar binary trees. Other important dendriform algebras include the shuffle algebra and the algebra of permutations of C. Malvenuto and C. Reutenauer [MR95] (also called free quasi-symmetric functions FQSym), where the classical shuffle product splits into two partial shuffles according to the provenance of the first letter:

$$\sqcup\!\sqcup = \prec + \succ, \qquad \text{where} \qquad xX \prec yY = x(X \sqcup\!\sqcup\, yY) \qquad \text{and} \qquad xX \succ yY = y(xX \sqcup\!\sqcup\, Y).$$

Further splitting of associative products. Several generalizations of dendriform algebras were recently introduced and studied, including tridendriform algebras [LR04], quadri-algebras [AL04, Foi15], ennea-algebras [Ler04], m-dendriform algebras [Nov14], polydendriform algebras [Gir16a, Gir16c], and k-twistiform algebras [Pil18]. These structures rely on various decompositions of an associative product into several operators, often motivated by the combinatorics of the shuffle product. For instance, there are two natural ways to split further the shuffle product into four partial shuffles, according to the provenance of:

(i) the first and last letters, defining a quadri-algebra structure of [AL04, Foi15], or
(ii) the first two letters, defining a 2-twistiform algebra structure of [Pil18].

While the corresponding Quad and Twist operads are not isomorphic, it turns out that they have the same Hilbert series, which is the Lagrange inverse of $\sum_{p\geq 1} p^2\, t^p$. In fact, this equi-enumeration extends to two infinite families of operads generalizing the Dend, Quad and Twist operads:

(i) the k-th Manin power of the dendriform operad (Quad is the Manin square of Dend),
(ii) the k-twistiform operads defined in [Pil18].

Their Hilbert series coincide and is the Lagrange inverse of $\sum_{p\geq 1} p^k\, t^p$. The objective of this paper is to explore further the twistiform operads in order to find a good algebraic interpretation of this equi-enumeration.

k-signaletic operads. Our approach via the Koszul duals of all these operads offers an enlightening perspective. The first step is to come back to J.-L. Loday's original interpretation of the diassociative operad. Consider a syntax tree $\mathfrak{t}$ on $\{\prec, \succ\}$ and imagine that a car starts at the root of $\mathfrak{t}$ and follows the signs in the nodes of $\mathfrak{t}$ until it reaches a leaf. The diassociative operad is then the quotient of the free operad on $\{\prec, \succ\}$ by the equivalence relation identifying syntax trees of the same arity in which the car reaches the same destination (see Section 3.1).

This interpretation then naturally generalizes to operads with 2^k operations $\{\prec, \succ\}^k$. Namely, for a syntax tree on $\{\prec, \succ\}^k$, we imagine that k cars start at the root of $\mathfrak{t}$ and follow the signs in the nodes of $\mathfrak{t}$ until they reach the leaves. We define the *k-signaletic operad* as the quotient of the free operad on $\{\prec, \succ\}^k$ by the equivalence relation identifying syntax trees with the same arity and in which the k cars reach the same destinations (see Section 3). As the equivalence classes are determined by the destinations of the k cars, and all destination vectors are possible, the Hilbert series of the k-signaletic operad is obviously given by $\sum_{p \ge 1} p^k \, t^k$.

Of course, the destinations of the cars crucially depend on the *circulation rule* telling each car which signal it should consider in each node. There are two natural possible circulation rules:

(i) *Parallel*: All k cars start together, and the i-th car always follows the indication given by the i-th letter of the traffic signal at each node.
(ii) *Series*: The k cars start one after the other, and each car always follows the indication given by the leftmost remaining letter of the traffic signal at each node and erases it.

This results in two non-isomorphic but obviously equi-enumerated versions of the k-signaletic operad. The parallel version is the k-th Manin power of the diassociative operad, while the series version is a somewhat twisted k-th Manin power of the diassociative operad.

Besides the two possible circulation rules, we can also impose two *ordering rules*:

(i) *Tidy*: All signals not used by a car must point to the left,
(ii) *Messy*: No further constraints on the remaining signals.

Again, the resulting operads are non-isomorphic but equi-enumerated. For instance, when $k = 1$, the tidy signaletic operad is an analogue of the dual duplicial operad, while the messy signaletic operad is the diassociative operad. We therefore obtain four versions of the k-signaletic operad: $\mathsf{TSig}_k^{\parallel}$, $\mathsf{MSig}_k^{\parallel}$, $\mathsf{TSig}_k^{\ddagger}$, $\mathsf{MSig}_k^{\ddagger}$ (parallel or series, tidy or messy).

It turns out that these k-signaletic operads are all quadratic and Koszul operads (see Section 3.4). Remarkably, our proof is independent of which version we consider. Namely, we orient the k-signaletic quadratic relations according to the k-signaletic Tamari lattice, an order generalizing the classical Tamari lattice to syntax trees on $\{\prec, \succ\}^k$. We then show that the resulting k-signaletic rewriting system converges to certain right k-signaletic combs. The Koszulity then follows from a classical result of the operad theory.

k-citelangis operads. At that stage, it is natural to consider the Koszul duals of the k-signaletic operads, which we call *k-citelangis operads* (see Section 4). Again, k-citelangis operads arise in four flavours: $\mathsf{TCit}_k^{\parallel}$, $\mathsf{MCit}_k^{\parallel}$, $\mathsf{TCit}_k^{\ddagger}$, $\mathsf{MCit}_k^{\ddagger}$ (parallel or series, tidy or messy). From their presentation, one recognizes relevant operads:

(i) the messy parallel k-citelangis operad is the k-th Manin power of the dendriform operad,
(ii) the tidy parallel k-citelangis operad is the k-th Manin power of a twisted duplicial operad,
(iii) the messy series k-citelangis operad is a twisted k-th Manin power of the dendriform operad, and coincides with the k-twistiform operad of [Pil18],
(iv) the tidy series k-citelangis operad is a twisted k-th Manin power of a twisted duplicial operad.

While the k-citelangis operads are not isomorphic, they are all equi-enumerated: their Hilbert series is the Lagrange inverse of $\sum_{p \ge 1} p^k \, t^p$ (see Section 4.3.1). This gives in particular the algebraic interpretation of the equi-enumeration between the Quad and Twist operads observed earlier.

By Koszul duality, it is not difficult to orient the quadratic relations in these operads into a converging k-citelangis rewriting system. The normal forms of this system are syntax trees where each node imposes some constraints only to its right child. Therefore, the Hilbert series of the k-citelangis operads can be interpreted as a fixed point of the substitution into the generating function of the right combs that are in normal form. Moreover, the latter has a natural expression in terms of the power of the transition matrix M_k of the automaton where each operator of $\{\prec, \succ\}^k$ points towards its possible children. While simple to express, the resulting matrices M_k seem very interesting (see Section 4.3.3): for instance, we conjecture that their minimal polynomials are multiples of the Eulerian polynomial $\mathrm{Eul}_k(t) \coloneqq \sum_j \left\langle {k \atop j} \right\rangle t^{j+1}$ (where the Eulerian number $\left\langle {k \atop j} \right\rangle$ is the number of permutations of $\mathfrak{S}_k$ with precisely j descents). These polynomials are ubiquitous in this paper since

$$\sum_{p \ge 1} p^k \, t^p = \frac{\mathrm{Eul}_k(t)}{(1-t)^{k+1}} \, .$$

Actions on multipermutations and free k-citelangis algebras. Once the presentations of (the four versions of) k-citelangis operads are established, we look for natural actions of these operads and combinatorial models for their free algebras (see Section 5).

As already mentioned, this has been done for the Dend and Quad operads. Namely, the algebra of permutations FQSym has a dendriform algebra structure where the shuffle product decomposes into two operations depending on the provenance of the first letter of the result [Lod01], and a quadri-algebra structure where the shuffle product decomposes into four operations depending on the provenance of the first and last letters of the result [AL04]. Moreover, it was shown that the resulting algebras are free [Lod01, Foi15, Von15], so that the subalgebras generated by the permutation 1 for Dend and the permutation 12 for Quad provide combinatorial models for the corresponding operads. Unfortunately, the parallel world is quite limited: these actions cannot be extended to messy parallel k-citelangis structures for $k \ge 2$ because a permutation has only two ends!

In contrast, FQSym has a natural messy series k-citelangis algebra structure for any $k \ge 1$, where the shuffle product decomposes into 2^k operations depending on the provenance of the first k letters of the result [Pil18]. Nevertheless, the question of the freeness of the resulting algebras were left widely open in [Pil18], and we were lacking combinatorial models for the k-twistiform operads. This paper settles these two problems.

To start with, we need to revisit the combinatorics of the parallel 2-citelangis operads (see Section 5.1). We first observe that (a version of) the tidy parallel 2-citelangis operad also naturally acts on permutations: the four operators act by concatenation, except that they select the first and last letters depending on their first and last signs. The resulting tidy parallel 2-citelangis algebra is free on the permutations that admit no bounded cut, *i.e.* a value γ such that except the first and last letters, all letters smaller than γ arrive before all letters larger than γ in the permutation. We then observe a triangularity relation between the tidy and messy parallel 2-citelangis operations on permutations: namely, the result of a tidy operation is given by the lexicographically minimal permutation among the result of the corresponding messy operation. This triangularity transports the freeness of the tidy parallel 2-citelangis algebra to that of the messy parallel 2-citelangis algebra on permutations, thus providing an alternative proof of the results of [Foi15, Von15].

This approach to the parallel 2-citelangis operads provides a solid prototype to tackle the series k-citelangis operads (see Section 5.2). We indeed define a similar action of the tidy series k-citelangis operad on k-permutations (*i.e.* permutations of a multiset where each value is repeated k times). The resulting tidy series k-citelangis algebra is free on the k-permutations that admit no k-rooted cut, *i.e.* a value γ such that except the first k letters, all letters smaller than γ arrive before all letters larger than γ in the permutation. By the same triangularity argument, we obtain that the messy series k-citelangis algebra on permutations defined in [Pil18] is free.

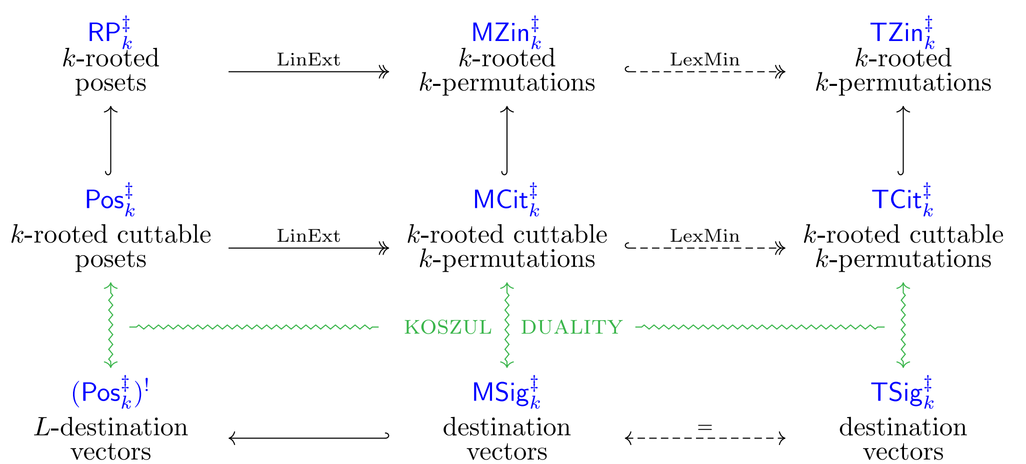
Operations on posets and poset operads. As the messy k-citelangis operations on permutations are defined as partial shuffle products, their results are sums over all linear extensions of certain k-posets (*i.e.* partial orders on a multiset where each value is repeated k times). The k-posets are bounded for the parallel action, and k-rooted for the series action (meaning that they have a chain of k successive minimal elements). The messy k-citelangis operations can be described directly on these k-posets, but the resulting operations do not satisfy all messy k-citelangis relations. It motivates the introduction of operads $\mathsf{BP}_2^{\parallel}$ on bounded 2-posets and $\mathsf{RP}_k^{\ddagger}$ on k-rooted k-posets defined by these operations (see Sections 5.1.5 and 5.2.5). The resulting operads are quadratic and Koszul, with a unique quadratic relation. The Hilbert series of their Koszul dual is thus $t + 2^k t^2 + t^3$, which by Lagrange inversion enables us to compute their own Hilbert series.

k-Zinbiel operads. Finally, motivated by the actions of the series k-citelangis operads on k-rooted cuttable k-permutations, we define the series *k-Zinbiel operad* $\mathsf{MZin}_k^{\ddagger}$ on all k-rooted k-permutations (see Section 5.2.7). This operad completes our operad zoo: the linear extensions map provides a natural morphism from the k-rooted poset operad $\mathsf{RP}_k^{\ddagger}$ to the k-Zinbiel operad $\mathsf{MZin}_k^{\ddagger}$, and the messy series k-citelangis operad $\mathsf{MCit}_k^{\ddagger}$ is a suboperad of the k-Zinbiel operad $\mathsf{MZin}_k^{\ddagger}$. We also study a similar operad in the parallel situation when $k = 2$.

Open questions. Our work motivates many interesting research directions and open questions, some of which are briefly discussed in Section 7. A short summary:

- Are there general notions of twisted Manin products of operads that would enlighten series signaletic and citelangis operads in a similar way the classical Manin products explain the parallel signaletic and citelangis operads?
- What are the properties of the k-Zinbiel operads as symmetric operads? Are they Koszul? What are their duals?
- What is the natural generalization of tridendriform algebras in parallel and series? Is there a natural interpretation in terms of traffic rules for their Koszul dual operads?
- Is there a relevant generalization of the duplicial operad in the series situation? While the parallel situation allows quite some freedom to extend both the duplicial and twisted duplicial operads, the series situation seems more limited, but might reveal an interesting operad.
- Is it possible to interpolate between the tidy and messy versions of our operads with q-analogues, in the same way the q-deformation of the shuffle product interpolates between the classical shuffle and the concatenation products?

Diagrammatic summary. Finally, we believe that the following diagram gives a good summary of the combinatorial maps between all the series operads (a similar diagram holds for the parallel operads when $k = 2$). In this diagram, plain arrows are operad morphisms, while dashed arrows are just bijections of normal forms. The different operads and combinatorial objects in this diagram should become clear along the text.



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2. Preliminaries

This section recalls some algebraic and combinatorial preliminaries and fixes notations. To start with, for an integer n, we denote $[n] := \{1, 2, \dots, n\}$.

2.1. **Operads.** An operad is an algebraic structure abstracting a type of algebras. We give here the formal definitions needed in this paper. We refer to [LV12, Mar08] for classical references on operads, and to [Cha08, Gir18, Gir11] for more combinatorial approches. Note that we work with non-symmetric operads.

Definition 2.1. *A (unital, non-symmetric) operad (in the category of vector spaces) is a graded $\mathbb{K}$-vector space $\mathcal{O} = \bigoplus_{p\geq 1} \mathcal{O}(p)$ endowed with a distinguished element $\mathbb{1} \in \mathcal{O}(1)$ (called unit of $\mathcal{O}$) and linear maps (called partial compositions of $\mathcal{O}$)*

$$\circ_i : \mathcal{O}(p) \otimes \mathcal{O}(q) \to \mathcal{O}(p+q-1) \qquad \text{for } p, q \geq 1 \text{ and } i \in [p]$$

such that the following three relations hold:

(unitality)	$\mathbb{1} \circ_1 \mathfrak{p} = \mathfrak{p} = \mathfrak{p} \circ_i \mathbb{1}$	*for all* $\mathfrak{p} \in \mathcal{O}(p),\ i \in [p]$,
(series comp.)	$(\mathfrak{p} \circ_i \mathfrak{q}) \circ_{i+j-1} \mathfrak{r} = \mathfrak{p} \circ_i (\mathfrak{q} \circ_j \mathfrak{r})$	*for all* $\mathfrak{p} \in \mathcal{O}(p),\ \mathfrak{q} \in \mathcal{O}(q),\ \mathfrak{r} \in \mathcal{O}(r),\ i \in [p],\ j \in [q]$,
(parallel comp.)	$(\mathfrak{p} \circ_i \mathfrak{q}) \circ_{j+q-1} \mathfrak{r} = (\mathfrak{p} \circ_j \mathfrak{r}) \circ_i \mathfrak{q}$	*for all* $\mathfrak{p} \in \mathcal{O}(p),\ \mathfrak{q} \in \mathcal{O}(q),\ \mathfrak{r} \in \mathcal{O}(r),\ i < j \in [p]$.

The elements of $\mathcal{O}(p)$ are called operations of arity p.

Alternatively, one can define the operad $\mathcal{O}$ by its *compositions maps*

$$\circ : \mathcal{O}(p) \otimes \big(\mathcal{O}(q_1) \otimes \cdots \otimes \mathcal{O}(q_p)\big) \to \mathcal{O}(q_1 + \cdots + q_p) \qquad \text{for } p, q_1, \dots, q_p \geq 1$$

linearly defined for $\mathfrak{p} \in \mathcal{O}(p)$ and $\mathfrak{q}_1 \in \mathcal{O}(q_1), \dots, \mathfrak{q}_p \in \mathcal{O}(q_p)$ by

$$\mathfrak{p} \circ (\mathfrak{q}_1 \otimes \cdots \otimes \mathfrak{q}_p) = (((\mathfrak{p} \circ_p \mathfrak{q}_p) \circ_{p-1} \mathfrak{q}_{p-1}) \dots) \circ_1 \mathfrak{q}_1.$$

(Note that partial compositions can be retrieved from the global composition maps using the unit).

Definition 2.2. *An operad morphism is a unital graded linear map $\phi : \mathcal{O} \to \mathcal{O}'$ between two operads which commutes with partial compositions:*

$$\phi(\mathfrak{p} \circ_i \mathfrak{q}) = \phi(\mathfrak{p}) \circ_i \phi(\mathfrak{q}) \text{ for all } \mathfrak{p} \in \mathcal{O}(p), \mathfrak{q} \in \mathcal{O}(q) \text{ and } i \in [p].$$

Definition 2.3. *An operad ideal of $\mathcal{O}$ is a graded subspace $\mathcal{I} = \bigoplus_{p\geq 1} \mathcal{I}(p)$ of $\mathcal{O}$ such that $\mathfrak{p} \circ_i \mathfrak{q} \in \mathcal{I}$ and $\mathfrak{q} \circ_j \mathfrak{p} \in \mathcal{I}$ for any $\mathfrak{p} \in \mathcal{O}(p)$, $\mathfrak{q} \in \mathcal{I}(q)$, $i \in [p]$ and $j \in [q]$.*

One can then define as usual the quotient $\mathcal{O}/\mathcal{I}$ which is itself an operad. We denote by $\langle \mathfrak{R} \rangle$ the operad ideal generated by a graded subspace $\mathfrak{R}$ of $\mathcal{O}$, *i.e.* the smallest operad ideal of $\mathcal{O}$ containing $\mathfrak{R}$.

Definition 2.4. *The Hilbert series of an operad $\mathcal{O}$ is the formal power series $\mathcal{H}_{\mathcal{O}}(t)$ encoding the dimensions of $\mathcal{O}(p)$ for $p \geq 1$:*

$$\mathcal{H}_{\mathcal{O}}(t) := \sum_{p\geq 1} \dim \mathcal{O}(p)\, t^p.$$

Definition 2.5. *An algebra over an operad $\mathcal{O}$ is a $\mathbb{K}$-vector space Alg endowed with an action $\cdot : \mathcal{O}(p) \otimes \mathsf{Alg}^{\otimes p} \to \mathsf{Alg}$ satisfying the relations imposed by the compositions in $\mathcal{O}$, meaning that for all $\mathfrak{p} \in \mathcal{O}(p)$, $\mathfrak{q} \in \mathcal{O}(q)$, $a_1 \otimes \cdots \otimes a_{p+q-1} \in \mathsf{Alg}^{\otimes p+q-1}$ and $i \in [p]$,*

$$(\mathfrak{p} \circ_i \mathfrak{q}) \cdot (a_1 \otimes \cdots \otimes a_{p+q-1}) = \mathfrak{p} \cdot (a_1 \otimes \cdots \otimes a_{i-1} \otimes \mathfrak{q} \cdot (a_i \otimes \cdots \otimes a_{i+q-1}) \otimes a_{i+q} \otimes \cdots \otimes a_{p+q-1}).$$

In other words, the elements of the operad $\mathcal{O}$ are linear operations on Alg, and the partial compositions on $\mathcal{O}$ correspond to partial compositions in Alg. In particular, there is a notion of free algebra over $\mathcal{O}$.

Definition 2.6. *The free algebra over an operad $\mathcal{O}$ generated by a set S is the $\mathbb{K}$-vector space $\mathsf{FAlg} := \bigoplus_{p\geq 1} \mathcal{O}(p) \otimes \mathbb{K}S^{\otimes p}$ endowed with the action $\cdot : \mathcal{O}(p) \otimes \mathsf{FAlg}^{\otimes p} \to \mathsf{FAlg}$ defined for all $\mathfrak{p} \in \mathcal{O}(p)$, $\mathfrak{q}_1 \in \mathcal{O}(q_1), \dots, \mathfrak{q}_p \in \mathcal{O}(q_p)$, and $v_1 \in \mathbb{K}S^{\otimes q_1}, \dots, v_p \in \mathbb{K}S^{\otimes q_p}$ by*

$$\mathfrak{p} \cdot \big((\mathfrak{q}_1 \otimes v_1) \otimes \cdots \otimes (\mathfrak{q}_p \otimes v_p)\big) := \big(\mathfrak{p} \circ (\mathfrak{q}_1 \otimes \cdots \otimes \mathfrak{q}_p)\big) \otimes (v_1 \otimes \cdots \otimes v_p).$$

In particular, the free algebra over $\mathcal{O}$ generated by a singleton is the same vector space as the operad $\mathcal{O}$ itself.

2.2. **Syntax trees, free operads, presentations, and rewriting systems.** Operads are about ways to compose operations. To manipulate these compositions and the relations between them, we need the fundamental tool of syntax trees.

Definition 2.7. *A syntax tree over a graded set $\mathfrak{B} := \bigsqcup_{p \geq 1} \mathfrak{B}(p)$ is a rooted planar tree $\mathfrak{t}$ where each internal node of arity p is labeled by an element of $\mathfrak{B}(p)$. The arity of $\mathfrak{t}$ is its number of leaves. We denote by $\mathbf{Trees}(\mathfrak{B})(p)$ the set of syntax trees of arity p and $\mathbf{Trees}(\mathfrak{B}) := \bigsqcup_{p \geq 1} \mathbf{Trees}(\mathfrak{B})(p)$.*

Definition 2.8. *The free operad over a graded set $\mathfrak{B} := \bigsqcup_{p \geq 1} \mathfrak{B}(p)$ is the operad*

$$\mathbf{Free}(\mathfrak{B}) := \bigoplus_{p \geq 1} \mathbf{Free}(\mathfrak{B})(p),$$

where:

- *the $\mathbf{Free}(\mathfrak{B})(p)$ is the $\mathbb{K}$-vector space generated by syntax trees on $\mathfrak{B}$ of arity p,*
- *the partial composition $\circ_i$ is the linear map defined on two syntax trees $\mathfrak{s}$ and $\mathfrak{t}$ by grafting the root of $\mathfrak{t}$ on the i-th leaf of $\mathfrak{s}$, and extended by linearity to $\mathbf{Free}(\mathfrak{B})$, and*
- *the unit is the tree with no internal node and only one leaf.*

As its name suggests, the free operad is the free object in the category of non-symmetric operads so that any operad can be obtained by a quotient of a free operad.

Definition 2.9. *A presentation of an operad $\mathcal{O}$ is a pair $(\mathfrak{B}, \mathfrak{R})$ where $\mathfrak{B} := \bigsqcup_{p \geq 1} \mathfrak{B}(p)$ is a graded set and $\mathfrak{R}$ is a subspace of the free operad $\mathbf{Free}(\mathfrak{B})$ such that $\mathcal{O}$ is isomorphic to the quotient $\mathbf{Free}(\mathfrak{B})/\langle\mathfrak{R}\rangle$. The elements of $\mathfrak{B}$ are called generators while the elements of $\mathfrak{R}$ are called relations of the presentation.*

Definition 2.10. *An operad is called binary if it admits a presentation $(\mathfrak{B}, \mathfrak{R})$ in which the generators $\mathfrak{B}$ all have arity 2. A binary operad is quadratic if it admits a presentation $(\mathfrak{B}, \mathfrak{R})$ in which the relations $\mathfrak{R}$ only involve trees with two nodes (that is operad elements of arity 3).*

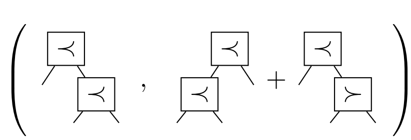
In this paper, we only work with binary quadratic operads. To compute with presentations, it is often useful to use rewriting.

Definition 2.11. *A quadratic rewriting rule is a pair $(\mathfrak{s}, \mathfrak{q})$ formed by a syntax tree $\mathfrak{s}$ of $\mathbf{Trees}(\mathfrak{B})(3)$ and a linear combination $\mathfrak{q}$ in $\mathbf{Free}(\mathfrak{B})(3)$. A quadratic rewriting system over $\mathfrak{B}$ is a collection of quadratic rewriting rules. We say that a tree $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B})$ rewrittes into an element $\mathfrak{p} \in \mathbf{Free}(\mathfrak{B})$ by this system and we write $\mathfrak{t} \to \mathfrak{p}$ if there exists an integer i, a tree $\mathfrak{u}$ of arity at least i, a rule $(\mathfrak{s}, \mathfrak{q})$, and a triple of trees $(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)$ such that*

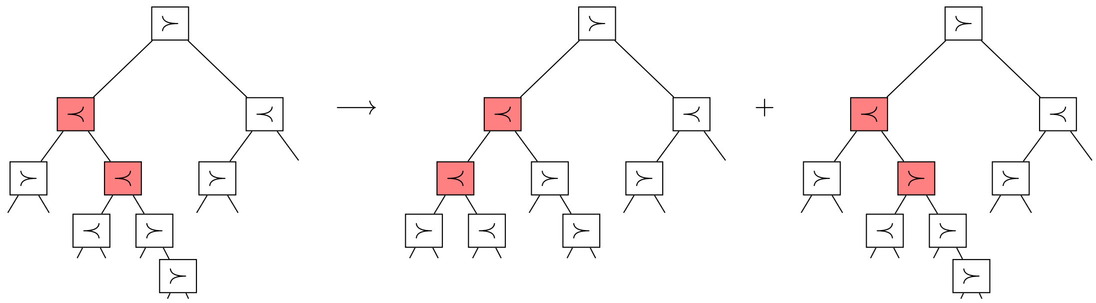
$$\mathfrak{t} = \mathfrak{u} \circ_i \big(\mathfrak{s} \circ (\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)\big) \qquad \text{and} \qquad \mathfrak{p} = \mathfrak{u} \circ_i \big(\mathfrak{q} \circ (\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)\big).$$

We say that $\mathfrak{s}$ is the pattern that was rewritten to $\mathfrak{q}$. Extending by linearity and abusing notation we still write $\mathfrak{p} \to \mathfrak{q}$ this binary relation on $\mathbf{Free}(\mathfrak{B})$. We moreover write $\overset{*}{\to}$ its reflexive and transitive closure.

Example 2.12. We show here an example of an application of a rewriting rule. We fix $\mathfrak{B} := \{\prec, \succ\}$ and we suppose that the system contains the following rewriting rule:



Then the following is a possible rewriting:



Definition 2.13. *A* normal form *of a rewriting system is a tree that is not rewritable by the rewriting system, which means that it does not contain any pattern* $\mathfrak{s}$ *of a rewriting rule* $(\mathfrak{s}, \mathfrak{q})$.

Definition 2.14. *A rewriting system is*

- terminating *if there are no infinite rewriting sequences, so that any tree can be finitely rewritten into a linear combination of normal forms,*
- confluent *if for any* $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B})$ *and* $\mathfrak{p}, \mathfrak{q} \in \mathbf{Free}(\mathfrak{B})$ *such that* $\mathfrak{t} \to \mathfrak{p}$ *and* $\mathfrak{t} \to \mathfrak{q}$, *there exists* $\mathfrak{r} \in \mathbf{Free}(\mathfrak{B})$ *such that* $\mathfrak{p} \xrightarrow{*} \mathfrak{r}$ *and* $\mathfrak{q} \xrightarrow{*} \mathfrak{r}$, *so that any tree rewrites to at most one combination of normal forms,*
- convergent *when it is both terminating and confluent, so that any tree rewrites as a unique linear combination of normal forms.*

Any convergent rewriting system defines a presentation where the relation space $\mathfrak{R}$ is spanned by $\mathfrak{s} - \mathfrak{q}$ for all rules $(\mathfrak{s}, \mathfrak{q})$ in the system. In this case, the quotient $\mathbf{Free}(\mathfrak{B})/\langle\mathfrak{R}\rangle$ can be identified with the linear span of the normal forms of the rewriting system.

2.3. **Koszulity and Koszul duality.** Koszul operads where defined in term of a certain complex. In this paper we will only use the technique introduced by [Hof10] consisting in exhibiting a so-called Poincaré – Birkhoff – Witt basis. Moreover, we will only use this tools in the specific context of set operads (*i.e.* where the right hand side of each rewriting rule consists in a single tree). This allows the following reformulation of [DK10] (see also [Gir16b]), which we take as definition.

Definition 2.15. *A set operad* $\mathcal{O}$ *is* Koszul *if it admits a quadratic presentation whose relations can be oriented into a convergent rewriting system. Moreover, the set of normal forms of the rewriting system is called a* Poincaré – Birkhoff – Witt basis *of* $\mathcal{O}$.

Next comes a notion of Koszul dual $\mathcal{O}^!$ of a quadratic operad $\mathcal{O}$, which has particularly nice features when $\mathcal{O}$ is Koszul. Consider the free operad $\mathbf{Free}(\mathfrak{B})$ on $\mathfrak{B}$. The homogeneous component of degree 3 of $\mathbf{Free}(\mathfrak{B})$ can be endowed with a scalar product $\langle\cdot|\cdot\rangle$ defined by:

$$\langle\, \mathfrak{a}_1 \circ_i \mathfrak{a}_2 \,|\, \mathfrak{b}_1 \circ_j \mathfrak{b}_2 \,\rangle := \begin{cases} 1 & \text{if } i = j = 1 \text{ and } \mathfrak{a}_1 = \mathfrak{b}_1, \mathfrak{a}_2 = \mathfrak{b}_2, \\ -1 & \text{if } i = j = 2 \text{ and } \mathfrak{a}_1 = \mathfrak{b}_1, \mathfrak{a}_2 = \mathfrak{b}_2, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.16. *The* Koszul dual *of a quadratic operad* $\mathcal{O}$ *presented by* $(\mathfrak{B}, \mathfrak{R})$ *is the quadratic operad* $\mathcal{O}^!$ *presented by* $(\mathfrak{B}, \mathfrak{R}^!)$, *whose relations* $\mathfrak{R}^!$ *are given by the orthogonal complement for* $\langle\cdot|\cdot\rangle$ *of the relations* $\mathfrak{R}$ *in* $\mathbf{Free}(\mathfrak{B})$.

Remark 2.17. A particularly relevant setting is that of set operads. Namely, assume that the set of relations is given by $\mathfrak{R} := \mathfrak{R}_1 \cup \mathfrak{R}_2$ where $\mathfrak{R}_1 \subseteq \{\mathfrak{a} \circ_i \mathfrak{b} = 0 \mid \mathfrak{a}, \mathfrak{b} \in \mathfrak{B},\ i \in [2]\}$ and $\mathfrak{R}_2 \subseteq \{\mathfrak{a} \circ_1 \mathfrak{b} = \mathfrak{c} \circ_2 \mathfrak{d} \mid \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathfrak{B}\}$. Then $\mathfrak{R}^! = \bar{\mathfrak{R}}_1 \cup \mathfrak{R}_2$ where $\bar{\mathfrak{R}}_1$ is the complement of $\mathfrak{R}_1$ in $\{\mathfrak{a} \circ_i \mathfrak{b} = 0 \mid \mathfrak{a}, \mathfrak{b} \in \mathfrak{B},\ i \in [2]\}$. In other words, to get the Koszul dual of an operad with such a set of relations, we keep the relations of the form "left tree = right tree" and we complement the relations of the form "tree = 0".

The dual of a Kozul operad is always Koszul. Moreover, the Hilbert series of Koszul dual Koszul operads are related by Lagrange inversion as stated in the following theorem.

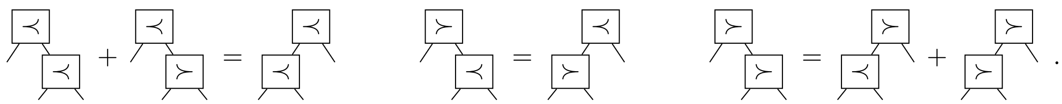
Theorem 2.18. *The Hilbert series of two Koszul dual Koszul operads* $\mathcal{O}$ *and* $\mathcal{O}^!$ *satisfy*

$$\mathcal{H}_{\mathcal{O}}(-\mathcal{H}_{\mathcal{O}^!}(-t)) = t.$$

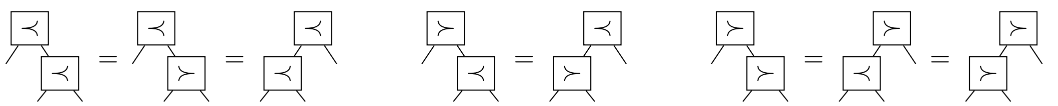
2.4. **Six particular operads.** This paper deals with generalizations of the following three specific pairs of Koszul dual operads.

2.4.1. *Dendriform and diassociative operads.*

Definition 2.19. *The* dendriform operad Dend *is the quadratic operad over* $\mathfrak{B} := \{\prec, \succ\}$ *defined by the three linear relations:*



Definition 2.20. *The* diassociative operad Diass *is the quadratic operad over* $\mathfrak{B} := \{\prec, \succ\}$ *defined by the five relations:*



Proposition 2.21 ([Lod01, LV12])**.** *(i) The dendriform and diassociative operads are Koszul.*

(ii) The dendriform and diassociative operads are Koszul dual operads.

(iii) The diassociative operad is isomorphic to the operad $(\mathcal{P}, \mathbb{1}, (\circ_i))$ *with p-th homogeneous component* $\mathcal{P}(p) := \mathbb{K}\{(\!| r |\!)_p \mid r \in [p]\}$*, neutral element* $\mathbb{1} = (\!| 1 |\!)_1$ *and where the composition* $\circ_i$ *is defined by*

$$(\!| r |\!)_p \circ_i (\!| s |\!)_q = \begin{cases} (\!| r |\!)_{p+q-1} & \text{if } r < i, \\ (\!| r+s-1 |\!)_{p+q-1} & \text{if } r = i, \\ (\!| r+q-1 |\!)_{p+q-1} & \text{if } r > i. \end{cases}$$

(iv) The Hilbert series of the dendriform and diassociative operads are given by

$$\mathcal{H}_{\mathsf{Dend}}(t) = \sum_{p \ge 1} C_p \, t^p = \frac{1 - \sqrt{1-4t}}{2t} - 1 \qquad \text{and} \qquad \mathcal{H}_{\mathsf{Diass}}(t) = \sum_{p \ge 1} p \, t^p = \frac{t}{(1-t)^2},$$

where $C_p := \frac{1}{p+1}\binom{2p}{p}$ *denotes the p-th Catalan number.*

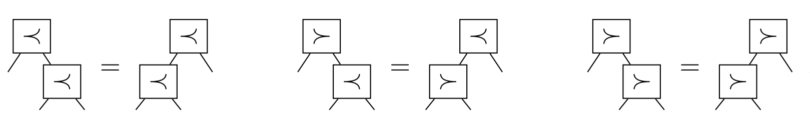
Remark 2.22. The shuffle algebra (see Definition 2.49) can be endowed with a structure of dendriform algebra [Lod01] defined for two words xX and yY by

$$xX \prec yY = x(X \sqcup\!\sqcup yY) \qquad \text{and} \qquad xX \succ yY = y(xX \sqcup\!\sqcup Y).$$

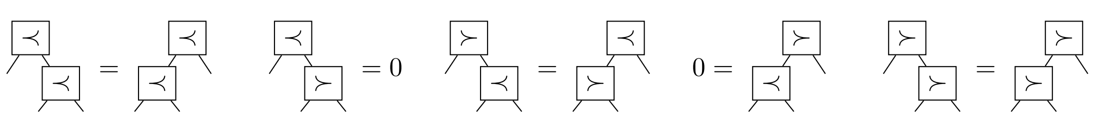
Similarly, C. Malvenuto and C. Reutenauer's Hopf algebra FQSym on permutations can be endowed with a structure of dendriform algebra by splitting the shifted shuffle product (see Definition 2.50).

2.4.2. *Duplicial and dual duplicial operads.*

Definition 2.23. *The* duplicial operad $\mathsf{Dup}^!$ *is the quadratic operad over* $\mathfrak{B} := \{\prec, \succ\}$ *defined by the three relations:*



Definition 2.24. *The* dual duplicial operad Dup *is the quadratic operad over* $\mathfrak{B} := \{\prec, \succ\}$ *defined by the five linear relations:*



Proposition 2.25 ([Lod06, LV12]). *(i) The duplicial and dual duplicial operads are Koszul.*

(ii) The duplicial and dual duplicial operads are Koszul dual operads.

(iii) The duplicial operad is isomorphic to the operad $(\mathcal{P}, \mathbb{1}, (\circ_i))$ with p-th homogeneous component $\mathcal{P}(p) := \mathbb{K}\{(\!| r |\!)_p \mid r \in [p]\}$, neutral element $\mathbb{1} = (\!| 1 |\!)_1$ and where the composition $\circ_i$ is defined by

$$(\!| r |\!)_p \circ_i (\!| s |\!)_q = \begin{cases} 0 & \text{if } r < i \text{ and } s \neq 1, \\ (\!| r |\!)_{p+q-1} & \text{if } r < i \text{ and } s = 1, \\ (\!| r+s-1 |\!)_{p+q-1} & \text{if } r = i, \\ 0 & \text{if } r > i \text{ and } s \neq q, \\ (\!| r+q-1 |\!)_{p+q-1} & \text{if } r > i \text{ and } s = q. \end{cases}$$

(iv) The Hilbert series of the duplicial and dual duplicial operads are given by

$$\mathcal{H}_{\mathsf{Dup}}(t) = \sum_{p \geq 1} C_p\, t^p = \frac{1 - \sqrt{1-4t}}{2t} - 1 \qquad \text{and} \qquad \mathcal{H}_{\mathsf{Dup}^!}(t) = \sum_{p \geq 1} p\, t^p = \frac{t}{(1-t)^2},$$

where $C_p := \frac{1}{p+1}\binom{2p}{p}$ denotes the p-th Catalan number.

Remark 2.26. Any operation $\mathfrak{b} := \mu \prec + \nu \succ$ with $\mu, \nu \in \mathbb{K}$ in the operad $\mathsf{Dup}^!$ is associative. In other words, $\prec$ and $\succ$ define a compatible associative structure.

2.4.3. *Leibniz and Zinbiel operads.* Leibniz algebras are natural generalizations of Lie algebras with non antisymmetric brackets, and Zinbiel algebras play a salient role in the analysis of divided power algebras [Lod06, LV12]. The Zinbiel operad is usually defined as the symmetric Koszul dual of the Leibniz symmetric operad. As a symmetric operad, it is generated by $\prec$ with the unique relation $(x \prec y) \prec z = x \prec (y \prec z + z \prec y)$. In this paper, we use the non-symmetric Zinbiel operad Zinb, defined combinatorially as follows [CHNT08].

Definition 2.27. *The (non-symmetric) Zinbiel operad Zinb is the operad whose arity n component $\mathsf{Zinb}(n)$ has for basis the set of permutations $\mathsf{Perm}(n)$ and where the composition is given as follows. Let σ and τ be two permutations of degree m and n and let $i \in [m]$. Write $\sigma = \lambda\, i\, \mu$ and $\tau = f\,\theta$ where f is the first letter. Then the composition is given by*

$$\sigma \circ_i \tau = \lambda[i,n]\, f[i-1]\, \big(\mu[i,n] \sqcup\!\sqcup\, \theta[i-1]\big)\,,$$

where the shuffle $\sqcup\!\sqcup$ is formally defined in Definition 2.49 and the usual operad shifting uses the notation of Definition 2.43 and Definition 2.44. In this combinatorial realization, the generator $\prec$ of the presentation is given by $\prec := 12$.

It was finally observed *e.g.* in [CHNT08, Example 4.2], that the dendriform operad is the suboperad of the (non-symmetric) Zinbiel operad (same $\mathsf{Zinb}(n)$ and composition but the action of the symmetric group is simply forgotten) generated by $\prec := 12$ and $\succ := 21$.

2.5. **Manin products.** The original Manin product where defined for quadratic algebras and generalized by B. Vallette [Val08] to any category endowed with two coherent monoidal products which include, as a particular case, non-symmetric operads. As an application, he showed that

$$\mathsf{Quad} = \mathsf{Dend} \,\square\, \mathsf{Dend} = \mathsf{Dend} \,\blacksquare\, \mathsf{Dend}.$$

We will generalize this statement to some of our operads in Sections 3.2 and 4.1. As the exposition in [Val08] is too general for our purposes, we follow the former version of [Foi15] still available on `arxiv:1411.6501v1`.

Consider two operads $\mathcal{O} := \mathbf{Free}(\mathfrak{B})/\langle \mathfrak{R} \rangle$ and $\mathcal{O}' := \mathbf{Free}(\mathfrak{B}')/\langle \mathfrak{R}' \rangle$. Then there is a natural inclusion $\Psi : \mathbf{Free}(\mathfrak{B} \otimes \mathfrak{B}') \to \mathbf{Free}(\mathfrak{B}) \otimes \mathbf{Free}(\mathfrak{B}')$, which sends a syntax tree on $\mathfrak{B} \otimes \mathfrak{B}'$ to a pair formed by a syntax tree on $\mathfrak{B}$ and a syntax tree on $\mathfrak{B}'$.

Definition 2.28. *The white Manin product $\mathcal{O} \,\square\, \mathcal{O}'$ is the operad with generators $\mathfrak{B} \otimes \mathfrak{B}'$ and relations $\mathfrak{R}'' \subseteq \mathbf{Free}(\mathfrak{B} \otimes \mathfrak{B}')$ defined by*

$$\mathfrak{R}'' := \Psi^{-1}\big(\mathfrak{R} \otimes \mathbf{Free}(\mathfrak{B}') + \mathbf{Free}(\mathfrak{B}) \otimes \mathfrak{R}'\big).$$

The black Manin product is defined by Koszul duality as $\mathcal{O} \,\blacksquare\, \mathcal{O}' := \big(\mathcal{O}^! \,\square\, \mathcal{O}'^!\big)^!$.

For quadratic operads the white product can be computed without relying on presentation by the following characterization, which is also closer to the original definition of V. Gizburg and M. Kapranov in the case of symmetric operads [GK94].

Proposition 2.29 ([arxiv:1411.6501v1, Proposition 2])**.** *For two quadratic operads $\mathcal{O}, \mathcal{O}'$, the white Manin product $\mathcal{O} \square \mathcal{O}'$ is the suboperad of the tensor operad $\mathcal{O} \otimes \mathcal{O}'$ generated by the homogeneous component of degree 2.*

2.6. **Eulerian numbers and polynomials.** We now focus on Eulerian numbers, which are underlying all the present study. Indeed, from a purely combinatorial point of view, the main result of the present paper is to give a combinatorial interpretation of the Lagrange inverse of the generating series $\sum_p p^k t^p$ which involve Eulerian numbers. These numbers were introduced by L. Euler in the context of differential calculus.

Definition 2.30. *For $j < k$, the* Eulerian number $\left\langle {k \atop j} \right\rangle$ *is the number of permutations σ of $\mathfrak{S}_k$ with precisely j descents (i.e. positions $i \in [k]$ such that $\sigma_i > \sigma_{i+1}$). The k-th* Eulerian polynomial *is the polynomial*

$$\mathrm{Eul}_k(t) := \sum_{j=0}^{k-1} \left\langle {k \atop j} \right\rangle t^{j+1} = \sum_{\sigma \in \mathfrak{S}_k} t^{|\operatorname{Des}(\sigma)|+1}.$$

Example 2.31. Table 1 gathers the first Eulerian numbers. The first Eulerian polynomials are:

$$\mathrm{Eul}_1(t) = t, \quad \mathrm{Eul}_2(t) = t^2 + t, \quad \mathrm{Eul}_3(t) = t^3 + 4t^2 + t, \quad \mathrm{Eul}_4(t) = t^4 + 11t^3 + 11t^2 + t, \quad \dots$$

We will use the following classical identity on Eulerian numbers.

Lemma 2.32 ([Wor83])**.** *For any $p, k \in \mathbb{N}$, we have $p^k = \displaystyle\sum_{j=0}^{k-1} \left\langle {k \atop j} \right\rangle \binom{j+p}{k}$.*

Proof. It is shown by induction using the identity $\left\langle {k \atop j} \right\rangle = (j+1)\left\langle {k-1 \atop j} \right\rangle + (k-j)\left\langle {k-1 \atop j-1} \right\rangle$, see *e.g.* [Sta12] (the insertion of k in a permutation of size $k-1$ creates a new descent if and only if it is performed at an ascent position). Another approach is to observe that the standardization provides a bijection between the words on $[p]$ with k letters and the pairs (σ, π) where σ is a permutations of $\mathfrak{S}_k$ and π a composition of p (with possible empty parts) refining the descent composition of σ. □

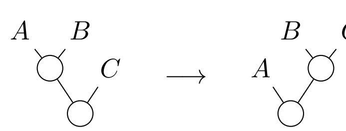
Lemma 2.32 translates to an identity of generating functions, and to a differential recursion.

Proposition 2.33. *For any $k \in \mathbb{N}$, we have $\displaystyle\sum_{p \geq 1} p^k t^p = \frac{\mathrm{Eul}_k(t)}{(1-t)^{k+1}}$.*

Proposition 2.34. *For any $k \in \mathbb{N}$, we have $\mathrm{Eul}_k(t) = t(1-t)\mathrm{Eul}'_{k-1}(t) + tk\mathrm{Eul}_{k-1}(t)$.*

2.7. **Tamari lattice.** The other classical combinatorial ingredient of the present paper is a lattice on the set of binary trees of a given arity, defined by D. Tamari [Tam51]. We recall some basic facts.

Definition 2.35. *A* right rotation *in a binary tree is a substitution of the form*



where A, B and C are three binary subtrees and the substitution is performed at any node of the tree (not necessarily at the root). The Tamari lattice *is the partial order on binary trees with n internal nodes whose cover relations are given by right rotations.*

The Tamari lattice is known to be a lattice. Its Hasse diagram (graph of cover relations) is the skeleton (graph of vertices and edges, see [Zie98]) of an $(n-1)$-dimensional polytope, called the associahedron, see *e.g.* [Lod04]. We have represented in Figure 1 the Tamari lattice for $n = 4$ and in Figure 2 J.-L. Loday's 2- and 3-dimensional associahedra.

$k \backslash j$	0	1	2	3	4	5	6	7	8	9
1	1									
2	1	1								
3	1	4	1							
4	1	11	11	1						
5	1	26	66	26	1					
6	1	57	302	302	57	1				
7	1	120	1191	2416	1191	120	1			
8	1	247	4293	15619	15619	4293	247	1		
9	1	502	14608	88234	156190	88234	14608	502	1	
10	1	1013	47840	455192	1310354	1310354	455192	47840	1013	1

TABLE 1. The Eulerian numbers $\left\langle {k \atop j} \right\rangle$ for $0 \le j < k \le 10$.

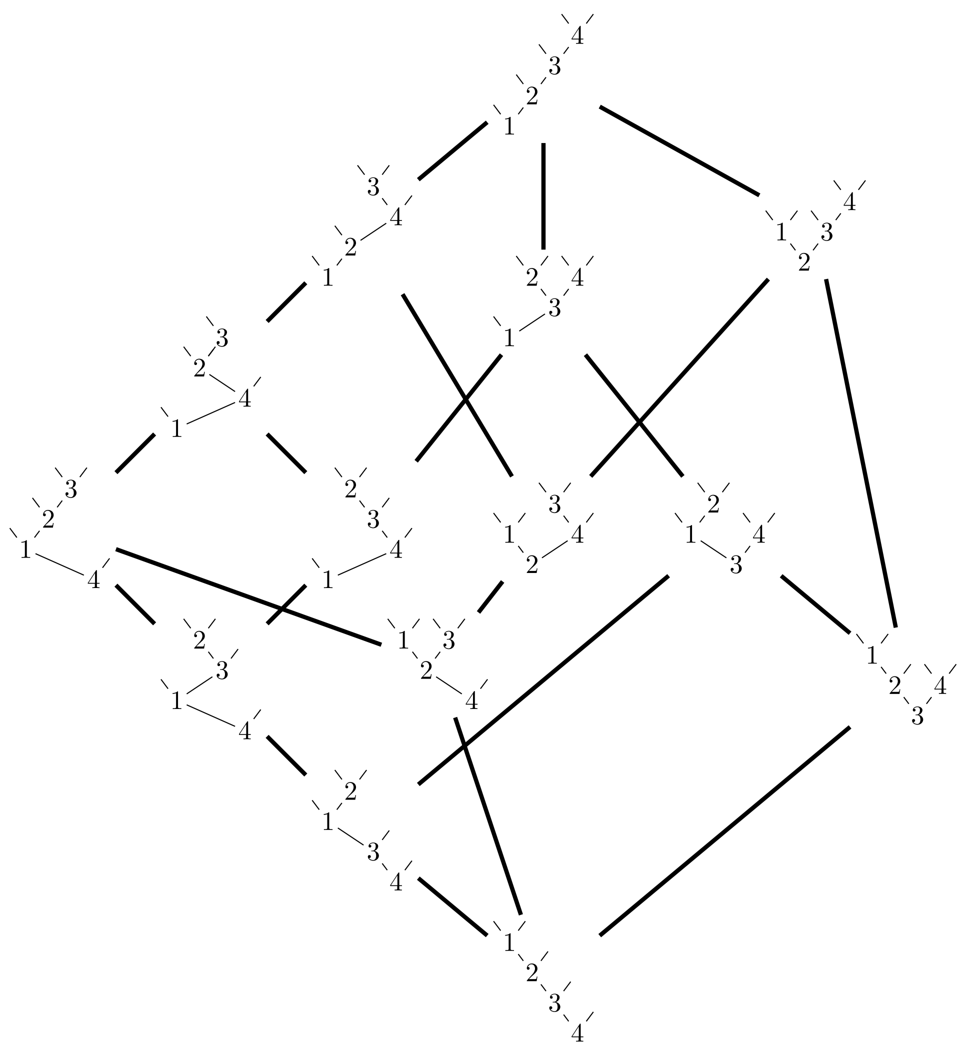


FIGURE 1. The Tamari lattice on binary trees. See Definition 2.35.

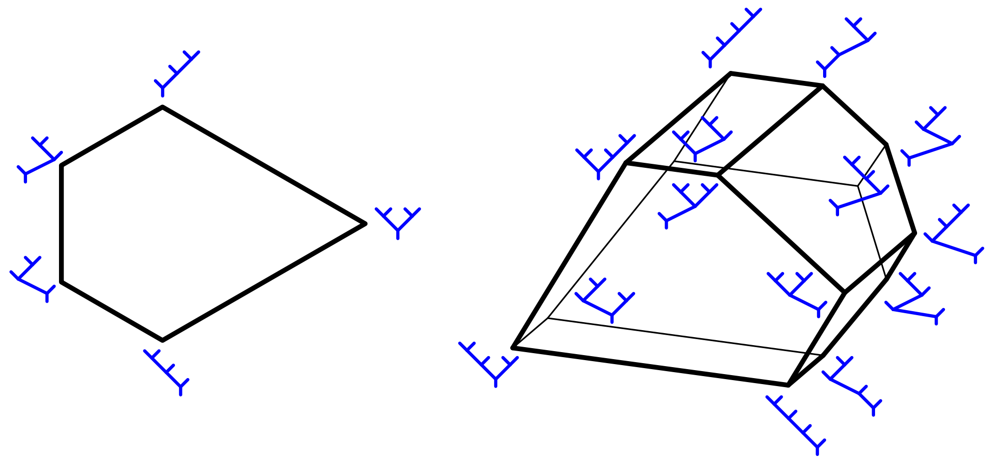


FIGURE 2. J.-L. Loday's associahedra [Lod04] of dimension 2 (left) and 3 (right).

2.8. **Multipermutations and multiposets.** We finally introduce some combinatorial objects and tools needed to define actions of our operads in Section 5.

2.8.1. *Multisets.* We call *multiset* a set

$$M := \{1_1, 1_2, \dots, 1_{a_1}, 2_1, 2_2, \dots, 2_{a_2}, \dots, m_1, m_2, \dots, m_{a_m}\}$$

with several copies of each element that we distinguish with indices when necessary. For brevity, we use the monomial notation $M := 1^{\{a_1\}} 2^{\{a_2\}} \dots m^{\{a_m\}}$ where the exponents are surrounded by curly brackets to avoid any ambiguity. We denote by $|M| := \sum_{i \in [m]} a_i$ the cardinality of M, by $\max(M) := m$ its largest element, and by $\operatorname{supp}(M) := \{i \in [m] \mid a_i \neq 0\}$ the support of M. It will be convenient to have the following restriction, shift and standardization operators. These operators are classically used to define the product and coproduct of Hopf algebras and the compositions of operads.

Definition 2.36. *For a multiset $M := 1^{\{a_1\}} \dots m^{\{a_m\}}$ and a subset $L := \{\ell_1 < \dots < \ell_p\}$ of $[m]$, we define the restricted multiset $M^{|L} := 1^{\{a_{\ell_1}\}} \dots p^{\{a_{\ell_p}\}}$. For w in $M^{|L}$ we let $\overline{w} := \ell_w$ denote the corresponding element in M.*

Definition 2.37. *For a multiset $M := 1^{\{a_1\}} \dots m^{\{a_m\}}$ and an integer j, we define the shifted multiset $M[j] := (1+j)^{\{a_1\}} \dots (m+j)^{\{a_m\}}$. For w in $M[j]$, we let $\overline{w} := w - j$ denote the corresponding element of M.*

Definition 2.38. *For a multiset $M := 1^{\{a_1\}} \dots m^{\{a_m\}}$ and two integers $i \le m$ and j, we define the multiset $M[i,j] := 1^{\{a_1\}} \dots (i-1)^{\{a_{i-1}\}} (i+1+j-1)^{\{a_{i+1}\}} \dots (m+j-1)^{\{a_m\}}$. In other words, we erase all values i and replace values $v > i$ by $v + j - 1$. For w in $M[i,j]$, we let $\overline{w}$ denote the corresponding element in M, that is*

$$\overline{w} := \begin{cases} w & \text{if } w \le i-1, \\ w - j + 1 & \text{if } w \ge i + j. \end{cases}$$

Note that the notation $\overline{x}$ of the three previous definition is ambiguous on purpose. Its intended meaning should be always clear from the context.

Definition 2.39. *The standardization of a multiset $M := 1^{\{a_1\}} 2^{\{a_2\}} \dots m^{\{a_m\}}$ is the bijection* Std *from M to $[|M|]$ defined by*

$$\operatorname{Std}(i_u) := u + \sum_{j<i} a_j.$$

For example, consider the multiset $M = 1^{\{3\}}2^{\{1\}}4^{\{2\}} = \{1_1, 1_2, 1_3, 2_1, 4_1, 4_2\}$. Then we have $M^{|\{1,4\}} = 1^{\{3\}}2^{\{2\}}$, $M[3] = 4^{\{3\}}5^{\{1\}}7^{\{2\}}$, $M[2,4] = 1^{\{3\}}7^{\{2\}}$, $\mathrm{Std}(2_1) = 4$ and $\mathrm{Std}(4_2) = 6$.

Remark 2.40. Note that there are natural relations between these operators. Among others:

$$(M^{|L})^{|X} = M^{|\{\ell_x \mid x\in X\}}, \quad (M^{|L})[j] = (M[j])^{|\{\ell+j \mid \ell\in L\}} \quad \text{and} \quad (M[i,j])[k] = (M[k])[i+k,j].$$

2.8.2. *Multipermutations.* We consider a permutation of a finite set X as a word whose letters are the elements of X.

Definition 2.41. *A multipermutation of a multiset $M := 1^{\{a_1\}}2^{\{a_2\}}\dots m^{\{a_m\}}$ is a permutation of M where the copies of the same integer appear in natural order, meaning that i_u is to the left of i_v for all $i \in [m]$ and $0 < u < v \le a_i$. Since the different copies appear in natural order, there is no need to distinguish between them when writing a multipermutation.*

For two integers m and $k > 0$, a k-permutation of degree m is a multipermutation of the multiset $1^{\{k\}}2^{\{k\}}\dots m^{\{k\}}$. We denote by Perm_k the set of k-permutations, by $\mathsf{Perm}_k(m)$ those of degree m, and by $\mathsf{FQSym}_k = \bigoplus_{m>0} \mathbb{K}\mathsf{Perm}_k(m)$ the graded $\mathbb{K}$-vector space whose m-th homogeneous component has basis $\mathsf{Perm}_k(m)$.

The cardinality of $\mathsf{Perm}_k(m)$ is a multinomial coefficient:

$$|\mathsf{Perm}_k(m)| = \binom{mk}{k^{\{m\}}} = \frac{(mk)!}{(k!)^m}.$$

For example, the word 132123 is one of the 90 2-permutations of degree 3, the word 321312132 is one of the 1680 3-permutations of degree 3, and the word 31421324 is one of the 2520 2-permutations of degree 4. We now define the restriction, shift and standardization operators on multipermutations, similar to Definitions 2.36 to 2.39.

Definition 2.42. *For a multipermutation μ of M and a subset $L := \{\ell_1 < \dots < \ell_p\}$ of $[\max(M)]$, the restricted multipermutation $\mu^{|L}$ of $M^{|L}$ is obtained from μ by keeping only the values of μ that belong to L, and replacing ℓ_w by w.*

Definition 2.43. *For a multipermutation μ of M and an integer j, the shifted multipermutation $\mu[j]$ on $M[j]$ is defined by $\mu[j]_p = \mu_p + j$ for any $p \in [|M|]$.*

Definition 2.44. *For a multipermutation μ of M and two integers $i \le \max(M)$ and j, the multipermutation $\mu[i,j]$ of $M[i,j]$ is obtained from μ by erasing all values i and replacing all values $v > i$ by $v + j - 1$.*

Definition 2.45. *The standardization of a multipermutation μ of M is the permutation $\mathrm{Std}(\mu)$ of $[|M|]$ defined by $\mathrm{Std}(\mu)_p = \mathrm{Std}(\mu_p)$ for any $p \in [|M|]$.*

For example, consider the multipermutation $\mu = 31421324$. Then we have $\mu^{|\{2,3,4\}|} = 231213$, $\mu[3] = 64754657$, $\mu[2,4] = 617167$, and $\mathrm{Std}(\mu) = 51732648$.

Remark 2.46. As in Remark 2.57, there are natural relations between these operators, including:

$$(\mu^{|L})^{|X} = \mu^{|\{\ell_x \mid x\in X\}}, \quad (\mu^{|L})[j] = (\mu[j])^{|\{\ell+j \mid \ell\in L\}} \quad \text{and} \quad (\mu[i,j])[k] = (\mu[k])[i+k,j].$$

The standardization clearly defines a bijection from the set of k-permutations of degree m to the set of k-monotone permutations of $[mk]$, *i.e.* permutations σ of $[mk]$ such that the letters $ki, ki+1, \dots, k(i+1)-1$ appear form left to right in σ for all $i \in [m]$. Equivalently, σ appears in the m-iterated shifted shuffle (see Definition 2.50) of the identity permutation of degree k. Although we could work with monotone permutations, we believe that most statements are more natural on multipermutations.

2.8.3. *Concatenation and shifted concatenation.* We quickly recall the concatenation product on words and the shifted concatenation product on permutations.

Definition 2.47. *Let $\mathcal{A}$ be a finite alphabet and $\mathcal{A}^n$ denote the set of words on $\mathcal{A}$ of length n. The free associative algebra over $\mathcal{A}$ is the graded $\mathbb{K}$-vector space $\mathcal{A}^* := \bigoplus_{n\in\mathbb{N}} \mathbb{K}\mathcal{A}^n$ endowed with the concatenation $X \cdot Y := XY$.*

We denote by $\mathcal{A}^{\geq k} := \bigoplus_{n \geq k} \mathbb{K}\mathcal{A}^n$ which is stable by the concatenation product. Note that the concatenation product does not stabilizes the subspace of k-permutations. For this, we need a slight modification of this product obtained by shifting.

Definition 2.48. *The* shifted concatenation $\mu \bar{\cdot} \nu$ *of two multipermutations* μ *of* M *and* ν *of* N *is the permutation of* $M \cup N[m]$ *defined by* $\mu \bar{\cdot} \nu := \mu \cdot \nu[m]$, *where* $m := \max(M)$.

For example $123213 \bar{\cdot} 2112 = 1232135445$. Note that the shifted concatenation is associative and graded on FQSym_k. Indeed the concatenation of two multipermutations of degrees m and n is of degree $m + n$.

2.8.4. *Shuffle and shifted shuffle.* We now quickly recall the shuffle product on words and the shifted shuffle product on permutations.

Definition 2.49. *Let* $\mathcal{A}$ *be a finite alphabet and* $\mathcal{A}^n$ *denote the set of words on* $\mathcal{A}$ *of length* n. *The* shuffle algebra *over* $\mathcal{A}$ *is the graded* $\mathbb{K}$*-vector space* $\mathsf{Shuffle}(\mathcal{A}) := \bigoplus_{n \in \mathbb{N}} \mathbb{K}\mathcal{A}^n$ *endowed with the* shuffle product *defined inductively by* $X \sqcup\!\sqcup \varepsilon := X$, $\varepsilon \sqcup\!\sqcup Y := Y$ *and* $xX \sqcup\!\sqcup yY := x(X \sqcup\!\sqcup yY) + y(xX \sqcup\!\sqcup Y)$ *for any two words* X, Y *on* $\mathcal{A}$.

We denote by $\mathsf{Shuffle}^{\geq k} := \bigoplus_{n \geq k} \mathbb{K}\mathcal{A}^n$ which is stable by the shuffle product. Note that the shuffle product does not stabilizes the subspace of k-permutations. For this, we need a slight modification of this product obtained by shifting.

Definition 2.50. *The* shifted shuffle $\mu \,\bar{\sqcup\!\sqcup}\, \nu$ *of two multipermutations* μ *of* M *and* ν *of* N *is the linear combination* $\mu \sqcup\!\sqcup \nu[m]$ *of multipermutations of* $M \cup N[m]$, *where* $m := \max(M)$.

Lemma 2.51. *The standardization* Std *and the shifted shuffle* $\bar{\sqcup\!\sqcup}$ *commute: for any two multipermutations* μ *and* ν, *we have* $\mathrm{Std}(\mu \,\bar{\sqcup\!\sqcup}\, \nu) = \mathrm{Std}(\mu) \,\bar{\sqcup\!\sqcup}\, \mathrm{Std}(\nu)$.

The shifted shuffle endows the vector space FQSym_k with a graded algebra structure that generalizes the algebra on permutations of C. Malvenuto and C. Reutenauer [MR95]. By standardization (see Definition 2.45), it is also a subalgebra of the classical algebra FQSym on permutations. For example

$$123213 \,\bar{\sqcup\!\sqcup}\, 2112 = 1232135445 + 1232153445 + \cdots (210 \text{ terms}) \cdots + 54415123213 + 5445123213.$$

2.8.5. *Multiposets.* Recall that a *poset* is a set endowed with a partial order (a reflexive, antisymmetric and transitive binary relation). We depict posets by their Hasse diagram which is the graph of their cover relations and we always orient them from bottom to top. The following definition is illustrated on Figure 3.

Definition 2.52. *A* multiposet *on a multiset* $M := 1^{\{a_1\}} 2^{\{a_2\}} \dots m^{\{a_m\}}$ *is a partial order* $\leq_M$ *on* M *where the copies of the same integer are naturally ordered, meaning that* $i_u \leq_M i_v$ *for all* $i \in [\max(M)]$ *and* $0 < u < v \leq a_i$. *Since the different copies are naturally ordered there is no need to distinguish between them when drawing the Hasse diagram of a multiposet. For two integers* m *and* $k > 0$, *a* k-poset *of degree* m *is a multiposet on the multiset* $1^{\{k\}} 2^{\{k\}} \dots m^{\{k\}}$.

We now define the restriction, shift and standardization operators on multiposets, similar to Definitions 2.36 to 2.39 and 2.42 to 2.45. These definitions are illustrated on Figure 4.

Definition 2.53. *For a multiposet* $\leq_M$ *on* M *and a subset* $L := \{\ell_1, \dots, \ell_p\}$ *of* $[\max(M)]$, *we define the* restricted *multiposet* $\leq_{M^{|L}}$ *on* $M^{|L}$ *by* $i_u \leq_{M^{|L}} j_v \iff (\ell_i)_u \leq_M (\ell_j)_v$.

Definition 2.54. *For a multiposet* $\leq_M$ *on* M *and an integer* j, *we define the* shifted *multiposet* $\leq_{M[j]}$ *on* $M[j]$ *by* $x_u \leq_{M[j]} y_v \iff x_u - j \leq_M y_v - j$.

Definition 2.55. *For a multiposet* $\leq_M$ *on* M *and two integers* $i \leq \max(M)$ *and* j, *we define the multiposet* $\leq_{M[i,j]}$ *on* $M[i,j]$ *by* $x \leq_{M[i,j]} y \iff \overline{x} \leq_M \overline{y}$ *where* $x \mapsto \overline{x}$ *is the injection from* $M[i,j]$ *to* M *described in Definition 2.38.*

Definition 2.56. *The* standardization *of a multiposet* $\leq_M$ *on* M *is the poset* $\leq_{\mathrm{Std}(M)}$ *of* $\mathrm{Std}(M)$ *defined by* $x \leq_{\mathrm{Std}(M)} y \iff \mathrm{Std}(x) \leq_M \mathrm{Std}(y)$.

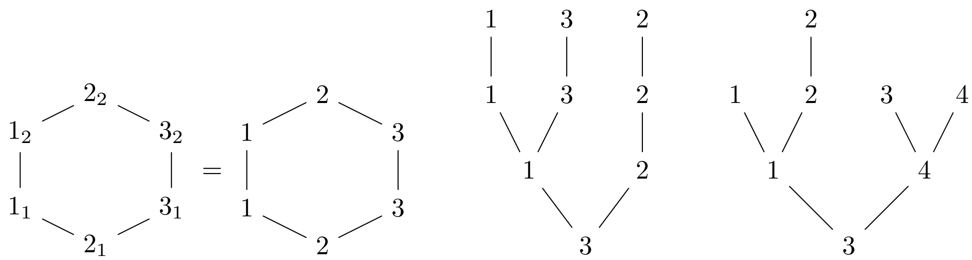


FIGURE 3. A 2-poset of degree 3 (left), a 3-poset of degree 3 (middle), and a 2-poset of degree 4 (right). See Definition 2.52.

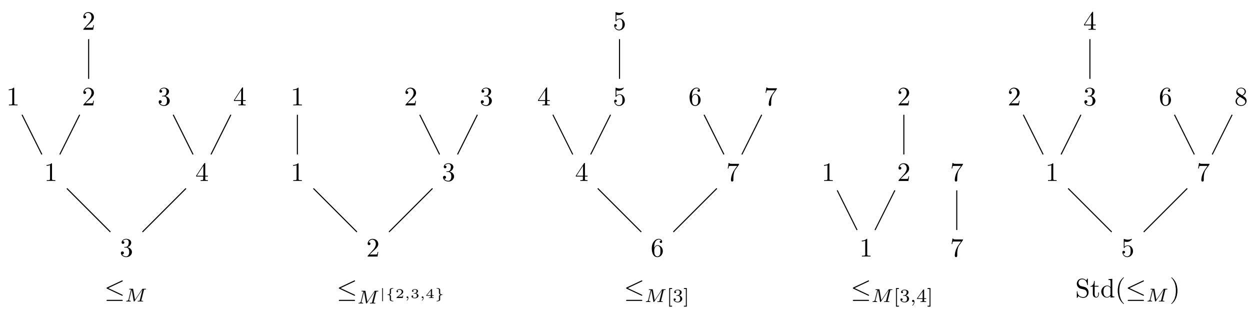


FIGURE 4. Shifts and standardization of multiposets. See Definitions 2.53 to 2.56.

Remark 2.57. As in Remark 2.57, there are natural relations between these operators, including:

$$\le_{(M^{|L})^{|X}} = \le_{M^{|\{\ell_x \mid x \in X\}}}, \quad \le_{(M^{|L})[j]} = \le_{(M[j])^{|\{\ell+j \mid \ell \in L\}}} \quad \text{and} \quad \le_{(M[i,j])[k]} = \le_{(M[k])[i+k,j]}.$$

We conclude with two relevant operations on multiposets.

Definition 2.58. *Consider two multiposets $\le_M$ and $\le_N$ on two disjoint multisets M and N (meaning $\mathrm{supp}(M) \cap \mathrm{supp}(N) = \varnothing$), and let $P := M \sqcup N$. We define*

- *the disjoint union $\le_M \sqcup \le_N$ to be the multiposet $\le_\sqcup$ on P where $x \le_\sqcup y$ if and only if either $x \in M$, $y \in M$ and $x \le_M y$, or $x \in N$, $y \in N$ and $x \le_N y$.*
- *the ordered sum $\le_M + \le_N$ to be the multiposet $\le_+$ on P where $x \le_+ y$ if and only if either $x \le_\sqcup y$, or $x \in M$ and $y \in N$.*

2.8.6. *Tree multiposets.* We are now interested in multiposets with special shapes, namely trees and forests. They can be defined recursively or directly in terms of the partial order.

Definition 2.59. *A multiposet $\le_M$ is a forest if $x \le_M z$ and $y \le_M z$ implies either $x \le_M y$ or $y \le_M x$ for all $x, y, z \in M$. It is a tree if it is a forest with a unique minimal element, called root.*

For example, the leftmost multiposet of Figure 3 is not a tree while the middle and rightmost multiposets of Figure 3 are. Clearly a multiposet is a forest (resp. a tree) if its Hasse diagram is. We therefore freely use the vocabulary of trees (such as children, subtree, ...) on tree multiposets.

Definition 2.60. *A tree multiposet is interval labelled if for any subtrees t and t' with the same parent, all labels that appear in t are strictly smaller than all labels that appear in t' or vice versa. In other words, the labels that appear in distinct descendant subtrees of a given node belong to disjoint intervals.*

For instance, the middle tree multiposet of Figure 3 is not interval labelled (since the left descendant tree of the root contains 1 and 3 while the right descendant tree of the root contains 2) while the right tree multiposet of Figure 3 is interval labelled.

2.8.7. *Linear extensions.* A multiposet $\leq_M$ on M is *linear* if it is a total order, meaning that any two elements of M are comparable for $\leq_M$. Note that the multipermutations of M are precisely the linear multiposets of M.

Definition 2.61. *A linear extension of a multiposet $\leq_M$ is a multipermutation σ that extend $\leq_M$, meaning that $x \leq_M y$ implies that x appears to the left of y in σ for all $x, y \in M$. We denote by $\mathcal{L}(\leq_M)$ the set of linear extensions of the multiposet $\leq_M$.*

For example, the multipermutations 231132, 321312132 and 31421324 are linear extensions of the multiposets of Figure 3. The following statement is illustrated by the examples of the previous sections.

Lemma 2.62. *The linear extensions operator $\mathcal{L}$ commutes with the restriction, the shift and the standardization: for any multiposet $\leq_M$, any subset $L \subseteq [\max(M)]$ and any integers $i \leq \max(M)$ and j, we have*

$$\mathcal{L}(\leq_{M^{|L}}) = \{\sigma^{|L} \mid \sigma \in \mathcal{L}(\leq_M)\} \qquad \mathcal{L}(\leq_{M[j]}) = \{\sigma[j] \mid \sigma \in \mathcal{L}(\leq_M)\}$$
$$\mathcal{L}(\leq_{M[i,j]}) = \{\sigma[i,j] \mid \sigma \in \mathcal{L}(\leq_M)\} \qquad \textit{and} \qquad \mathcal{L}(\mathrm{Std}(\leq_M)) = \{\,\mathrm{Std}(\mu) \mid \mu \in \mathcal{L}(\leq_M)\}.$$

The last point of this lemma enables us to transport classical properties of linear extensions of posets to linear extensions of multiposets. For instance, the hook length formula enables us to count the number of linear extensions of a tree multiposet.

Lemma 2.63 ([BW91])**.** *The number of linear extensions of a tree multiposet $\leq_M$ is given by*

$$\frac{|M|!}{\prod_{x \in M} |M_x|}$$

where M_x denotes the subtree rooted at $x \in M$.

For instance, the number of linear extensions of the middle and rightmost tree multiposets of Figure 3 are $9!/(9 \cdot 5 \cdot 3 \cdot 2^3) = 336$ and $8!/(8 \cdot 4 \cdot 3 \cdot 2) = 210$ respectively. We leave the following statement to the reader.

Lemma 2.64. *For any two multiposets $\leq_M$ and $\leq_N$ on two disjoints multisets M and N (meaning $\mathrm{supp}(M) \cap \mathrm{supp}(N) = \varnothing$),*

$$\mathcal{L}(\leq_M \sqcup \leq_N) = \mathcal{L}(\leq_M) \,\text{ш}\, \mathcal{L}(\leq_N) = \bigcup_{\substack{\sigma \in \mathcal{L}(\leq_M) \\ \tau \in \mathcal{L}(\leq_N)}} \sigma \,\text{ш}\, \tau,$$
$$\mathcal{L}(\leq_M + \leq_N) = \mathcal{L}(\leq_M) \cdot \mathcal{L}(\leq_N) = \{\sigma \cdot \tau \mid \sigma \in \mathcal{L}(\leq_M),\ \tau \in \mathcal{L}(\leq_N)\},$$

where ш denotes the shifted shuffle product of Definition 2.50 and $\cdot$ denotes the shifted concatenation product of Definition 2.48.

3. Signaletic operads

This paper essentially relies on an interpretation of the diassociative and dual duplicial operads in terms of traffic signals in an arborescent road. This interpretation then admits four natural extensions that we call messy/tidy series/parallel signaletic operads.

3.1. **Signaletic interpretation of the diassociative and dual duplicial operads.** We consider that a syntax tree $\mathfrak{t}$ on the operators $\{\prec, \succ\}$ is an arborescent road where each branching node is occupied by a traffic signal $\boxed{\prec}$ or $\boxed{\succ}$. A car arrives at the root of $\mathfrak{t}$ and drives through the tree $\mathfrak{t}$ following at each branching node the direction indicated by the traffic signal. The car ends at a certain leaf of $\mathfrak{t}$ that we call the *destination* of $\mathfrak{t}$. This procedure is illustrated on three syntax trees in Figure 5.

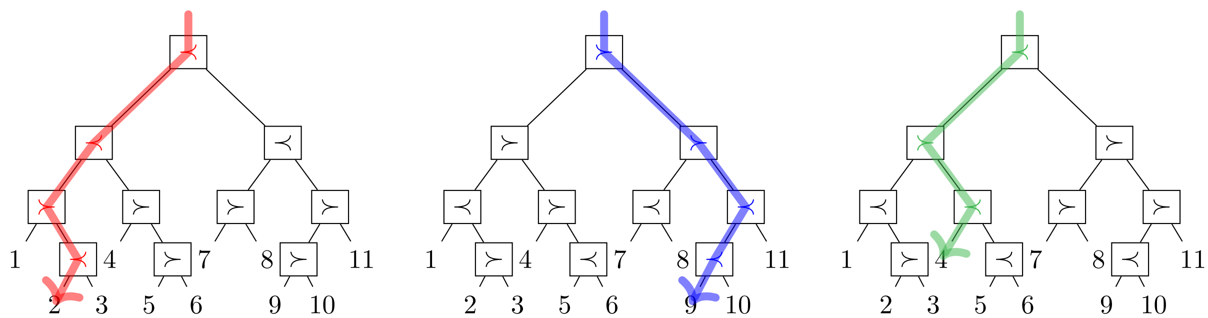
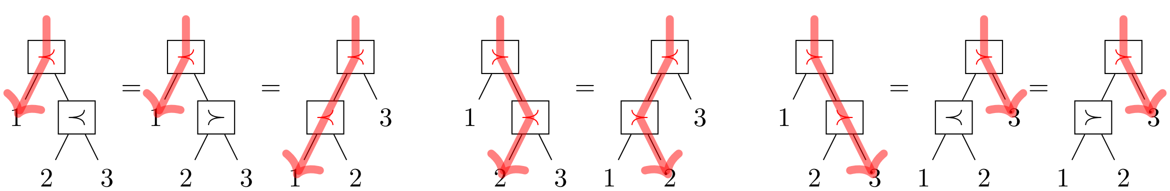


Figure 5. Interpretation of the diassociative operad in terms of syntax tree traversal.

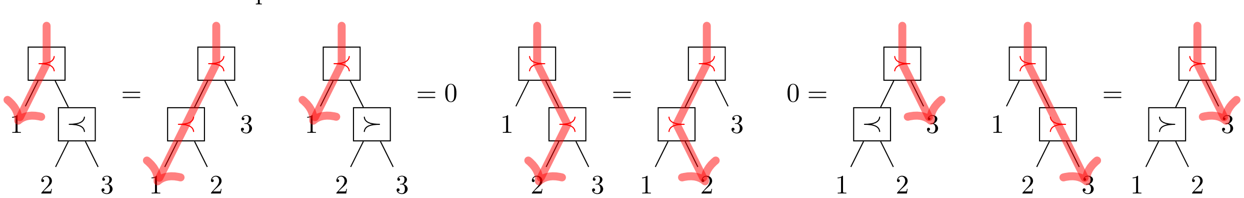
Observe that the diassociative relations are compatible with the destination:



This shows that two equivalent syntax trees have the same destination. The reverse statement can be shown using normal forms as will be generalized in Theorem 3.40.

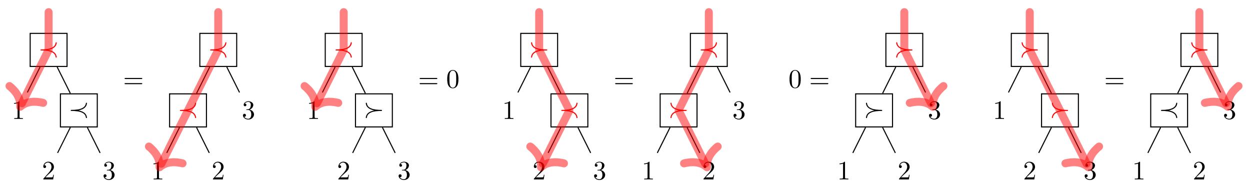
Proposition 3.1 ([Lod01])**.** *Two syntax trees on $\{\prec, \succ\}$ with the same arity represent the same operation in the diassociative operad if and only if they have the same destination.*

Consider now the dual duplicial operad. We observe that its relations are also compatible with the destination. The difference is that a syntax tree vanishes if some signal outside the route of the car does not point towards this route:



Proposition 3.2. *A syntax tree $\mathfrak{t}$ on $\{\prec, \succ\}$ vanishes in the dual duplicial operad except if all signals outside the route of the car point toward this route. Two non-vanishing syntax trees on $\{\prec, \succ\}$ represent the same operation in the dual duplicial operad if and only if they have the same destination.*

For our purposes, we need to consider a slight modification of the dual duplicial operad. We still consider a syntax tree on $\{\prec, \succ\}$ and we let a car traverse the tree, but we now impose that all signals outside the route of the car point to the left. We will see as a special case of Section 3.2.2 that this defines a quadratic operad $\mathsf{Dup}^!_\prec$ over $\{\prec, \succ\}$ defined by the five linear relations:



The opposite rule, forcing all signals not located on the route to point to the right, also yields an operad $\mathsf{Dup}^!_\succ$. We let $\mathsf{Dup}^!_\odot := \mathsf{Dup}^!$ denote the classical dual duplicial operad, where the signals not located on the route point towards the route. In contrast, note that the rule forcing all signals not located on the route to point away from the route does not define an operad. We will see as a special case of Section 3.2.2 that the three operads $\mathsf{Dup}^!_\odot$, $\mathsf{Dup}^!_\prec$ and $\mathsf{Dup}^!_\succ$ are quadratic and Koszul.

In Sections 3.2 and 3.3, we define and study two natural generalizations of this signaletic interpretation of the diassociative and dual duplicial operads. Namely, we fix an integer $k \ge 1$, we consider a syntax tree $\mathfrak{t}$ on the operators $\{\prec, \succ\}^k$, and we assume that k cars arrive at the root of $\mathfrak{t}$. These k cars will drive through the tree $\mathfrak{t}$ following the directions indicated by the traffic signal at each branching node in two different ways:

- *Parallel*: The k cars all start together at the root of $\mathfrak{t}$, and the i-th car always follows the indication given by the i-th letter of the traffic signal at each branching node. This corresponds to a white Manin product of the diassociative and duplicial operads.
- *Series*: The k cars start one after the other at the root of $\mathfrak{t}$, and each car always follows the indication given by the leftmost remaining letter of the traffic signal at each branching node and erases it.

These two circulation rules are illustrated in Figures 6 and 7. Moreover, we can impose the following additional constraints:

- *Tidy*: All signals not used by a car must point to the left (as in the twisted duplicial operad).
- *Messy*: No further constraints on the remaining signals (as in the diassociative operad).

We will see that the two circulation rules (parallel and series) and the two ordering rules (messy or tidy) define four families of operads. Although not isomorphic, these four families of operads have identical Hilbert series and very similar behaviors. We will actually observe that many proof techniques can be applied to both families irrespective of the series or parallel circulation rule, and of the tidy or messy constraints.

In fact, these two circulation rules can also be mixed as quickly discussed in Section 6.2. We have preferred to first discuss them separately to simplify the presentation.

3.2. **Parallel signaletic operads.** We start with the parallel signaletic operads, and we treat separately the messy and tidy situations. Although these operads are just white Manin powers of the diassociative and duplicial operads, we use an elementary combinatorial presentation that will allow to define similarly series signaletic operads with a slight modification of the circulation rule.

3.2.1. *Messy parallel signaletic operads.* Fix an integer $k \ge 0$ and consider a syntax tree $\mathfrak{t}$ on the operators $\mathfrak{B}_k := \{\prec, \succ\}^k$ with n leaves that we label $1, \dots, n$ from left to right. Assume that k cars $c_1, \dots, c_k$ arrive simultaneously at the root of $\mathfrak{t}$ and that the j-th car c_j always follows the indication given by the j-th letter of the traffic signal at each branching node. We call *parallel routes* the paths followed by the k cars in the syntax tree $\mathfrak{t}$. Finally, each car c_j ends at a certain leaf labeled ℓ_j and we call *parallel destination vector* of $\mathfrak{t}$ the vector $(\!| \ell_1, \dots, \ell_k |\!)_n$. See Figure 6.

We say that two syntax trees $\mathfrak{t}, \mathfrak{t}'$ on $\mathfrak{B}_k$ with the same arity are *messy parallel k-signaletic equivalent* and we write $\mathfrak{t} \approxeq^{\|} \mathfrak{t}'$ if they have the same parallel destination vector.

Proposition 3.3. *The messy parallel k-signaletic equivalence is compatible with grafting of syntax trees: $\mathfrak{t} \circ_i \mathfrak{s} \approxeq^{\|} \mathfrak{t}' \circ_i \mathfrak{s}'$ for any syntax trees $\mathfrak{t} \approxeq^{\|} \mathfrak{t}'$ of arity p and $\mathfrak{s} \approxeq^{\|} \mathfrak{s}'$ of arity q, and any $i \in [p]$.*

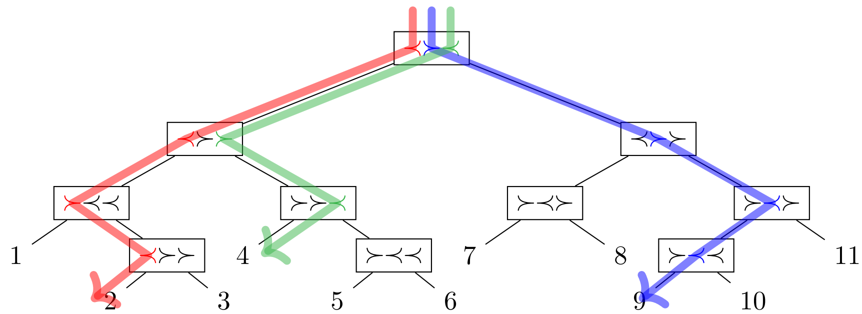


FIGURE 6. Traversing the syntax tree in parallel. The parallel routes are marked, and the parallel destination vector is $(\!|2, 9, 4|\!)_{11}$.

Proof. Consider the j-th car c_j and denote by p_j its parallel destination in t and by q_j its parallel destination in $\mathfrak{s}$. Then its parallel destination r_j in $\mathsf{t} \circ_i \mathfrak{s}$ is given by

$$\mathsf{r}_j = \begin{cases} \mathsf{p}_j & \text{if } \mathsf{p}_j < i, \\ \mathsf{p}_j + \mathsf{q}_j - 1 & \text{if } \mathsf{p}_j = i, \\ \mathsf{p}_j + q - 1 & \text{if } \mathsf{p}_j > i. \end{cases} \tag{1}$$

□

Here are some examples of Equation (1): for $\mathsf{p} := (\!|2, 1, 4, 2, 2|\!)_5$ and $\mathsf{q} := (\!|3, 1, 6, 1, 2|\!)_6$, we have $\mathsf{p} \circ_1 \mathsf{q} = (\!|7, 1, 9, 7, 7|\!)_{10}$, $\mathsf{p} \circ_2 \mathsf{q} = (\!|4, 1, 9, 2, 3|\!)_{10}$, $\mathsf{p} \circ_3 \mathsf{q} = \mathsf{p} \circ_4 \mathsf{q} = (\!|2, 1, 9, 2, 2|\!)_{10}$, and $\mathsf{p} \circ_5 \mathsf{q} = (\!|2, 1, 4, 2, 2|\!)_{10}$.

Definition 3.4. *The messy parallel k-signaletic operad $\mathsf{MSig}^{\parallel}_k$ is the quotient of the free operad on $\mathfrak{B}_k := \{\prec, \succ\}^k$ by the messy parallel k-signaletic equivalence.*

We can immediately observe the connection with the Manin products of Section 2.5.

Proposition 3.5. *For any two integers k and l, we have*

$$\mathsf{MSig}^{\parallel}_k = \mathsf{Diass}^{\square k} = \mathsf{Diass}^{\blacksquare k} \qquad \textit{and} \qquad \mathsf{MSig}^{\parallel}_{k+l} = \mathsf{MSig}^{\parallel}_k \,\square\, \mathsf{MSig}^{\parallel}_l.$$

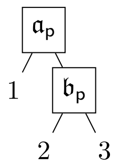
Proof. The operad $\mathsf{MSig}^{\parallel}_k(n)$ has for basis the destination vectors $(\!|\ell_1, \dots \ell_k|\!)_n$. The concatenation of destination vectors shows that $\mathsf{MSig}^{\parallel}_{k+l} = \mathsf{MSig}^{\parallel}_k \otimes \mathsf{MSig}^{\parallel}_l$ as vector spaces. Moreover, the messy parallel signaletic equivalence on $\mathfrak{B}_{k+l}$ is clearly the transitive closure of the tensor product of the messy parallel signaletic equivalences on $\mathfrak{B}_k$ and $\mathfrak{B}_l$. This immediately shows that $\mathsf{MSig}^{\parallel}_k = \mathsf{Diass}^{\square k}$. The other equality follow from the fact that $\mathsf{Dend} \,\square\, \mathsf{Dend} = \mathsf{Dend} \blacksquare \mathsf{Dend}$, which was shown in [Val08]. □

Definition 2.20 and Propositions 2.21, 2.29 and 3.5 imply that the messy parallel k-signaletic operad $\mathsf{MSig}^{\parallel}_k$ is quadratic and Koszul. We will see an alternative proof in Section 3.4, with an argument uniform for all k-signaletic operads (it applies to both messy and tidy and to both parallel and series). We now extract from the definitions the quadratic relations which provide a presentation of $\mathsf{MSig}^{\parallel}_k$.

Definition 3.6. *We call messy parallel k-signaletic relations all the quadratic relations of $\mathsf{MSig}^{\parallel}_k$, that is all the relations of the form $\mathsf{t}_1 = \mathsf{t}_2$ where t_1 and t_2 are two syntax trees with two nodes (i.e. of arity 3) sharing the same parallel destination vector.*

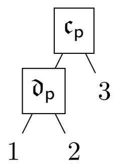
These relations can be explicitly described for any k as follows. For any destination vector $\mathsf{p} \in [3]^k$, the messy parallel k-signaletic operad $\mathsf{MSig}^{\parallel}_k$ satisfies the quadratic relation $\mathsf{t}_1 = \mathsf{t}_2$

for any two syntax trees $\mathfrak{t}_1$ and $\mathfrak{t}_2$ of one of the following forms:



where $(\mathfrak{a}_{\mathsf{p}})_i = \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \succ & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases}$ and $(\mathfrak{b}_{\mathsf{p}})_i = \begin{cases} \prec & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases}$ for all $i \in [k]$,

or

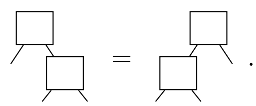


where $(\mathfrak{c}_{\mathsf{p}})_i = \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \prec & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases}$ and $(\mathfrak{d}_{\mathsf{p}})_i = \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \succ & \text{if } \mathsf{p}_i = 2 \end{cases}$ for all $i \in [k]$.

Note that there are 2^{1+2k} syntax trees of arity 3 on $\mathfrak{B}_k$ but only 3^k parallel destination vectors (*i.e.* messy parallel k-signaletic equivalence classes). Therefore, there are $2^{1+2k} - 3^k$ independent messy parallel k-signaletic relations among syntax trees of arity 3 on $\mathfrak{B}_k$.

Example 3.7 (Messy parallel 0-, 1- and 2-signaletic relations)**.** The messy parallel 0-signaletic relation is the associative relation:

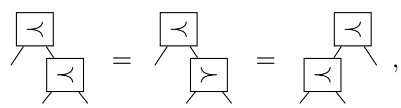
($\mathsf{MSig}^{\|}$.)



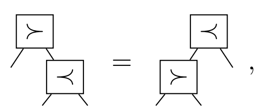
In other words, the messy parallel 0-signaletic operad $\mathsf{MSig}^{\|}_0$ is just the associative operad As.

The messy parallel 1-signaletic relations are the 5 diassociative relations:

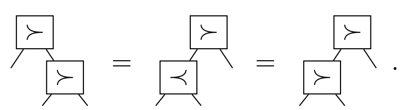
($\mathsf{MSig}^{\|}$ 1)



($\mathsf{MSig}^{\|}$ 2)



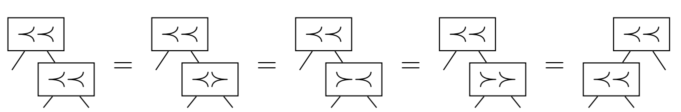
($\mathsf{MSig}^{\|}$ 3)



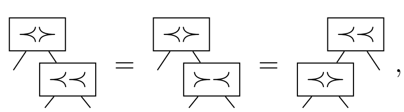
In other words, the messy parallel 1-signaletic operad $\mathsf{MSig}^{\|}_1$ is just the diassociative operad Diass.

The messy parallel 2-signaletic relations are the following 23 relations:

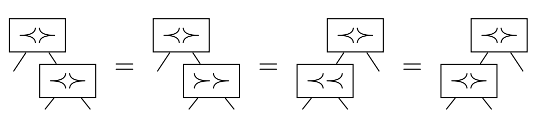
($\mathsf{MSig}^{\|}$ 11)



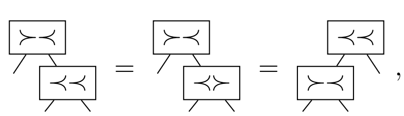
($\mathsf{MSig}^{\|}$ 12)

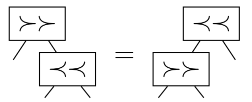


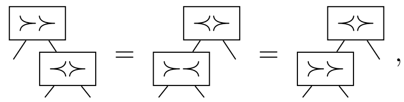
($\mathsf{MSig}^{\|}$ 13)

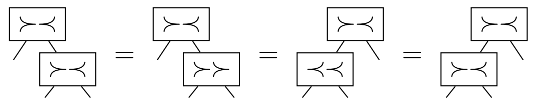


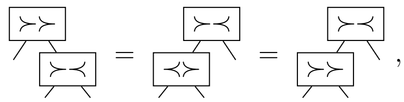
($\mathsf{MSig}^{\|}$ 21)

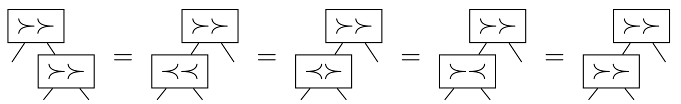


(MSig$^{\|}$ 22) 

(MSig$^{\|}$ 23) 

(MSig$^{\|}$ 31) 

(MSig$^{\|}_k$ 32) 

(MSig$^{\|}$ 33) 

Note that these relations were already considered as the dual of the Quad operad, see for instance [Foi15, Proposition 3]. The translation between our notations and that of [Foi15, Proposition 3] is the following:

$$\prec\prec\, = \nwarrow, \qquad \prec\succ\, = \nearrow, \qquad \succ\prec\, = \swarrow, \qquad \text{and} \qquad \succ\succ\, = \searrow.$$

Remark 3.8. For any subset I of $[k]$ of cardinal $\ell \le k$, there is a surjective restriction operad morphism $\mathsf{Res}_I : \mathsf{MSig}^{\|}_k \to \mathsf{MSig}^{\|}_\ell$ sending a destination vector $\mathsf{p} = (\!| \mathsf{p}_1, \dots, \mathsf{p}_k |\!)_n$ to the destination subvector $\mathsf{Res}_I(\mathsf{p}) := (\!| \mathsf{p}_i \mid i \in I |\!)_n$. It is equivalently defined on operators by $\mathsf{Res}_I(\mathfrak{b}) := (\mathfrak{b}_i)_{i \in I}$. This is actually a general fact for Manin products of non-symmetric set operads (see Section 2.5 for more details). Indeed, each (non-symmetric) set operad $\mathcal{O}$ has a morphism m to the non-symmetric associative operad As sending any operation to the unique operation of As with the same arity. Noting that As is the neutral element for the Manin product, we observe that Res_I is nothing but the following Manin product of morphisms:

$$\mathsf{Res}_I = \mathsf{m}_1 \,\square\, \mathsf{m}_2 \,\square\, \dots \,\square\, \mathsf{m}_k \qquad \text{where} \qquad \mathsf{m}_i = \begin{cases} \mathsf{id} & \text{if } i \in I, \\ \mathsf{m} & \text{otherwise.} \end{cases}$$

Remark 3.9. Conversely, since both operators $\prec$ and $\succ$ are associative in $\mathsf{MSig}^{\|}_1$, for any $i \in [k+1]$ and $\mathfrak{a} \in \{\prec, \succ\}$, the map $\mathsf{Ins}^{\mathfrak{a}}_i : \mathsf{MSig}^{\|}_k \to \mathsf{MSig}^{\|}_{k+1}$ inserting $\mathfrak{a}$ at position i in every nodes is an injective operad morphism. On destination vectors of degree n it amounts to respectively insert 1 or n at position i. Similarly to the previous Remark 3.8, this is inserting $\mathsf{As} \to \mathsf{MSig}^{\|}_1$ in a Manin product of identity morphisms.

3.2.2. *Tidy parallel signaletic operads.* We now impose additional constraints on the trafic signals not contained in the parallel routes. We say that a syntax tree $\mathfrak{t}$ on $\mathfrak{B}_k$ is *tidy parallel* if all the traffic signals not contained in its parallel routes **point to the left**. Otherwise, we say that $\mathfrak{t}$ is *messy parallel*. We define the *tidy parallel destination vector* of a syntax tree $\mathfrak{t}$ to be its parallel destination vector if $\mathfrak{t}$ is tidy parallel, and to be 0 if $\mathfrak{t}$ is messy parallel. Finally, we say that two syntax trees $\mathfrak{t}, \mathfrak{t}'$ on $\mathfrak{B}_k$ are *tidy parallel k-signaletic equivalent* and we write $\mathfrak{t} \equiv^{\|} \mathfrak{t}'$ if they have the same tidy parallel destination vector. In particular, all messy parallel syntax trees are equivalent to the zero tree 0.

Proposition 3.10. *The tidy parallel k-signaletic equivalence is compatible with grafting of syntax trees: $\mathfrak{t} \circ_i \mathfrak{s} \equiv^{\|} \mathfrak{t}' \circ_i \mathfrak{s}'$ for any syntax trees $\mathfrak{t} \equiv^{\|} \mathfrak{t}'$ of arity p and $\mathfrak{s} \equiv^{\|} \mathfrak{s}'$ of arity q, and any $i \in [p]$.*

Proof. Let p and q denote the tidy parallel destination vectors of $\mathfrak{t}$ and $\mathfrak{s}$ respectively. Then the parallel series destination vector r of $\mathfrak{t} \circ_i \mathfrak{s}$ is 0 unless $\mathsf{q}_\ell = 1$ for all $j \in [k]$ such that $\mathsf{p}_j \neq i$, and if so, is equal to the messy parallel destination vector, that is for all $j \in [k]$,

$$\mathsf{r}_j = \begin{cases} \mathsf{p}_j & \text{if } \mathsf{p}_j < i, \\ \mathsf{p}_j + \mathsf{q}_j - 1 & \text{if } \mathsf{p}_j = i, \\ \mathsf{p}_j + q - 1 & \text{if } \mathsf{p}_j > i. \end{cases} \tag{2}$$

□

Here are some examples of Equation (2): for $\mathsf{p} := (\!|2, 1, 4, 2, 2|\!)_5$ and $\mathsf{q} := (\!|3, 1, 1, 4, 1|\!)_4$, we have $\mathsf{p} \circ_1 \mathsf{q} = \mathsf{p} \circ_3 \mathsf{q} = \mathsf{p} \circ_4 \mathsf{q} = \mathsf{p} \circ_5 \mathsf{q} = 0$ and $\mathsf{p} \circ_2 \mathsf{q} = (\!|4, 1, 7, 5, 2|\!)_8$.

Definition 3.11. *The* tidy parallel k-signaletic operad $\mathsf{TSig}_k^{\|}$ *is the quotient of the free operad on* $\mathfrak{B}_k := \{\prec, \succ\}^k$ *by the tidy parallel k-signaletic equivalence.*

Similarly to Proposition 3.5, we can immediately observe the connection with the white Manin product of Section 2.5.

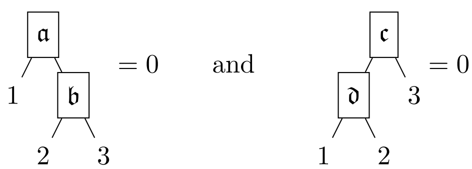
Proposition 3.12. *For any two integers k and l, we have*

$$\mathsf{TSig}_k^{\|} = \mathsf{Dup}_{\prec}^{! \,\square k} \qquad \textit{and} \qquad \mathsf{TSig}_{k+l}^{\|} = \mathsf{TSig}_k^{\|} \,\square\, \mathsf{TSig}_l^{\|}.$$

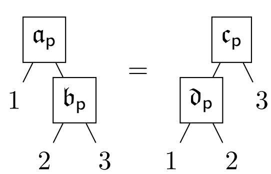
This implies that the tidy parallel k-signaletic operad $\mathsf{TSig}_k^{\|}$ is quadratic and Koszul. As already mentioned, we will see a uniform argument in Section 3.4. We now extract from the definitions the quadratic relations which provide a presentation of $\mathsf{TSig}_k^{\|}$.

Definition 3.13. *We call* tidy parallel k-signaletic relations *all the quadratic relations of* $\mathsf{TSig}_k^{\|}$*, that is all the relations of the form $\mathfrak{t} = 0$ where $\mathfrak{t}$ is a messy parallel syntax tree with two nodes (i.e. of arity 3), and all the relations of the form $\mathfrak{t}_1 = \mathfrak{t}_2$ where $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two tidy syntax trees with two nodes (i.e. of arity 3) sharing the same parallel destination vector.*

These relations can be explicitly described for any k as follows. First, the tidy parallel k-signaletic operad $\mathsf{TSig}_k^{\|}$ satisfies the quadratic relations



for any $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathfrak{B}_k$ such that $\mathfrak{a}_i = \prec$ and $\mathfrak{b}_i = \succ$ for some $i \in [k]$, and $\mathfrak{c}_j = \succ$ and $\mathfrak{d}_j = \succ$ for some $j \in [k]$. Second, for any destination vector $\mathsf{p} \in [3]^k$, the tidy parallel k-signaletic operad $\mathsf{TSig}_k^{\|}$ satisfies the quadratic relation



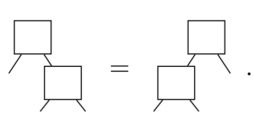
where $\mathfrak{a}_\mathsf{p}, \mathfrak{b}_\mathsf{p}, \mathfrak{c}_\mathsf{p}, \mathfrak{d}_\mathsf{p} \in \mathfrak{B}_k$ are defined by

$$(\mathfrak{a}_\mathsf{p})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \succ & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases} \quad (\mathfrak{b}_\mathsf{p})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \prec & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases} \quad (\mathfrak{c}_\mathsf{p})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \prec & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases} \quad (\mathfrak{d}_\mathsf{p})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \succ & \text{if } \mathsf{p}_i = 2 \\ \prec & \text{if } \mathsf{p}_i = 3 \end{cases}$$

for all $i \in [k]$. Again, there are $2^{1+2k} - 3^k$ independent tidy parallel k-signaletic relations among syntax trees of arity 3 on $\mathfrak{B}_k$.

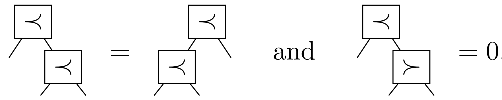
Example 3.14 (Tidy parallel 0-, 1- and 2-signaletic relations)**.** The tidy parallel 0-signaletic relation is the associative relation:

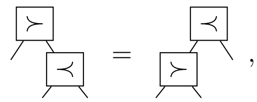
($\mathsf{TSig}^{\|}$.)

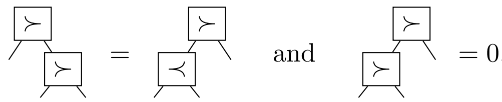


In other words, the tidy parallel 0-signaletic operad $\mathsf{TSig}_0^{\parallel}$ is just the associative operad As.

The tidy parallel 1-signaletic relations are the 5 twisted dual duplicial relations:

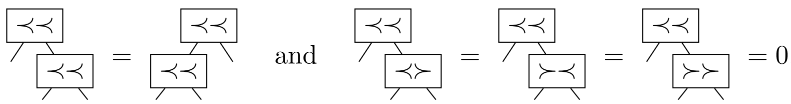
($\mathsf{TSig}^{\parallel}$ 1) 

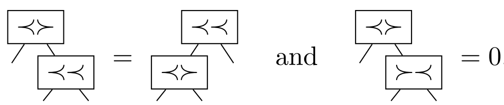
($\mathsf{TSig}^{\parallel}$ 2) 

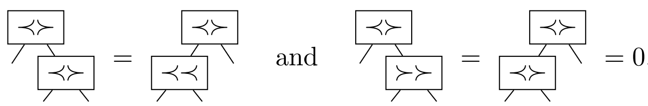
($\mathsf{TSig}^{\parallel}$ 3) 

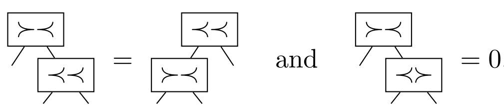
In other words, the tidy parallel 1-signaletic operad $\mathsf{TSig}_1^{\parallel}$ is just the twisted dual duplicial operad $\mathsf{Dup}^!_{\prec}$.

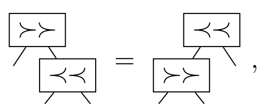
The tidy parallel 2-signaletic relations are the following 23 relations:

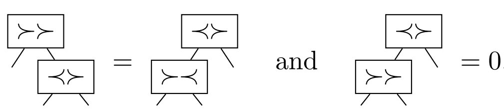
($\mathsf{TSig}^{\parallel}$ 11) 

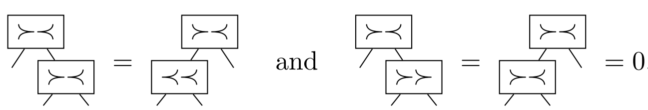
($\mathsf{TSig}^{\parallel}$ 12) 

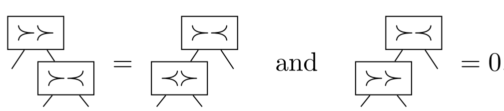
($\mathsf{TSig}^{\parallel}$ 13) 

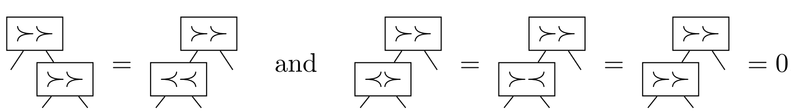
($\mathsf{TSig}^{\parallel}$ 21) 

($\mathsf{TSig}^{\parallel}$ 22) 

($\mathsf{TSig}^{\parallel}$ 23) 

($\mathsf{TSig}^{\parallel}$ 31) 

($\mathsf{TSig}^{\parallel}$ 32) 

($\mathsf{TSig}^{\parallel}$ 33) 

Remark 3.15. Similarly to Remark 3.8, for any subset I of $[k]$ of cardinal $\ell \le k$, there is a surjective restriction operad morphism $\mathsf{Res}_I : \mathsf{TSig}_k^{\parallel} \to \mathsf{TSig}_\ell^{\parallel}$ sending a destination vector $\mathsf{p} = (\!| \mathsf{p}_1, \dots, \mathsf{p}_k |\!)_n$ to the destination subvector $\mathsf{Res}_I(\mathsf{p}) := (\!| \mathsf{p}_i \mid i \in I |\!)_n$.

Remark 3.16. Similarly to Remark 3.9, since the to two operations $\prec$ and $\prec + \succ$ are associative in $\mathsf{TSig}_1^{\parallel}$, for any $i \in [k+1]$ and $\mathfrak{a} \in \{\prec, \prec + \succ\}$, the map $\mathsf{Ins}_i^{\mathfrak{a}} : \mathsf{TSig}_k^{\parallel} \to \mathsf{TSig}_{k+1}^{\parallel}$ inserting $\mathfrak{a}$ at position i in every nodes is an injective operad morphism. On destination vectors, the map $\mathsf{Ins}_i^{\prec}$ inserts a 1 at position i, while the map $\mathsf{Ins}_i^{\prec+\succ}$ is given by

$$\mathsf{Ins}_i^{\prec+\succ}(\llparenthesis r_1, \dots, r_k \rrparenthesis_n) = \sum_{d \in [n]} \llparenthesis r_1, \dots, r_{i-1}, d, r_i, \dots, r_k \rrparenthesis_n.$$

Remark 3.17. In this parallel setting, we have in fact much more freedom on the additional constraints that we impose for the tidy situation. Consider a constraint word $\mathbf{c}$ with k letters in $\{\prec, \odot \succ\}$. We say that a syntax tree $\mathfrak{t}$ on $\mathfrak{B}_k$ is ***c**-tidy parallel* if at each node, the i-th traffic signal either is contained in the i-th parallel route of $\mathfrak{t}$, or points to the left if $\mathbf{c}_i = \prec$, to the right if $\mathbf{c}_i = \succ$, and towards the i-th parallel route of $\mathfrak{t}$ if $\mathbf{c}_i = \odot$. The resulting $\mathbf{c}$-tidy parallel signaletic equivalence is then clearly compatible with grafting. Indeed, for two syntax trees $\mathfrak{t}$ and $\mathfrak{s}$ of arities p and q with $\mathbf{c}$-tidy parallel destination vectors p and q, the $\mathbf{c}$-tidy parallel destination vector r of $\mathfrak{t} \circ_i \mathfrak{s}$ is 0 unless

- for all $j \in [k]$ such that $\mathsf{p}_j < i$, we have $\mathsf{q}_j = 1$ if $\mathbf{c}_j \in \{\odot, \prec\}$ and $\mathsf{q}_j = q$ if $\mathbf{c}_j = \succ$,
- for all $j \in [k]$ such that $\mathsf{p}_j > i$, we have $\mathsf{q}_j = 1$ if $\mathbf{c}_j = \prec$ and $\mathsf{q}_j = q$ if $\mathbf{c}_j \in \{\odot, \succ\}$,

and if so, is equal to the messy parallel destination vector. This defines the ***c**-tidy parallel signaletic operad* $\mathsf{TSig}_{\mathbf{c}}^{\parallel}$ as in Definition 3.11. In other words, for any words $\mathbf{c}, \mathbf{d} \in \{\prec, \odot, \succ\}$, we have

$$\mathsf{TSig}_{\mathbf{c}}^{\parallel} = \bigsqcup_{i \in |\mathbf{c}|} \mathsf{Dup}_{\mathbf{c}_i}^{!} \qquad \text{and} \qquad \mathsf{TSig}_{\mathbf{c} \cdot \mathbf{d}}^{\parallel} = \mathsf{TSig}_{\mathbf{c}}^{\parallel} \,\square\, \mathsf{TSig}_{\mathbf{d}}^{\parallel}.$$

We have decided to present the tidy parallel k-signaletic operad with the constraint $\mathbf{c} = \prec^k$ to simplify the presentation and since this will be the only possible option in series. However, note that we will use an action of the Koszul dual of the $\prec\succ$-tidy parallel signaletic operad in Section 5.1.

3.3. **Series signaletic operads.** We now consider the series signaletic operads, and we treat separately the messy and tidy situations.

3.3.1. *Messy series signaletic operads.* Fix an integer $k \geq 0$ and consider a syntax tree $\mathfrak{t}$ on the operators $\{\prec, \succ\}^k$ with n leaves that we label $1, \dots, n$ from left to right. Assume now that k cars $c_1, \dots, c_k$ arrive sequencially at the root of $\mathfrak{t}$ and that the j-th car c_j always follows the indication given by the leftmost remaining letter of the traffic signal at each branching node and erases this letter. We call *series routes* the paths followed by the k cars in the syntax tree $\mathfrak{t}$. Finally, each car c_j ends at a certain leaf labeled ℓ_j and we call *series destination vector* of $\mathfrak{t}$ the vector $\llparenthesis \ell_1, \dots, \ell_k \rrparenthesis_n$. See Figure 7.

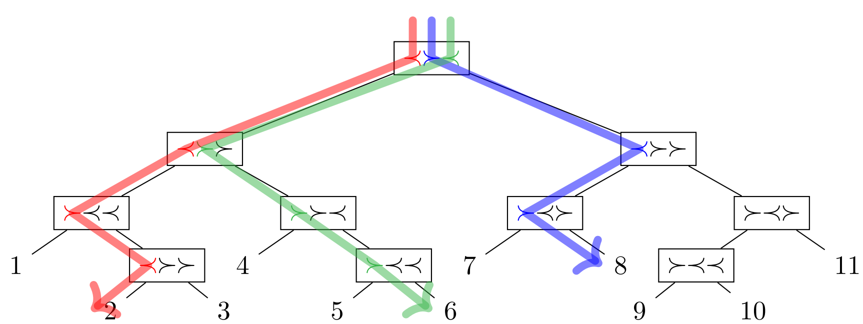


FIGURE 7. Traversing the syntax tree in series. The series routes are marked, and the series destination vector is $\llparenthesis 2, 8, 6 \rrparenthesis_{11}$.

We say that two syntax trees $\mathfrak{t}, \mathfrak{t}'$ on $\mathfrak{B}_k$ with the same arity are *messy series k-signaletic equivalent* and we write $\mathfrak{t} \approx^{\ddagger} \mathfrak{t}'$ if they have the same series destination vector.

Proposition 3.18. *The messy series k-signaletic equivalence is compatible with grafting of syntax trees:* $\mathfrak{t} \circ_i \mathfrak{s} \approx^{\ddagger} \mathfrak{t}' \circ_i \mathfrak{s}'$ *for any syntax trees* $\mathfrak{t} \approx^{\ddagger} \mathfrak{t}'$ *of arity p and* $\mathfrak{s} \approx^{\ddagger} \mathfrak{s}'$ *of arity q, and any $i \in [p]$.*

Proof. Consider the j-th car c_j and denote by p_j its series destination in $\mathfrak{t}$ and by q_j its series destination in $\mathfrak{s}$. Then its series destination r_j in $\mathfrak{t} \circ_i \mathfrak{s}$ is given by

$$\mathsf{r}_j = \begin{cases} \mathsf{p}_j & \text{if } \mathsf{p}_j < i, \\ \mathsf{p}_j + \mathsf{q}_{|\{\ell \le j \mid \mathsf{p}_\ell = i\}|} - 1 & \text{if } \mathsf{p}_j = i, \\ \mathsf{p}_j + q - 1 & \text{if } \mathsf{p}_j > i. \end{cases} \tag{3}$$ □

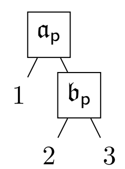
Here are some examples of Equation (3): for $\mathsf{p} := (\!|2,1,4,2,2|\!)_5$ and $\mathsf{q} := (\!|3,1,6,1,2|\!)_6$, we have $\mathsf{p} \circ_1 \mathsf{q} = (\!|7,3,9,7,7|\!)_{10}$, $\mathsf{p} \circ_2 \mathsf{q} = (\!|4,1,9,2,7|\!)_{10}$, $\mathsf{p} \circ_3 \mathsf{q} = (\!|2,1,9,2,2|\!)_{10}$, $\mathsf{p} \circ_4 \mathsf{q} = (\!|2,1,6,2,2|\!)_{10}$, and $\mathsf{p} \circ_5 \mathsf{q} = (\!|2,1,4,2,2|\!)_{10}$.

Definition 3.19. *The messy series k-signaletic operad $\mathsf{MSig}^{\ddagger}_k$ is the quotient of the free operad on $\mathfrak{B}_k := \{\prec, \succ\}^k$ by the messy series k-signaletic equivalence.*

As already mentioned, we will see in Section 3.4 that the messy series k-signaletic operad $\mathsf{MSig}^{\ddagger}_k$ is quadratic and Koszul. We now extract from the definitions the quadratic relations which provide a presentation of $\mathsf{MSig}^{\ddagger}_k$.

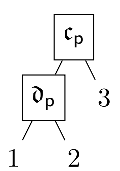
Definition 3.20. *We call messy series k-signaletic relations all the quadratic relations of $\mathsf{MSig}^{\ddagger}_k$, that is all the relations of the form $\mathfrak{t}_1 = \mathfrak{t}_2$ where $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two syntax trees with two nodes (i.e. of arity 3) sharing the same series destination vector.*

These relations can be explicitly described for any k as follows. For any destination vector $\mathsf{p} \in [3]^k$, the messy series k-signaletic operad $\mathsf{MSig}^{\ddagger}_k$ satisfies the quadratic relation $\mathfrak{t}_1 = \mathfrak{t}_2$ for any two syntax trees $\mathfrak{t}_1$ and $\mathfrak{t}_2$ of one of the following forms:



where $(\mathfrak{a}_\mathsf{p})_i = \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \succ & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases}$ and $(\mathfrak{b}_\mathsf{p})_i = \begin{cases} \prec & \text{if } (\mathsf{p}^{\{2,3\}})_i = 2 \\ \succ & \text{if } (\mathsf{p}^{\{2,3\}})_i = 3 \end{cases}$ for all $i \in [k]$,

or

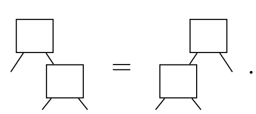


where $(\mathfrak{c}_\mathsf{p})_i = \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \prec & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases}$ and $(\mathfrak{d}_\mathsf{p})_i = \begin{cases} \prec & \text{if } (\mathsf{p}^{\{1,2\}})_i = 1 \\ \succ & \text{if } (\mathsf{p}^{\{1,2\}})_i = 2 \end{cases}$ for all $i \in [k]$,

where for any $L \subseteq [3]$, we have denoted by p^L the subword of p consisting only of the letters which belong to L, and for any $x \in L$ the condition $(\mathsf{p}^L)_i = x$ implicitly implies that p^L has length at least i. Again, there are $2^{1+2k} - 3^k$ independent messy series k-signaletic relations among syntax trees of arity 3 on $\mathfrak{B}_k$.

Example 3.21 (Messy series 0-, 1- and 2-signaletic relations)**.** The messy series 0-signaletic relation is the associative relation:

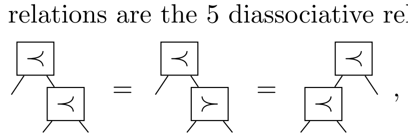
($\mathsf{MSig}^{\ddagger}$.)



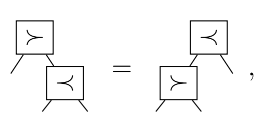
In other words, the messy series 0-signaletic operad $\mathsf{MSig}^{\ddagger}_0$ is just the associative operad As.

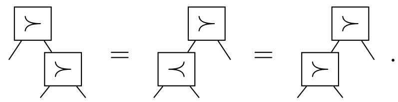
The messy series 1-signaletic relations are the 5 diassociative relations:

($\mathsf{MSig}^{\ddagger}$ 1)



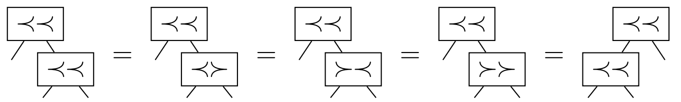
($\mathsf{MSig}^{\ddagger}$ 2)

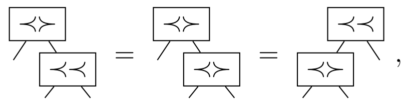


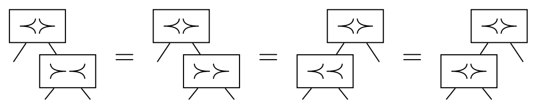
(MSig‡ 3) 

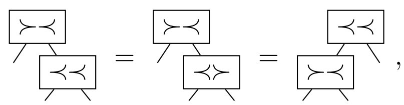
In other words, the messy series 1-signaletic operad $\mathsf{MSig}^\ddagger_1$ is just the diassociative operad Diass.

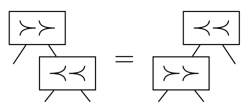
The messy series 2-signaletic relations are the following 23 relations:

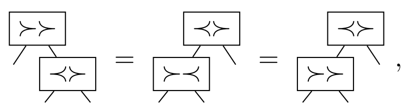
(MSig‡ 11) 

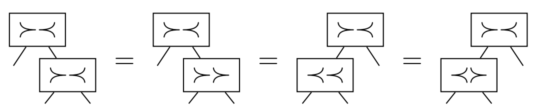
(MSig‡ 12) 

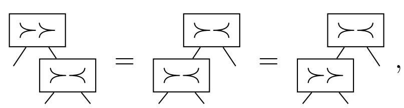
(MSig‡ 13) 

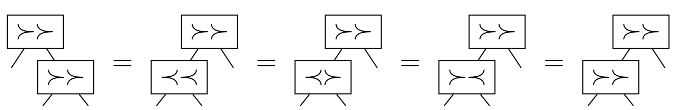
(MSig‡ 21) 

(MSig‡ 22) 

(MSig‡ 23) 

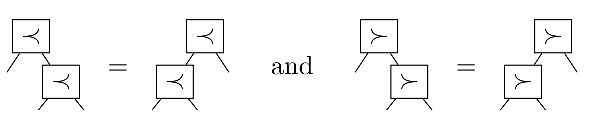
(MSig‡ 31) 

(MSig‡ 32) 

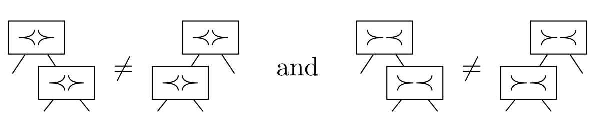
(MSig‡ 33) 

Remark 3.22. The only subset $I \subseteq [k]$ of cardinal ℓ for which the restriction Res_I defined in Remark 3.8 is a surjective morphism $\mathsf{Res}_I : \mathsf{MSig}^\ddagger_k \mapsto \mathsf{MSig}^\ddagger_l$ between messy series k-signaletic operads is the initial subset $[\ell]$. We denote this morphism by Res^k_ℓ. Its action on destination vectors or operators amounts to keep only the first ℓ coordinates.

Remark 3.23. In contrast to Remark 3.9, the map $\mathsf{Ins}^{\mathfrak{a}}_i$ inserting an operation $\mathfrak{a} \in \{\prec, \succ\}$ at position i in all nodes of a syntax tree is not an operad morphism $\mathsf{MSig}^\ddagger_k \mapsto \mathsf{MSig}^\ddagger_{k+1}$. Counter-examples are given by:



while



3.3.2. *Tidy series signaletic operads.* We now impose additional constraints on the trafic signals not contained in the series routes. We say that a syntax tree $\mathfrak{t}$ on $\mathfrak{B}_k$ is *tidy series* if all the traffic signals not contained in its series routes point to the left. Otherwise, we say that $\mathfrak{t}$ is *messy series*. We define the *tidy series destination vector* of a syntax tree $\mathfrak{t}$ to be its series destination vector if $\mathfrak{t}$ is tidy series, and to be 0 if $\mathfrak{t}$ is messy series. Finally, we say that two syntax trees $\mathfrak{t}, \mathfrak{t}'$ on $\mathfrak{B}_k$ are *tidy series k-signaletic equivalent* and we write $\mathfrak{t} \equiv^{\ddagger} \mathfrak{t}'$ if they have the same tidy series destination vector. In particular, all messy series syntax trees are equivalent to the zero tree 0.

Proposition 3.24. *The tidy series k-signaletic equivalence is compatible with grafting of syntax trees: $\mathfrak{t} \circ_i \mathfrak{s} \equiv^{\ddagger} \mathfrak{t}' \circ_i \mathfrak{s}'$ for any syntax trees $\mathfrak{t} \equiv^{\ddagger} \mathfrak{t}'$ of arity p and $\mathfrak{s} \equiv^{\ddagger} \mathfrak{s}'$ of arity q, and any $i \in [p]$.*

Proof. Let p and q denote the tidy series destination vectors of $\mathfrak{t}$ and $\mathfrak{s}$ respectively. For $i \in [p]$, we denote by $|\mathsf{p}|_i := |\{\ell \in [k] \mid \mathsf{p}_\ell = i\}|$ the number of occurrences of i in p. Then the tidy series destination vector r of $\mathfrak{t} \circ_i \mathfrak{s}$ is 0 unless $\mathsf{q}_\ell = 1$ for all $|\mathsf{p}|_i < \ell \le k$, and if so, is equal to the messy series destination vector, that is for all $j \in [k]$,

$$\mathsf{r}_j = \begin{cases} \mathsf{p}_j & \text{if } \mathsf{p}_j < i, \\ \mathsf{p}_j + \mathsf{q}_{|\{\ell \le j \mid \mathsf{p}_\ell = i\}|} - 1 & \text{if } \mathsf{p}_j = i, \\ \mathsf{p}_j + q - 1 & \text{if } \mathsf{p}_j > i. \end{cases} \tag{4}$$
□

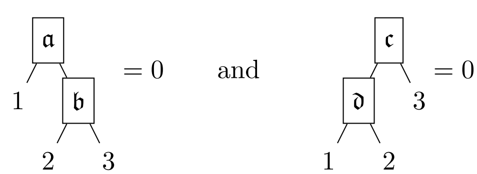
Here are some examples of Equation (4): for $\mathsf{p} := (\!| 2,1,4,2,4 |\!)_5$ and $\mathsf{q} := (\!| 3,5,1,1,1 |\!)_6$, we have $\mathsf{p} \circ_1 \mathsf{q} = \mathsf{p} \circ_3 \mathsf{q} = \mathsf{p} \circ_5 \mathsf{q} = 0$, $\mathsf{p} \circ_2 \mathsf{q} = (\!| 4,1,9,6,9 |\!)_{10}$, $\mathsf{p} \circ_4 \mathsf{q} = (\!| 2,1,6,2,8 |\!)_{10}$.

Definition 3.25. *The* tidy series k-signaletic operad $\mathsf{TSig}_k^{\ddagger}$ *is the quotient of the free operad on $\mathfrak{B}_k := \{\prec, \succ\}^k$ by the tidy series k-signaletic equivalence.*

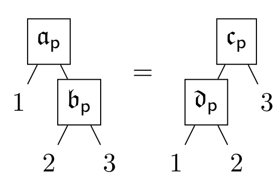
As already mentioned, we will see in Section 3.4 that the tidy series k-signaletic operad $\mathsf{TSig}_k^{\ddagger}$ is quadratic and Koszul. We now extract from the definitions the quadratic relations which provide a presentation of $\mathsf{TSig}_k^{\ddagger}$.

Definition 3.26. *We call* tidy series k-signaletic relations *all the quadratic relations of $\mathsf{TSig}_k^{\ddagger}$, that is all the relations of the form $\mathfrak{t} = 0$ where $\mathfrak{t}$ is a messy series syntax tree with two nodes (i.e. of arity 3), and all the relations of the form $\mathfrak{t}_1 = \mathfrak{t}_2$ where $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two tidy syntax trees with two nodes (i.e. of arity 3) sharing the same series destination vector.*

These relations can be explicitly described for any k as follows. First, the tidy series k-signaletic operad $\mathsf{TSig}_k^{\ddagger}$ satisfies the quadratic relations



for any $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathfrak{B}_k$ such that $\mathfrak{b}_i = \succ$ for some $i > |\mathfrak{a}|_\succ$, and $\mathfrak{d}_j = \succ$ for some $j > |\mathfrak{c}|_\prec$. Second, for any destination vector $\mathsf{p} \in [3]^k$, the tidy series k-signaletic operad $\mathsf{TSig}_k^{\ddagger}$ satisfies the quadratic relation



where $\mathfrak{a}_{\mathsf{p}}, \mathfrak{b}_{\mathsf{p}}, \mathfrak{c}_{\mathsf{p}}, \mathfrak{d}_{\mathsf{p}} \in \mathfrak{B}_k$ are defined by

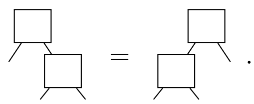
$$(\mathfrak{a}_{\mathsf{p}})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \succ & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases} \qquad (\mathfrak{b}_{\mathsf{p}})_i := \begin{cases} \prec & \text{if } |\mathsf{p}^{\{2,3\}}| < i \\ \prec & \text{if } (\mathsf{p}^{\{2,3\}})_i = 2 \\ \succ & \text{if } (\mathsf{p}^{\{2,3\}})_i = 3 \end{cases}$$

$$(\mathfrak{c}_{\mathsf{p}})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \prec & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases} \qquad (\mathfrak{d}_{\mathsf{p}})_i := \begin{cases} \prec & \text{if } (\mathsf{p}^{\{1,2\}})_i = 1 \\ \succ & \text{if } (\mathsf{p}^{\{1,2\}})_i = 2 \\ \prec & \text{if } |\mathsf{p}^{\{1,2\}}| < i \end{cases}$$

for all $i \in [k]$. Again for any $L \subseteq [3]$, we have denoted by p^L the subword of p consisting only of the letters which belong to L, and for any $x \in L$ the condition $(\mathsf{p}^L)_i = x$ implicitly assume that p^L has length at least i. Again, there are $2^{1+2k} - 3^k$ independent tidy series k-signaletic relations among syntax trees of arity 3 on $\mathfrak{B}_k$.

Example 3.27 (Tidy series 0-, 1- and 2-signaletic relations)**.** The tidy series 0-signaletic relation is the associative relation:

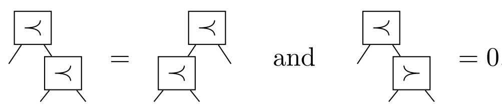
(TSig‡.)



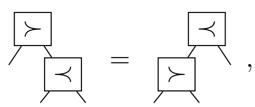
In other words, the tidy series 0-signaletic operad $\mathsf{TSig}_0^{\ddagger}$ is just the associative operad As.

The tidy series 1-signaletic relations are the 5 twisted dual duplicial relations:

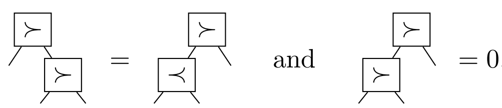
(TSig‡ 1)



(TSig‡ 2)



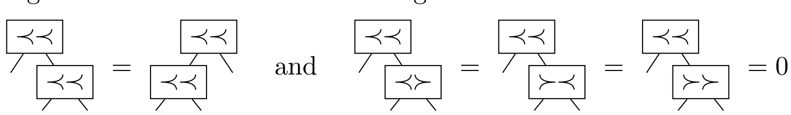
(TSig‡ 3)



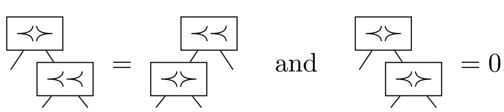
In other words, the tidy series 1-signaletic operad $\mathsf{TSig}_1^{\ddagger}$ is just the twisted dual duplicial operad $\mathsf{Dup}^!_{\prec}$.

The tidy series 2-signaletic relations are the following 23 relations:

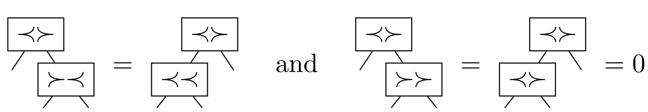
(TSig‡ 11)



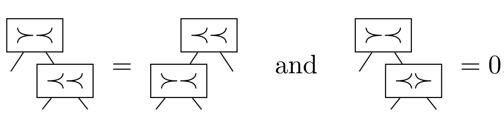
(TSig‡ 12)



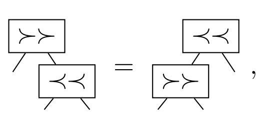
(TSig‡ 13)

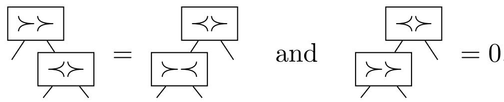


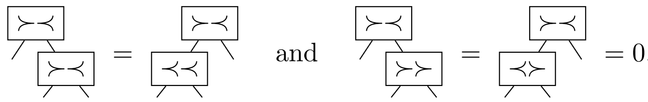
(TSig‡ 21)

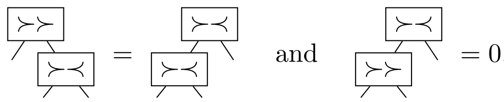


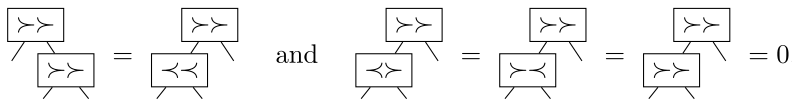
(TSig‡ 22)



($\mathsf{TSig}^{\ddagger}$ 23) 

($\mathsf{TSig}^{\ddagger}$ 31) 

($\mathsf{TSig}^{\ddagger}$ 32) 

($\mathsf{TSig}^{\ddagger}$ 33) 

Remark 3.28. As in Remark 3.22, there is a surjective restriction morphism $\mathsf{Res}^k_\ell : \mathsf{TSig}^{\ddagger}_k \mapsto \mathsf{TSig}^{\ddagger}_l$.

Remark 3.29. Similar to Remark 3.23, the maps $\mathsf{Ins}_i^{\prec}$ and $\mathsf{Ins}_i^{\prec + \succ}$ are not operad morphisms $\mathsf{TSig}^{\ddagger}_k \mapsto \mathsf{TSig}^{\ddagger}_{k+1}$. For instance, the reader can check that when $k = 1$, for any $i \in \{1, 2\}$ and $\mathfrak{a} \in \{\prec, \prec + \succ\}$, the morphism relation is not verified on $\succ \circ_1 \prec$.

3.4. **Hilbert series, presentation and Koszulity.** In this section, we give the Hilbert series and prove the quadratic presentation and Koszulity of all signaletic operads, using an approach which does not depend on whether the cars arrive in series or parallel, and whether the signals outside the routes of the cars are pointing or not to the left. We will therefore speak here about the signaletic operad and signaletic relations without further precision concerning series/parallel, nor messy/tidy. We therefore write Sig_k to denote without distinction any of the four k-signaletic operads $\mathsf{MSig}^{\|}_k$, $\mathsf{TSig}^{\|}_k$, $\mathsf{MSig}^{\ddagger}_k$, or $\mathsf{TSig}^{\ddagger}_k$.

3.4.1. *Hilbert series.* By definition, the basis of Sig_k is given by syntax trees on $\mathfrak{B}_k$ modulo the k-signaletic equivalence, and the composition is given by grafting syntax trees. Alternatively, we can represent the basis of Sig_k using destination vectors: namely, $\mathsf{Sig}_k = \bigoplus_{p \ge 1} \mathsf{Sig}_k(p)$ where $\mathsf{Sig}_k(p) = [p]^k$ represent all possible destination vectors in a syntax tree on $\mathfrak{B}_k$ with arity p, and the composition rule is described by the rules given in Equations (1) to (4). Moreover, all destination vectors can clearly be obtained from a syntax tree. This shows the following statement.

Corollary 3.30. *The Hilbert series of the (messy or tidy, parallel or series) k-signaletic operad Sig_k is given by*

$$\mathcal{H}_{\mathsf{Sig}_k}(t) = \sum_{p \ge 1} p^k t^p = \frac{\mathrm{Eul}_k(t)}{(1-t)^{k+1}},$$

where $\mathrm{Eul}_k(t) := \sum\limits_{p \ge 1} \left\langle {k \atop p} \right\rangle t^p$ *and* $\left\langle {k \atop p} \right\rangle$ *is the number of permutations of $\mathfrak{S}_k$ with precisely p descents.*

3.4.2. *Signaletic Tamari order.* In order to show that the k-signaletic operads are Koszul, we will need to orient the k-signaletic relations into a well-chosen rewriting system. For this, we will be guided by the following natural generalization of the Tamari lattice of Section 2.7 to the syntax trees on $\mathfrak{B}_k$.

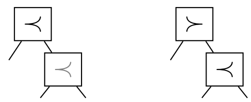
Definition 3.31. *The n-th k-signaletic Tamari order is the partial order on the set* $\mathbf{Trees}(\mathfrak{B}_k)(n)$ *of syntax trees of arity n defined by $\mathfrak{s} \le \mathfrak{t}$ if*

- *the shape of $\mathfrak{s}$ is Tamari strictly smaller than the shape of $\mathfrak{t}$*
- *or $\mathfrak{s}$ and $\mathfrak{t}$ have the same shape and each letter in $\mathfrak{s}$ is smaller than the corresponding letter in $\mathfrak{t}$ for the order $\succ \le \prec$.*

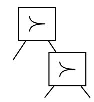
We will show in Lemma 3.34 that the following syntax trees are the maximums of the k-signaletic Tamari order in their signaletic equivalence classes.

Definition 3.32. *A* right k-signaletic comb *is a syntax tree on* $\mathfrak{B}_k := \{\prec, \succ\}^k$ *where the left child of each node is empty and each signal which is not visited by a car points to the left (i.e. it is tidy).*

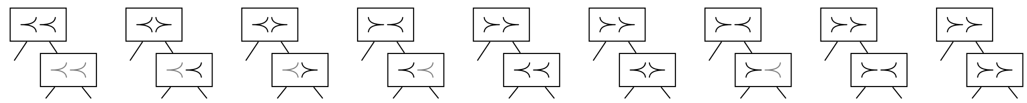
For instance, the quadratic right k-signaletic combs are



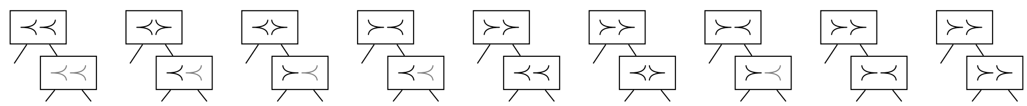
and



for the 1-signaletic operads,

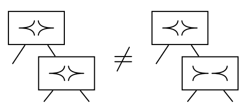


for the parallel 2-signaletic operads, and



for the series 2-signaletic operads. The unvisited signals are colored in gray, and they all point to the left.

Note that the right k-signaletic combs are distinct in the series or parallel signaletic operad:

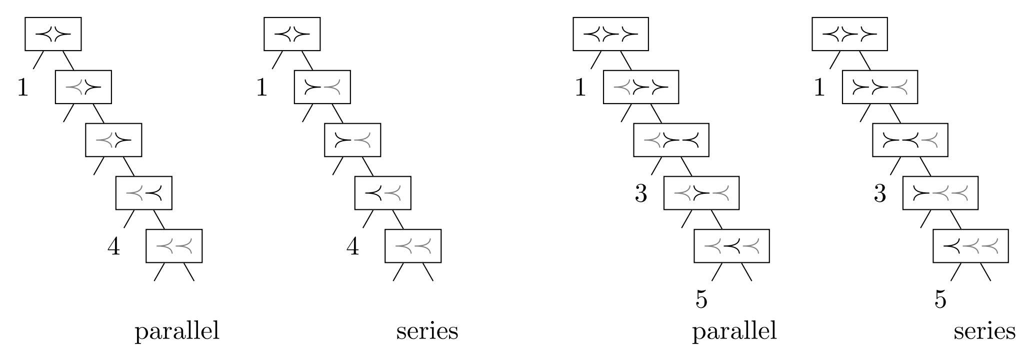


are respectively the parallel and series right 2-signaletic combs with destination vector $(\!|1,3|\!)_3$. However, the right k-signaletic combs are counted in both cases by the number p^k of possible signaletic destination vectors, as shown by the following statement.

Lemma 3.33. *The map sending the right k-signaletic combs to their destination vectors is bijective.*

Proof. For a given destination vector, the corresponding right k-signaletic comb is obtained by letting the k cars traverse the right comb in such a way that they reach their planned destination and finally complete the unvisited signals with $\prec$. $\square$

For example, the following picture shows respectively the series and parallel right k-signaletic combs corresponding to the destination vector $(\!|1,4|\!)_6$ and the destination vector $(\!|1,5,3|\!)_6$:



Again, the unvisited signals are colored in gray, and they all point to the left.

Lemma 3.34. *In each k-signaletic congruence class, the right k-signaletic comb is the unique k-signaletic Tamari maximum.*

Proof. Consider a syntax tree $\mathfrak{t}$ and a right k-signaletic comb $\mathfrak{c}$ with the same arity and destination vector. If $\mathfrak{t}$ is not a right comb, then the shape of $\mathfrak{t}$ is Tamari smaller than the shape of $\mathfrak{c}$, so that $\mathfrak{t}$ is k-signaletic Tamari smaller than $\mathfrak{c}$. Otherwise, since $\mathfrak{t}$ and $\mathfrak{c}$ have the same destination vector, the signals visited by the cars coincide in the two syntax trees. As all unvisited signals in $\mathfrak{c}$ point to the left, and $\succ \leq \prec$, we obtain that $\mathfrak{t}$ is k-signaletic Tamari smaller than $\mathfrak{c}$. □

3.4.3. *Rewriting system.* We now orient the k-signaletic relations of Definitions 3.6, 3.13, 3.20 and 3.26 according to the signaletic Tamari order of Section 3.4.2.

Definition 3.35. *The k-signaletic rewriting system is the set of rewriting rules $(\mathfrak{s}, \mathfrak{c})$ where $\mathfrak{s}$ and $\mathfrak{c}$ are two distinct quadratic trees with the same destination vector and $\mathfrak{c}$ is a right k-signaletic comb.*

Lemma 3.36. *For any rewriting $\mathfrak{t} \to \mathfrak{p}$, then $\mathfrak{t} < \mathfrak{p}$ in the signaletic Tamari order.*

Proof. Call $(\mathfrak{s}, \mathfrak{q})$ the rule that was applied. We distinguish two cases:

- If $\mathfrak{s}$ and $\mathfrak{q}$ have distinct shapes then $\mathfrak{s}$ and $\mathfrak{q}$ are left (resp. right) quadratic tree. Consequently, the shape of $\mathfrak{p}$ is obtained from the shape of $\mathfrak{t}$ by a right Tamari rotation so that $\mathfrak{t} < \mathfrak{p}$.
- If $\mathfrak{s}$ and $\mathfrak{q}$ have the same shape then they are both right quadratic trees and the rewriting changed some $\succ$ to $\prec$ in the bottom node. Again $\mathfrak{t} < \mathfrak{p}$. □

Corollary 3.37. *The rewriting system terminates.*

Proof. The rewriting system is strictly increasing for a finite order. □

3.4.4. *Normal forms.* According to our choice of rewriting system, the quadratic normal forms are precisely the quadratic right k-signaletic combs. We aim to prove that this property holds for any arity.

Lemma 3.38. *Any syntax tree on $\mathfrak{B}_k := \{\prec, \succ\}^k$ can be rewritten to a right k-signaletic comb by the k-signaletic rewriting system.*

Proof. First of all, if a node has an edge to a non-leaf left child, then there is a rewriting rule which rotates this edge. Therefore, any syntax tree can be rewritten to a syntax tree with the shape of a right comb. As a consequence, it is sufficient to show that any syntax tree $\mathfrak{t}$ whose shape is a right comb can be ultimately rewritten to the unique right k-signaletic comb with the same destination vector as $\mathfrak{t}$. The solution is to start from the root of the right comb, and to perform the rewritings from top to bottom. This replaces all the unvisited signs by left signs leading to the right k-signaletic comb. □

Corollary 3.39. *The right k-signaletic combs are precisely the normal forms of the k-signaletic quadratic rewriting system.*

Proof. This follows from Lemmas 3.34, 3.36 and 3.38. □

3.4.5. *Presentation and Koszulity.* We conclude this section with the presentation and Koszulity of the signaletic operads.

Theorem 3.40. *The k-signaletic operad Sig_k is quadratic and Koszul. In particular, the k-signaletic relations give a presentation of Sig_k and the right k-signaletic combs form a Poincaré – Birkhoff – Witt basis of Sig_k.*

Proof. By Corollary 3.39, the right k-signaletic combs are the normal forms of the k-signaletic quadratic rewriting system. Since the number right k-signaletic combs of arity p is the dimension of the homogeneous component of degree p of the k-signaletic operad, we obtain that the k-signaletic rewriting system is convergent. This shows that the k-signaletic operads are quadratic and Koszul by definition, and that the right k-signaletic combs form a Poincaré – Birkhoff – Witt basis. □

Remark 3.41. The result of Theorem 3.40 was already partially established for messy k-signaletic operads: see [Lod01] for the diassociative operad Diass (*i.e.* the 1-signaletic operad), [Foi15, Vat05] for the operad $\mathsf{Quad}^!$ (*i.e.* the messy parallel 2-signaletic operad, see Remark 4.3), and [Vat05] for higher black Manin products of the diassociative operad Diass (*i.e.* the messy parallel k-signaletic operads, see Section 2.5 and Proposition 3.5).

Remark 3.42. Note that in the tidy situation, we actually used a slight extension of the notion of set-operad. Namely, we included a zero element $\mathfrak{o}_d$ in $\mathcal{O}(d)$ for each degree d such that any composition involving some $\mathfrak{o}_d$ results in some $\mathfrak{o}_{d'}$. Then the Koszulity of the set-operad is equivalent to the Koszulity of the linearized operad when $\mathfrak{o}_d$ is actually the zero of the vector space $\mathcal{O}(d)$.

4. Citelangis operads

We now introduce the citelangis operads, which are defined as the Koszul duals of the signaletic operads presented in Section 3. In this section, we describe their presentation in terms of generators and relations and discuss properties of their Hilbert series. We show in particular that some citelangis operads where already considered in the literature: the messy parallel 2-citelangis algebras are the quadrialgebras of [AL04, Foi15] and the messy series k-citelangis algebras are the k-twistiform algebras of [Pil18]. Combinatorial models and actions of the citelangis operads will be discussed later in Section 5.

4.1. Parallel citelangis operads.

We start with the Koszul duals of the parallel signaletic operads, and we treat separately the messy and tidy situations.

4.1.1. *Messy parallel citelangis operads.* We start with the messy setting.

Definition 4.1. *The* messy parallel k-citelangis operad $\mathsf{MCit}_k^{\|}$ *is the Koszul dual of the messy parallel k-signaletic operad:*

$$\mathsf{MCit}_k^{\|} := (\mathsf{MSig}_k^{\|})^!$$

Proposition 4.2. *For any two integers k and l, we have*

$$\mathsf{MCit}_k^{\|} = \mathsf{Dend}^{\square k} = \mathsf{Dend}^{\blacksquare k} \qquad \text{and} \qquad \mathsf{MCit}_{k+l}^{\|} = \mathsf{MCit}_k^{\|} \blacksquare \mathsf{MCit}_l^{\|}.$$

Remark 4.3. Besides the well-known associative and dendriform algebras, some special cases of messy parallel k-citelangis algebras were specifically studied in the literature:

- for $k = 2$, the messy parallel 2-citelangis algebras are known as quadri-algebras, introduced by M. Aguiar and J.-L. Loday in [AL04] and studied in [Vat05, Foi15].
- for $k = 3$, the messy parallel 3-citelangis algebras are the octo-algebras of [Ler03].

By Theorem 3.40, the messy parallel k-citelangis operad $\mathsf{MCit}_k^{\|}$ is quadratic and Koszul. To describe its quadratic relations, it is convenient to consider sums of operations like $\rtimes\!\!\!\ltimes := \prec + \succ$.

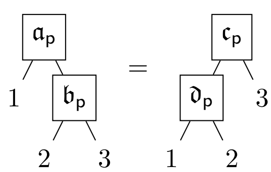
Proposition 4.4. *Consider the 3^k operations $\{\prec, \rtimes\!\!\!\ltimes, \succ\}^k$ related by*

$$\mathfrak{b} \rtimes\!\!\!\ltimes \mathfrak{b}' = \mathfrak{b} \prec \mathfrak{b}' + \mathfrak{b} \succ \mathfrak{b}'$$

for all $\mathfrak{b}, \mathfrak{b}' \in \{\prec, \rtimes\!\!\!\ltimes, \succ\}^*$ with $|\mathfrak{b}| + |\mathfrak{b}'| = k - 1$.

For any destination vector $\mathsf{p} \in [3]^k$, the messy parallel k-citelangis operad $\mathsf{MCit}_k^{\|}$ satisfies the quadratic relation

($\mathsf{MCit}^{\|}$ p)



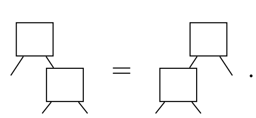
where the operators $\mathfrak{a}_\mathsf{p}, \mathfrak{b}_\mathsf{p}, \mathfrak{c}_\mathsf{p}, \mathfrak{d}_\mathsf{p} \in \{\prec, \rtimes\!\!\!\ltimes, \succ\}^k$ are defined by

$$(\mathfrak{a}_\mathsf{p})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \succ & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases} \quad (\mathfrak{b}_\mathsf{p})_i := \begin{cases} \rtimes\!\!\!\ltimes & \text{if } \mathsf{p}_i = 1 \\ \prec & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases} \quad (\mathfrak{c}_\mathsf{p})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \prec & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases} \quad (\mathfrak{d}_\mathsf{p})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \succ & \text{if } \mathsf{p}_i = 2 \\ \rtimes\!\!\!\ltimes & \text{if } \mathsf{p}_i = 3 \end{cases}$$

for any $i \in [k]$. Note that Equation ($\mathsf{MCit}^{\|}$ p) *involves $2^{|\mathsf{p}|_1}$ syntax trees of* **Trees**$(\mathfrak{B}_k)$ *on the left hand side and $2^{|\mathsf{p}|_3}$ syntax trees of* **Trees**$(\mathfrak{B}_k)$ *on the right hand side. We call these relations the* messy parallel k-citelangis relations.

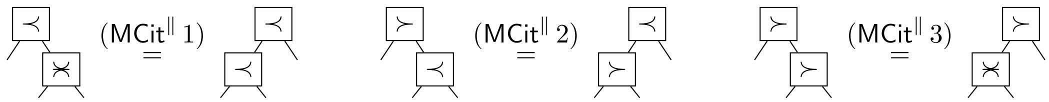
Example 4.5 (Messy parallel 0-, 1- and 2-citelangis relations)**.** The messy parallel 0-citelangis relation is the associative relation:

($\mathsf{MCit}^{\|}$.)



In other words, the messy parallel 0-citelangis operad $\mathsf{MCit}_0^{\parallel}$ is just the associative operad As.

The messy parallel 1-citelangis relations are the 3 dendriform relations:

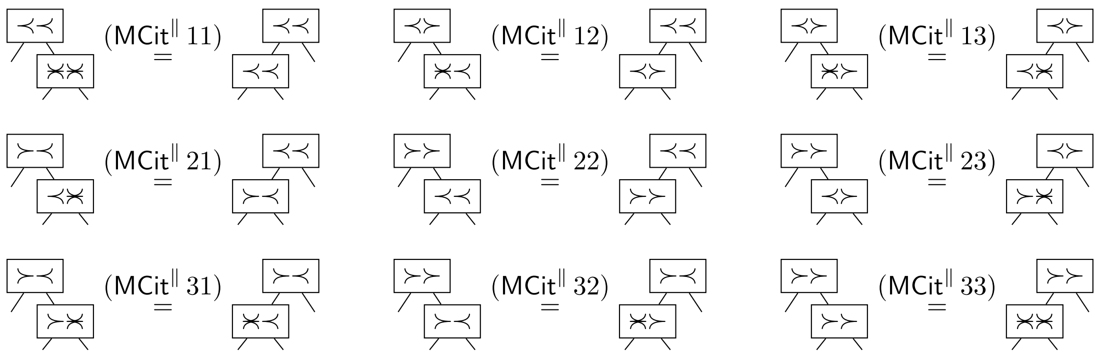


where

$$\bowtie = \prec + \succ.$$

In other words, the messy parallel 1-citelangis operad $\mathsf{MCit}_1^{\parallel}$ is just the dendriform operad Dend.

The messy parallel 2-citelangis relations are the following 9 relations:



where

$$\begin{aligned} \bowtie\prec &= \prec\prec + \succ\prec, & \bowtie\succ &= \prec\succ + \succ\succ, \\ \prec\bowtie &= \prec\prec + \prec\succ, & \succ\bowtie &= \succ\prec + \succ\succ, \\ \text{and} \quad \bowtie\bowtie &= \prec\bowtie + \succ\bowtie = \bowtie\prec + \bowtie\succ = \prec\prec + \prec\succ + \succ\prec + \succ\succ. \end{aligned}$$

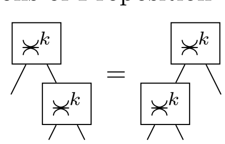
These relations are those of the Quad operad, see for instance [Foi15, Definition 1]. The translation between our notations and that of [Foi15, Definition 1] is the following:

$$\prec\prec = \nwarrow, \qquad \prec\succ = \nearrow, \qquad \succ\prec = \swarrow, \qquad \text{and} \qquad \succ\succ = \searrow.$$

Remark 4.6. For any subset $I = \{i_1 < \dots < i_\ell\}$ of $[k]$, the dual of the morphism Res_I defined in Remark 3.8 is an injective operad morphism $\mathsf{Res}_I^! : \mathsf{MCit}_\ell^{\parallel} \to \mathsf{MCit}_k^{\parallel}$. The image $\mathfrak{b} := \mathsf{Res}_I^!(\mathfrak{a})$ of an operator $\mathfrak{a}$ of $\mathsf{MCit}_\ell^{\parallel}$ verifies $\mathfrak{b}_{i_p} = \mathfrak{a}_p$ if $p \in [\ell]$ and $\mathfrak{b}_j = \bowtie$ if $j \in [k] \smallsetminus I$. Hence, for $\ell \le k$, there are $\binom{k}{\ell}$ natural messy parallel ℓ-citelangis structures in a messy parallel k-citelangis algebra.

In particular if $\ell = 1$, a messy parallel k-citelangis algebra $(\mathsf{Alg}, \mathfrak{B}_k)$ gives k different dendriform algebras $(\mathsf{Alg}, \prec_p, \succ_p)$ for the operations $\prec_p := \bowtie^{p-1}\prec\bowtie^{k-p}$ and $\succ_p := \bowtie^{p-1}\succ\bowtie^{k-p}$.

Another interesting particular case is $\ell = 0$ and therefore $I = \emptyset$. One then gets a morphism of operad $\mathsf{Res}_\emptyset^! : \mathsf{As} \to \mathsf{MCit}_k^{\parallel}$. In terms of algebra, this means that a messy parallel k-citelangis algebra $(\mathsf{Alg}, \mathfrak{B}_k)$ defines a structure of associative algebra $(\mathsf{Alg}, \bowtie^k)$. This can be seen by adding up all messy parallel k-citelangis relations of Proposition 4.4, obtaining



Reciprocally, we say that an associative algebra $(\mathsf{Alg}, \cdot)$ admits a *messy parallel k-citelangis structure* if it is possible to split the product $\cdot$ into 2^k operators $\mathfrak{B}_k$ defining a messy parallel k-citelangis algebra on Alg.

Remark 4.7. The dual of the morphism $\mathsf{Ins}_i^{\mathfrak{a}}$ where $\mathfrak{a} \in \{\prec, \succ\}$ defined in Remark 3.9 is a surjective morphism $(\mathsf{Ins}_i^{\mathfrak{a}})^! : \mathsf{MCit}_{k+1}^{\parallel} \to \mathsf{MCit}_k^{\parallel}$. It kills the operations whose i-th entry is not an $\mathfrak{a}$, and deletes the i-th entry in the other operations.

4.1.2. *Tidy parallel citelangis operads.* We now switch to the tidy setting.

Definition 4.8. *The* tidy parallel k-citelangis operad $\mathsf{TCit}^{\parallel}_k$ *is the Koszul dual of the tidy parallel k-signaletic operad:*

$$\mathsf{TCit}^{\parallel}_k := (\mathsf{TSig}^{\parallel}_k)^!$$

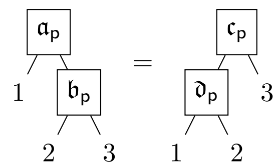
Proposition 4.9. *For any two integers k and l, we have*

$$\mathsf{TCit}^{\parallel}_k = \mathsf{Dup}_{\prec}{}^{\blacksquare k} \qquad \text{and} \qquad \mathsf{TCit}^{\parallel}_{k+l} = \mathsf{TCit}^{\parallel}_k \blacksquare \mathsf{TCit}^{\parallel}_l.$$

By Theorem 3.40, the tidy parallel k-citelangis operad $\mathsf{TCit}^{\parallel}_k$ is quadratic and Koszul. We now describe its quadratic relations.

Proposition 4.10. *For any destination vector $\mathsf{p} \in [3]^k$, the tidy parallel k-citelangis operad $\mathsf{TCit}^{\parallel}_k$ satisfies the quadratic relation*

$(\mathsf{TCit}^{\parallel}\ \mathsf{p})$



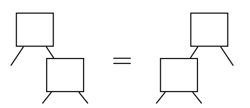
where the operators $\mathfrak{a}_{\mathsf{p}}, \mathfrak{b}_{\mathsf{p}}, \mathfrak{c}_{\mathsf{p}}, \mathfrak{d}_{\mathsf{p}} \in \{\prec, \rtimes\!\!\!\ltimes, \succ\}^k$ are defined by

$$(\mathfrak{a}_{\mathsf{p}})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \succ & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases} \quad (\mathfrak{b}_{\mathsf{p}})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \prec & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases} \quad (\mathfrak{c}_{\mathsf{p}})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \prec & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases} \quad (\mathfrak{d}_{\mathsf{p}})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \succ & \text{if } \mathsf{p}_i = 2 \\ \prec & \text{if } \mathsf{p}_i = 3 \end{cases}$$

for any $i \in [k]$. We call these relations the tidy parallel k-citelangis relations.

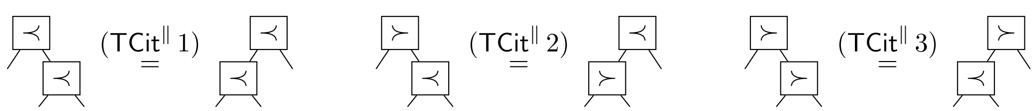
Example 4.11 (Tidy parallel 0-, 1- and 2-citelangis relations)**.** The tidy parallel 0-citelangis relation is the associative relation:

$(\mathsf{TCit}^{\parallel}\ .)$



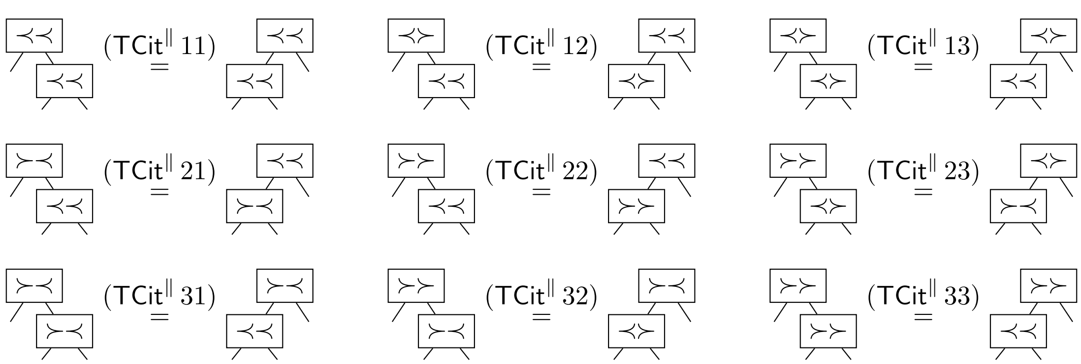
In other words, the tidy parallel 0-citelangis operad $\mathsf{TCit}^{\parallel}_0$ is just the associative operad As.

The tidy parallel 1-citelangis relations are the 3 twisted duplicial relations:



In other words, the tidy parallel 1-citelangis operad $\mathsf{TCit}^{\parallel}_1$ is just the twisted duplicial operad $\mathsf{Dup}_{\prec}$.

The tidy parallel 2-citelangis relations are the following 9 relations:



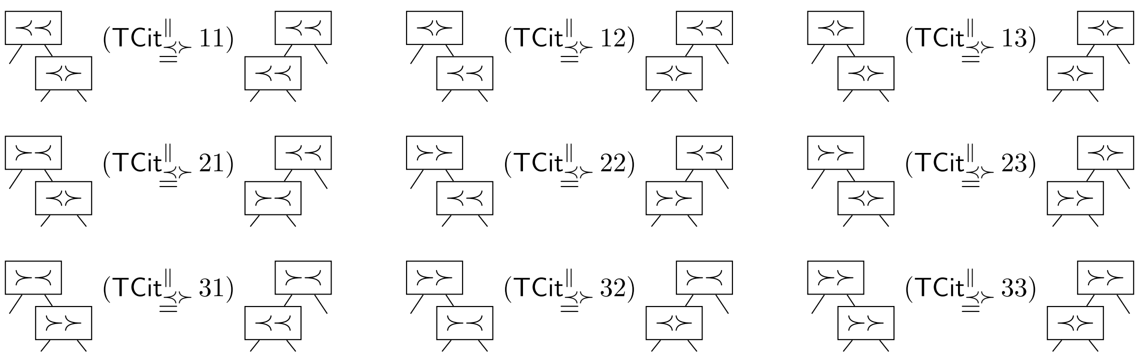
Remark 4.12. Similarly to Remark 4.6, for any subset I of $[k]$ of cardinal $\ell \le k$, the dual morphism of Res_I defined in Remark 3.15 is an injective operad morphism $\mathsf{Res}^!_I : \mathsf{TCit}^{\parallel}_\ell \to \mathsf{TCit}^{\parallel}_k$. The image $\mathfrak{b} := \mathsf{Res}^!_I(\mathfrak{a})$ of an operator $\mathfrak{a}$ of $\mathsf{TCit}^{\parallel}_\ell$ verifies $\mathfrak{b}_{i_p} = \mathfrak{a}_p$ if $p \in [\ell]$ and $\mathfrak{b}_j = \prec$ if $j \in [k] \smallsetminus I$.

Remark 4.13. The dual of the morphism $\mathsf{Ins}_i^{\mathfrak{a}}$ where $\mathfrak{a} \in \{\prec, \prec + \succ\}$ defined in Remark 3.16 is a surjective morphism $(\mathsf{Ins}_i^{\mathfrak{a}})^! : \mathsf{TCit}_{k+1}^{\|} \to \mathsf{TCit}_k^{\|}$. The map $(\mathsf{Ins}_i^{\prec})^!$ kills the operations whose i-th entry is not an $\prec$, and deletes the i-th entry in the other operations. The map $(\mathsf{Ins}_i^{\prec+\succ})^!$ just deletes the i-th entry in all the operations.

Remark 4.14. Following Remark 3.17, for any constraint word $\mathbf{c} \in \{\prec, \odot \succ\}^k$, we call ***c**-tidy parallel citelangis operad* $\mathsf{TCit}_{\mathbf{c}}^{\|}$ the Koszul dual of the $\mathbf{c}$-tidy parallel signaletic operad $\mathsf{TSig}_{\mathbf{c}}^{\|}$. In other words, for any words $\mathbf{c}, \mathbf{d} \in \{\prec, \odot, \succ\}$, we have

$$\mathsf{TCit}_k^{\|} = \blacksquare_{i \in |\mathbf{c}|} \mathsf{Dup}_{\mathbf{c}_i} \qquad \text{and} \qquad \mathsf{TCit}_{\mathbf{c}\cdot\mathbf{d}}^{\|} = \mathsf{TCit}_{\mathbf{c}}^{\|} \blacksquare \mathsf{TCit}_{\mathbf{d}}^{\|}.$$

Again, we have decided to present the tidy parallel k-citelangis operad with the constraint $\mathbf{c} = \prec^k$ to simplify the presentation and since this will be the only possible option in series. However, as we will use an action of the $\prec\!\succ$-tidy parallel citelangis operad in Section 5.1, let us present its 9 relations:



4.2. **Series citelangis operads.** We now consider the Koszul duals of the series signaletic operads, and we treat separately the messy and tidy situations.

4.2.1. *Messy series citelangis operads.* We start with the messy setting.

Definition 4.15. *The* messy series k-citelangis operad $\mathsf{MCit}_k^{\ddagger}$ *is the Koszul dual of the messy series k-signaletic operad:*

$$\mathsf{MCit}_k^{\ddagger} := (\mathsf{MSig}_k^{\ddagger})^!$$

By Theorem 3.40, the messy series k-citelangis operad $\mathsf{MCit}_k^{\ddagger}$ is quadratic and Koszul. To describe its quadratic relations, it is convenient to consider sums of operations like $\rtimes\!\!\!\ltimes := \prec + \succ$.

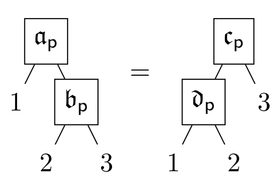
Proposition 4.16. *Consider the 3^k operations $\{\prec, \rtimes\!\!\!\ltimes, \succ\}^k$ related by*

$$\mathfrak{b} \rtimes\!\!\!\ltimes \mathfrak{b}' = \mathfrak{b} \prec \mathfrak{b}' + \mathfrak{b} \succ \mathfrak{b}'$$

for all $\mathfrak{b}, \mathfrak{b}' \in \{\prec, \rtimes\!\!\!\ltimes, \succ\}^*$ with $|\mathfrak{b}| + |\mathfrak{b}'| = k - 1$.

For any destination vector $\mathsf{p} \in [3]^k$, the messy series k-citelangis operad $\mathsf{MCit}_k^{\ddagger}$ satisfies the quadratic relation

$(\mathsf{MCit}^{\ddagger}\, \mathsf{p})$



where the operators $\mathfrak{a}_{\mathsf{p}}, \mathfrak{b}_{\mathsf{p}}, \mathfrak{c}_{\mathsf{p}}, \mathfrak{d}_{\mathsf{p}} \in \{\prec, \bowtie, \succ\}^k$ *are defined by*

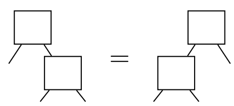
$$(\mathfrak{a}_{\mathsf{p}})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \succ & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases} \qquad (\mathfrak{b}_{\mathsf{p}})_i := \begin{cases} \bowtie & \text{if } |\mathsf{p}^{\{2,3\}}| < i \\ \prec & \text{if } (\mathsf{p}^{\{2,3\}})_i = 2 \\ \succ & \text{if } (\mathsf{p}^{\{2,3\}})_i = 3 \end{cases}$$

$$(\mathfrak{c}_{\mathsf{p}})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \prec & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases} \qquad (\mathfrak{d}_{\mathsf{p}})_i := \begin{cases} \prec & \text{if } (\mathsf{p}^{\{1,2\}})_i = 1 \\ \succ & \text{if } (\mathsf{p}^{\{1,2\}})_i = 2 \\ \bowtie & \text{if } |\mathsf{p}^{\{1,2\}}| < i \end{cases}$$

for any $i \in [k]$. *Again for any* $L \subseteq [3]$, *we have denoted by* p^L *the subword of* p *consisting only of the letters which belong to* L, *and for any* $x \in L$ *the condition* $(\mathsf{p}^L)_i = x$ *implicitly assume that* p^L *has length at least* i. *Note that Equation* ($\mathsf{MCit}^\ddagger\,\mathsf{p}$) *involves* $2^{|\mathsf{p}|_1}$ *syntax trees of* $\mathbf{Trees}(\mathfrak{B}_k)$ *on the left hand side and* $2^{|\mathsf{p}|_3}$ *syntax trees of* $\mathbf{Trees}(\mathfrak{B}_k)$ *on the right hand side. We call these relations the* messy series k-citelangis relations.

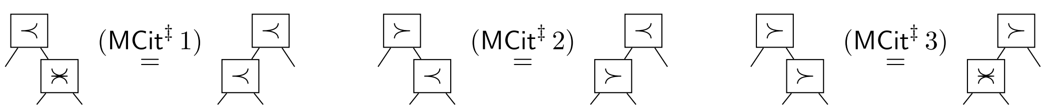
Example 4.17 (Messy series 0-, 1- and 2-citelangis relations)**.** The messy series 0-citelangis relation is the associative relation:

($\mathsf{MCit}^\ddagger$.)



In other words, the messy series 0-citelangis operad $\mathsf{MCit}^\ddagger_0$ is just the associative operad As.

The messy series 1-citelangis relations are the 3 dendriform relations:

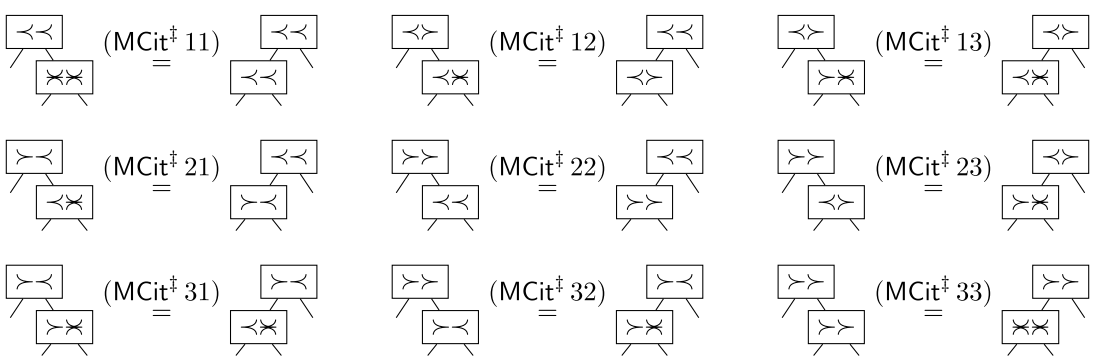


where

$$\bowtie \; = \; \prec + \succ.$$

In other words, the messy series 1-citelangis operad $\mathsf{MCit}^\ddagger_1$ is just the dendriform operad Dend.

The messy series 2-citelangis relations are the following 9 relations:

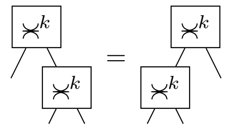


where

$$\bowtie\prec \; = \; \prec\prec + \succ\prec, \qquad \bowtie\succ \; = \; \prec\succ + \succ\succ,$$
$$\prec\bowtie \; = \; \prec\prec + \prec\succ, \qquad \succ\bowtie \; = \; \succ\prec + \succ\succ,$$
$$\text{and} \qquad \bowtie\bowtie \; = \; \prec\bowtie + \succ\bowtie \; = \; \bowtie\prec + \bowtie\succ \; = \; \prec\prec + \prec\succ + \succ\prec + \succ\succ.$$

Remark 4.18. Similarly to Remark 4.6, one can dualize Remark 3.22: For any $\ell \leq k$ there is a morphism $(\mathsf{Res}^k_\ell)^! : \mathsf{MCit}^\ddagger_\ell \mapsto \mathsf{MCit}^\ddagger_k$. In terms of algebras a messy series k-citelangis algebra for the operators $\{\prec, \succ\}^k$ is also a messy series ℓ-citelangis algebra for the operations $\left\{\mathfrak{b} \bowtie^{k-\ell} \;\middle|\; \mathfrak{b} \in \{\prec, \succ\}^\ell\right\}$.

In particular for $\ell = 0$, a messy series k-citelangis algebra $(\mathsf{Alg}, \mathfrak{B}_k)$ defines a structure of associative algebra $(\mathsf{Alg}, \bowtie^k)$. This can be seen by adding up all messy series k-citelangis relations of Proposition 4.16, obtaining



Reciprocally, we say that an associative algebra $(\mathsf{Alg}, \cdot)$ admits a *messy series k-citelangis structure* if it is possible to split the product $\cdot$ into 2^k operators $\mathfrak{B}_k$ defining a messy series k-citelangis algebra on Alg.

4.2.2. *Tidy series citelangis operads.* We now switch to the tidy setting.

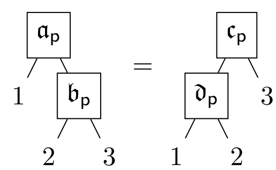
Definition 4.19. *The* tidy series k-citelangis operad $\mathsf{TCit}_k^\ddagger$ *is the Koszul dual of the tidy series k-signaletic operad:*

$$\mathsf{TCit}_k^\ddagger := (\mathsf{TSig}_k^\ddagger)^!$$

By Theorem 3.40, the tidy series k-citelangis operad $\mathsf{TCit}_k^\ddagger$ is quadratic and Koszul. We now describe its quadratic relations.

Proposition 4.20. *For any destination vector $\mathsf{p} \in [3]^k$, the tidy series k-citelangis operad $\mathsf{TCit}_k^\ddagger$ satisfies the quadratic relation*

(TCit‡ p)



where the operators $\mathfrak{a}_\mathsf{p}, \mathfrak{b}_\mathsf{p}, \mathfrak{c}_\mathsf{p}, \mathfrak{d}_\mathsf{p} \in \{\prec, \Join, \succ\}^k$ are defined by

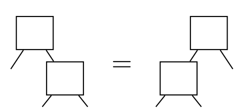
$$(\mathfrak{a}_\mathsf{p})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \succ & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases} \qquad (\mathfrak{b}_\mathsf{p})_i := \begin{cases} \prec & \text{if } (\mathsf{p}^{\{2,3\}})_i = 1 \\ \prec & \text{if } (\mathsf{p}^{\{2,3\}})_i = 2 \\ \succ & \text{if } |\mathsf{p}^{\{2,3\}}| < i \end{cases}$$

$$(\mathfrak{c}_\mathsf{p})_i := \begin{cases} \prec & \text{if } \mathsf{p}_i = 1 \\ \prec & \text{if } \mathsf{p}_i = 2 \\ \succ & \text{if } \mathsf{p}_i = 3 \end{cases} \qquad (\mathfrak{d}_\mathsf{p})_i := \begin{cases} \prec & \text{if } (\mathsf{p}^{\{1,2\}})_i = 1 \\ \succ & \text{if } (\mathsf{p}^{\{1,2\}})_i = 2 \\ \prec & \text{if } |\mathsf{p}^{\{1,2\}}| < i \end{cases}$$

for any $i \in [k]$. Again for any $L \subseteq [3]$, we have denoted by p^L the subword of p consisting only of the letters which belong to L, and for any $x \in L$ the condition $(\mathsf{p}^L)_i = x$ implicitly assume that p^L has length at least i. We call these relations the tidy series k-citelangis relations.

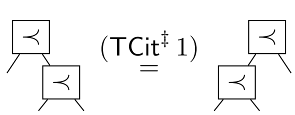
Example 4.21 (Tidy series 0-, 1- and 2-citelangis relations)**.** The tidy series 0-citelangis relation is the associative relation:

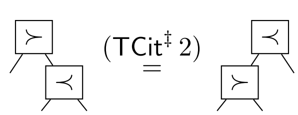
(TCit‡ .)

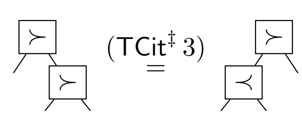


In other words, the tidy series 0-citelangis operad $\mathsf{TCit}_0^\ddagger$ is just the associative operad As.

The tidy series 1-citelangis relations are the 3 twisted duplicial relations:

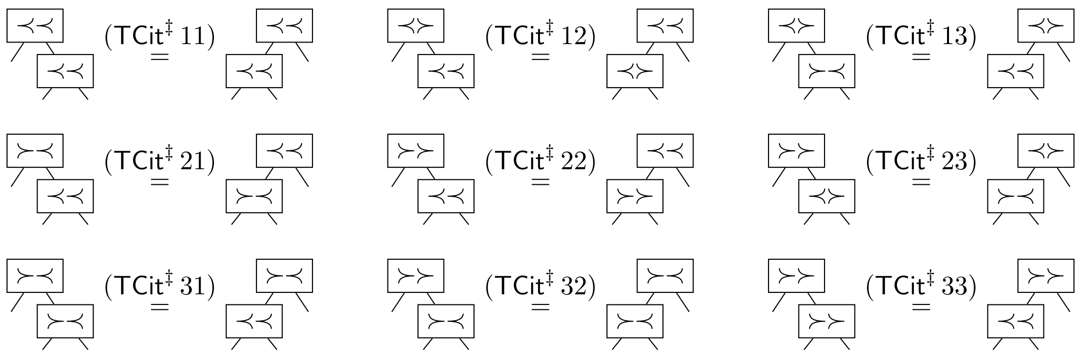






In other words, the tidy series 1-citelangis operad $\mathsf{TCit}_1^\ddagger$ is just the twisted duplicial operad $\mathsf{Dup}_\prec$.

The tidy series 2-citelangis relations are the following 9 relations:



Remark 4.22. Similar to Remark 4.18, there is a restriction morphism $(\mathsf{Res}_\ell^k)^! : \mathsf{TCit}_k^\ddagger \mapsto \mathsf{TCit}_l^\ddagger$.

4.3. **Hilbert series and normal forms.** In this section, we discuss numerologic properties of the citelangis operads. As for the signaletic operads (see Section 3.4.1), these properties do not really depend on whether the cars arrive in series or parallel, and whether the signals outside the routes of the cars are pointing or not to the left. We therefore write Cit_k to denote without distinction any of the four k-citelangis operads $\mathsf{MCit}_k^\parallel$, $\mathsf{TCit}_k^\parallel$, $\mathsf{MCit}_k^\ddagger$, or $\mathsf{TCit}_k^\ddagger$.

4.3.1. *Hilbert series.* We first discuss the Hilbert series of the k-citelangis operad Cit_k, based on Corollary 3.30 and Section 2.3.

Corollary 4.23. *The Hilbert series of the (messy or tidy, parallel or series) k-citelangis operad Cit_k satisfies the equation*

$$\mathrm{Eul}_k(-\mathcal{H}_{\mathsf{Cit}_k}(-t)) = t(1+\mathcal{H}_{\mathsf{Cit}_k}(-t))^{k+1}. \tag{5}$$

Therefore, the dimension $d_k(p) := \dim(\mathsf{Cit}_k(p))$ can be computed recursively from $d_k(1) = 1$ by

$$d_k(p) = \sum_{\substack{q_1,\dots,q_{k+1}\ge 0\\ q_1+\dots+q_{k+1}=p-1}} d_k(q_1)\cdots d_k(q_{k+1}) + \sum_{j=1}^{k-1}(-1)^{j+1}\left\langle {k \atop j} \right\rangle \sum_{\substack{q_1,\dots,q_{j+1}\ge 1\\ q_1+\dots+q_{j+1}=p}} d_k(q_1)\cdots d_k(q_{j+1}). \tag{6}$$

Alternatively, the dimension $d_k(p) := \dim(\mathsf{Cit}_k(p))$ is given by the summation formula

$$d_k(p) = \frac{1}{p}\sum_{\substack{i,j_1,\dots,j_{k-1}\ge 0\\ i+j'=p-1}} (-1)^{j+j'}\binom{(k+1)p}{i}\binom{p+j-1}{j}\binom{j}{j_1,\dots,j_{k-1}}\left\langle {k \atop 1}\right\rangle^{j_1}\cdots\left\langle {k \atop k-1}\right\rangle^{j_{k-1}}, \tag{7}$$

where $j = j_1+\dots+j_{k-1}$ and $j' = j_1+\dots+(k-1)j_{k-1}$.

Proof. Since the k-signaletic operad Sig_k and the k-citelangis operad Cit_k are Koszul dual Koszul operads by definition, their Hilbert series are Lagrange inverses by Theorem 2.18:

$$\mathcal{H}_{\mathsf{Sig}_k}(-\mathcal{H}_{\mathsf{Cit}_k}(-t)) = t.$$

Corollary 3.30 thus shows the functional equation of Equation (5).

Using this functional equation with $t = -u$, we obtain

$$\mathcal{H}_{\mathsf{Cit}_k}(u) = u(1+\mathcal{H}_{\mathsf{Cit}_k}(u))^{k+1} + \sum_{j=1}^{k-1}\left\langle {k \atop j}\right\rangle(-\mathcal{H}_{\mathsf{Cit}_k}(u))^{j+1}.$$

Comparing the coefficient of u^p in the two sides of this equality yields the recursive formula of Equation (6).

Finally, to obtain the summation formula of Equation (7), we use Lagrange's theorem: since $\mathcal{H}_{\mathsf{Cit}_k}(u) = u\Phi(\mathcal{H}_{\mathsf{Cit}_k}(u))$ with

$$\Phi(v) := \frac{-v\,(1+v)^{k+1}}{\mathrm{Eul}_k(-v)}$$

we obtain by Lagrange inversion theorem that

$$d_k(p) = [u^p]\,\mathcal{H}_{\mathsf{Cit}_k}(u) = \frac{1}{p}\,[v^{p-1}]\,\Phi(v)^p = \frac{1}{p}\sum_{i=0}^{p-1}[v^i]\,(1+v)^{(k+1)p}\cdot[v^{p-i-1}]\left(\frac{-v}{\mathrm{Eul}_k(-v)}\right)^p.$$

The result thus follows from the development

$$\left(\frac{-v}{\mathrm{Eul}_k(-v)}\right)^p = \left(1+\sum_{\ell=1}^{k-1}\left\langle {k \atop \ell}\right\rangle(-v)^\ell\right)^{-p} = \sum_{j\ge 0}(-1)^j\binom{p+j-1}{j}\left(\sum_{\ell=1}^{k-1}\left\langle {k \atop \ell}\right\rangle(-v)^\ell\right)^j$$

$$= \sum_{j_1,\dots,j_{k-1}\ge 0}(-1)^{j+j'}\binom{p+j-1}{j}\binom{j}{j_1,\dots,j_{k-1}}\left\langle {k \atop 1}\right\rangle^{j_1}\cdots\left\langle {k \atop k-1}\right\rangle^{j_{k-1}} v^{j'}$$

where $j = j_1+\dots+j_{k-1}$ and $j' = j_1+\dots+(k-1)j_{k-1}$. □

Remark 4.24. Observe that

$$\Phi(v) = \frac{1}{\sum_{p\ge 0} p^k\,(-v)^{p-1}} = \sum_{p\ge 0}\Bigg(\sum_{\substack{\ell\ge 0\\ c_1,\dots,c_\ell\ge 1\\ c_1+\dots+c_\ell=p}}(-1)^{p-\ell}(c_1+1)^k\cdots(c_\ell+1)^k\Bigg)v^p.$$

Example 4.25. For $k=1$, Equations (6) and (7) become

$$d_1(p) = \sum_{\substack{a,b\ge 0\\ a+b=p-1}} d_1(a)\,d_1(b) \qquad\text{and}\qquad d_1(p) = \frac{1}{p}\binom{2p}{p-1} = \frac{1}{p+1}\binom{2p}{p} = C_p.$$

For $k=2$, we have

$$(8)\quad d_2(p) = \sum_{\substack{a,b,c\ge 0\\ a+b+c=p-1}} d_2(a)\,d_2(b)\,d_2(c) + \sum_{\substack{a,b\ge 1\\ a+b=p}} d_2(a)\,d_2(b) \quad\text{and}\quad d_2(p) = \frac{1}{p}\sum_i\binom{3p}{i}\binom{2p-i-2}{p-i-1}.$$

These formulas appeared as conjectures in the original paper of M. Aguiar and J.-L. Loday on quadri-algebras [AL04]: the recursive formula is [AL04, Remark 4.7] while the summation formula is equivalent to [AL04, Conjecture 4.2] up to the change of variable $j = 2n-i+1$. These conjectures follow form the work of J. E. Vatne [Vat05] and L. Foissy in [Foi15].

For $k=3$ we have

$$d_3(p) = \sum_{\substack{a,b,c,d\ge 0\\ a+b+c+d=p-1}} d_3(a)\,d_3(b)\,d_3(c)\,d_3(d) - \sum_{\substack{a,b,c\ge 1\\ a+b+c=p}} d_3(a)\,d_3(b)\,d_3(c) + 4\sum_{\substack{a,b\ge 1\\ a+b=p}} d_3(a)\,d_3(b)$$

$$\text{and}\qquad d_3(p) = \frac{1}{p}\sum_{i,j}(-1)^{i+j+p+1}\binom{4p}{i}\binom{p+j-1}{j}\binom{j}{p-i-j-1}4^{2j+i-p+1}.$$

Table 2 gathers the values of $d_k(p)$ for $k\in[5]$ and $p\in[8]$.

$k \backslash p$	1	2	3	4	5	6	7	8
1	1	2	5	14	42	132	429	1430
2	1	4	23	156	1162	9192	75819	644908
3	1	8	101	1544	26190	474144	8975229	175492664
4	1	16	431	14256	525682	20731488	855780699	36512549680
5	1	32	1805	125984	9825222	820259712	71710602189	6481491238880

TABLE 2. The values of $d_k(p)$ for $k\in[5]$ and $p\in[8]$.

Remark 4.26. For $k = 2$, the numbers $d_2(p)$ count the number of rooted non-crossing connected arc diagrams on $p + 1$ points. To see the summation formula of Equation (8), decompose the diagrams according on whether the leftmost arc incident to the root is an isthme (then the deletion of this arc decomposes the diagram into 3 subdiagrams with at least one node) or not (then the deletion of this arc decomposes the diagram into 2 subdiagrams with at least two nodes). These decompositions are illustrated in Figure 8.

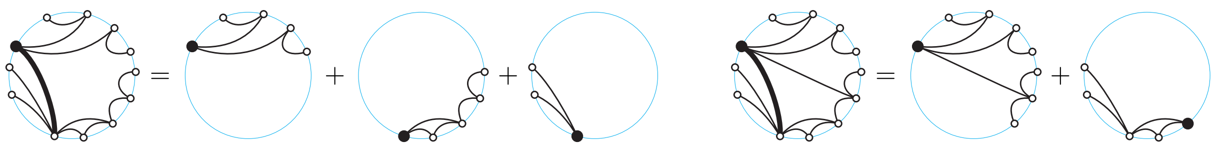
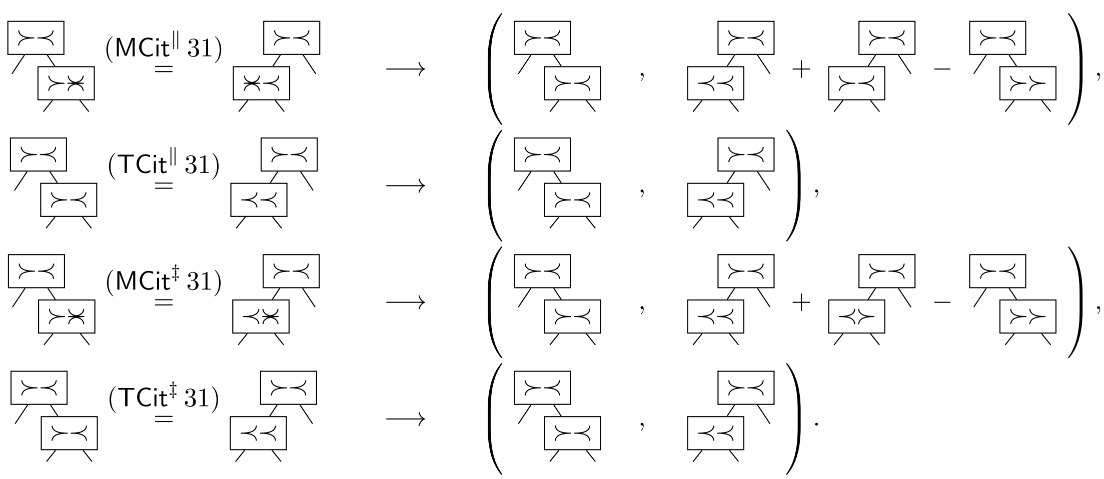


FIGURE 8. Decomposition of two rooted non-crossing connected arc diagrams, whose root and leftmost arc α incident to the root are bolded. We obtain three subdiagrams when α is an isthme (left), and two subdiagrams when α is not an isthme (right). See Remark 4.26.

4.3.2. *Rewriting system.* We now orient the k-citelangis relations of Propositions 4.4, 4.10, 4.16 and 4.20 to a convergent rewriting system. As in Section 3.4.3, we use the k-signaletic Tamari order defined in Definition 3.31, and the right k-signaletic combs.

Definition 4.27. *The k-citelangis rewriting system is the quadratic rewriting system where for each destination vector of $[3]^k$, we orient the corresponding k-citelangis relation so that it rewrites the corresponding quadratic right k-signaletic comb.*

Example 4.28. For the destination vector $(\!| 3, 1 |\!)$, the corresponding 2-citelangis relations (left) are transformed to the following rewriting rules (right):



Note that the k-citelangis rewriting system is oriented by the reverse k-signaletic Tamari lattice, thus opposite to the orientation for the k-signaletic rewriting system. By this choice of reverse orientation, the Koszulity of the k-signaletic rewriting system ensures that the k-citelangis rewriting system also converges.

4.3.3. *Normal forms and Eulerian matrices.* Since the k-citelangis rewriting system of Definition 4.27 rewrites precisely the quadratic right k-signaletic combs, its normal forms are the syntax trees of $\mathbf{Trees}(\mathfrak{B}_k)$ that do not contain a quadratic right k-signaletic comb. Note that these normal forms are the same for the tidy and the messy k-citelangis rewriting system. In contrast, these normal forms are not the same in series and in parallel, but their generating series are the same. These generating series are the Hilbert series $\mathcal{H}_{\mathsf{Cit}_k}(t)$ of the k-citelangis operads.

Call *admissible* the syntax trees that do not contain a quadratic right k-signaletic comb. This condition imposes some restrictions on the right child of each node, but none on the left child. This observation has two immediate consequences:

(1) We can decompose an admissible syntax tree as an admissible right comb where we attach a (possibly empty) admissible syntax tree as the left child of each node.
(2) The admissible right combs can be constructed by a simple automaton with transitions corresponding to all admissible quadratic right combs. The transition matrix of this automaton is described below.

We explore enumerative consequences of these two points in the rest of this section.

Denote by $\mathcal{R}_k(t)$ the generating series of right combs on $\mathbf{Trees}(\mathfrak{B}_k)$ (*i.e.* syntax trees where no node has a left child) that do not contain a quadratic right k-signaletic comb, where the variable t count their number of nodes. For instance, we have

$$R_1(t) = 1 + 2t + t^2 = (1+t)^2,$$
$$R_2(t) = 1 + 4t + 7t^2 + 8t^3 + 8t^4 + \cdots = \frac{(1+t)^3}{1-t},$$
$$R_3(t) = 1 + 8t + 37t^2 + 144t^3 + 540t^4 + 2016t^5 + \cdots = \frac{(1+t)^4}{1-4t+t^2},$$
$$R_4(t) = 1 + 16t + 175t^2 + 1760t^3 + 17456t^4 + 172832t^5 + \cdots = \frac{(1+t)^5}{1-11t+11t^2-t^3}.$$

Proposition 4.29. *The generating series $\mathcal{H}_{\mathsf{Cit}_k}(t)$ and $\mathcal{R}_k(t)$ satisfy $\mathcal{H}_{\mathsf{Cit}_k}(t) = t \cdot \mathcal{R}_k\big(\mathcal{H}_{\mathsf{Cit}_k}(t)\big)$.*

Proof. We have seen that we can inductively decompose any admissible syntax tree as an admissible right comb where we attach a (possibly empty) admissible syntax tree as the left child of each node. The generating series are thus related by composition: $\mathcal{H}_{\mathsf{Cit}_k}(t) = t \cdot \mathcal{R}_k\big(\mathcal{H}_{\mathsf{Cit}_k}(t)\big)$. □

Corollary 4.30. *The generating series $\mathcal{R}_k(t)$ is given by $\mathcal{R}_k(t) \cdot \mathrm{Eul}_k(-t) = -t \cdot (1+t)^{k+1}$.*

Proof. This follows from Corollary 4.23 and Proposition 4.29. □

We now interpret $\mathcal{R}_k(t)$ in terms of the coefficients of the iterated powers of the transition matrix of an automaton. Consider the automaton whose nodes are the operators of $\mathfrak{B}_k$ and with a transition $\mathfrak{a} \to \mathfrak{b}$ if the right comb with root $\mathfrak{a}$ and child $\mathfrak{b}$ is admissible. Let M_k denote the transition matrix of this automaton, *i.e.* the $(\mathfrak{B}_k \times \mathfrak{B}_k)$-matrix with coefficients $(\mathsf{M}_k)_{\mathfrak{a}\mathfrak{b}} = 1$ if there is a transition $\mathfrak{a} \to \mathfrak{b}$ and 0 otherwise. Since the automaton constructs the admissible right combs, we obtain the following statement.

Proposition 4.31. *The generating series $\mathcal{R}_k(t)$ is given by* $\mathcal{R}_k(t) = 1 + \sum_{p \ge 1} \sum_{\mathfrak{a},\mathfrak{b} \in \mathfrak{B}_k} (\mathsf{M}_k^{p-1})_{\mathfrak{a}\mathfrak{b}} \, t^p$.

Proof. The p-th power of the transition matrix encodes all right combs of arity $p+1$. □

Note that the transitions are the same in the tidy and in the messy setting. In contrast, they differ in the series and parallel settings:

(1) For the parallel k-citelangis operad, the transition matrix $\mathsf{M}_k^{\|}$ is the $(\mathfrak{B}_k \times \mathfrak{B}_k)$-matrix whose $(\mathfrak{a}, \mathfrak{b})$-coefficient is given by

$$(\mathsf{M}_k^{\|})_{\mathfrak{a}\mathfrak{b}} = \begin{cases} 1 & \text{if there is } i \in [k] \text{ such that } \mathfrak{a}_i = \prec \text{ while } \mathfrak{b}_i = \succ, \\ 0 & \text{otherwise.} \end{cases}$$

for any $\mathfrak{a} \in \mathfrak{B}_k$ and $\mathfrak{b} \in \mathfrak{B}_k$. For example,

$$\mathsf{M}_1^{\parallel} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{matrix} \prec \\ \succ \end{matrix} \qquad \mathsf{M}_2^{\parallel} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \prec\prec \\ \prec\succ \\ \succ\prec \\ \succ\succ \end{matrix} \qquad \mathsf{M}_3^{\parallel} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \prec\prec\prec \\ \prec\prec\succ \\ \prec\succ\prec \\ \succ\prec\prec \\ \prec\succ\succ \\ \succ\prec\succ \\ \succ\succ\prec \\ \succ\succ\succ \end{matrix}$$

(2) For the series k-citelangis operad, the transition matrix $\mathsf{M}_k^{\ddagger}$ is the $(\mathfrak{B}_k \times \mathfrak{B}_k)$-matrix whose $(\mathfrak{a}, \mathfrak{b})$-coefficient is given by

$$(\mathsf{M}_k^{\ddagger})_{\mathfrak{a}\mathfrak{b}} = \begin{cases} 1 & \text{if there is } i > |\mathfrak{a}|_{\succ} \text{ such that } \mathfrak{b}_i = \succ, \\ 0 & \text{otherwise.} \end{cases}$$

for any $\mathfrak{a} \in \mathfrak{B}_k$ and $\mathfrak{b} \in \mathfrak{B}_k$. For example,

$$\mathsf{M}_1^{\ddagger} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{matrix} \prec \\ \succ \end{matrix} \qquad \mathsf{M}_2^{\ddagger} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \prec\prec \\ \prec\succ \\ \succ\prec \\ \succ\succ \end{matrix} \qquad \mathsf{M}_3^{\ddagger} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \prec\prec\prec \\ \prec\prec\succ \\ \prec\succ\prec \\ \succ\prec\prec \\ \prec\succ\succ \\ \succ\prec\succ \\ \succ\succ\prec \\ \succ\succ\succ \end{matrix}$$

These matrices seem particularly relevant and closely related to Eulerian polynomials. Let us mention in particular the following conjectures.

Conjecture 4.32. *The minimal and characteristic polynomials of the matrix* $\mathsf{M}_k^{\parallel}$ *are*

$$\pi_{\mathsf{M}_k^{\parallel}}(t) = (-1)^k \cdot t \cdot (t+1)^{k-1} \cdot \mathrm{Eul}_k(-t), \qquad \chi_{\mathsf{M}_k^{\parallel}}(t) = (-1)^k \cdot t \cdot (t+1)^{\left\langle {k \atop 1} \right\rangle} \cdot \mathrm{Eul}_k(-t),$$

$$\pi_{\mathsf{M}_k^{\ddagger}}(t) = (-1)^k \cdot t \cdot \mathrm{Eul}_k(-t), \qquad \textit{and} \qquad \chi_{\mathsf{M}_k^{\ddagger}}(t) = (-1)^k \cdot t^{1+\left\langle {k \atop 1} \right\rangle} \cdot \mathrm{Eul}_k(-t).$$

For example, the minimal and characteristic polynomials of the matrices above are

$$\pi_{\mathsf{M}_1^{\parallel}}(t) = t^2, \qquad \pi_{\mathsf{M}_2^{\parallel}}(t) = t^4 - t^2, \qquad \pi_{\mathsf{M}_3^{\parallel}}(t) = t^6 - 2t^5 - 6t^4 - 2t^3 + t^2,$$
$$\chi_{\mathsf{M}_1^{\parallel}}(t) = t^2, \qquad \chi_{\mathsf{M}_2^{\parallel}}(t) = t^4 - t^2, \qquad \chi_{\mathsf{M}_3^{\parallel}}(t) = t^8 - 9t^6 - 16t^5 - 9t^4 + t^2,$$

and

$$\pi_{\mathsf{M}_1^{\ddagger}}(t) = t^2, \qquad \pi_{\mathsf{M}_2^{\ddagger}}(t) = t^3 - t^2, \qquad \pi_{\mathsf{M}_3^{\ddagger}}(t) = t^4 - 4t^3 + t^2,$$
$$\chi_{\mathsf{M}_1^{\ddagger}}(t) = t^2, \qquad \chi_{\mathsf{M}_2^{\ddagger}}(t) = t^4 - t^3, \qquad \chi_{\mathsf{M}_3^{\ddagger}}(t) = t^8 - 4t^7 + t^6.$$

On the examples of matrices $\mathsf{M}_k^{\parallel}$ and $\mathsf{M}_k^{\ddagger}$ above, we have gathered together rows and columns according to the number of $\succ$ signs of the corresponding operator of $\mathfrak{B}_k$. We observe that when we sum the columns of $\mathsf{M}_k^{\parallel}$ or $\mathsf{M}_k^{\ddagger}$ according to these groups, all entries in a group of rows are identical. The following lemma gives a more precise statement.

Lemma 4.33. *For* $0 \le i, j \le k$ *and* $\mathfrak{a} \in \mathfrak{B}_k$ *with* $|\mathfrak{a}|_{\succ} = i$, *we have* $\displaystyle\sum_{\substack{\mathfrak{b} \in \mathfrak{B}_k \\ |\mathfrak{b}|_{\succ} = j}} (\mathsf{M}_k)_{\mathfrak{a}\mathfrak{b}} = \binom{k}{j} - \binom{i}{j}.$

Proof. Let $0 \le i, j \le k$ and consider $\mathfrak{a}, \mathfrak{b} \in \mathfrak{B}_k$ with $|\mathfrak{a}|_\succ = i$ and $|\mathfrak{b}|_\succ = j$. From the definitions of $(\mathsf{M}_k^{\|})_{\mathfrak{a}\mathfrak{b}}$ and $(\mathsf{M}_k^{\ddagger})_{\mathfrak{a}\mathfrak{b}}$ above, we derive that

(1) $(\mathsf{M}_k^{\|})_{\mathfrak{a}\mathfrak{b}} = 0 \iff$ the j positions of $\succ$ in $\mathfrak{b}$ are chosen among the i positions of $\succ$ in $\mathfrak{a}$,
(2) $(\mathsf{M}_k^{\ddagger})_{\mathfrak{a}\mathfrak{b}} = 0 \iff$ the j positions of $\succ$ in $\mathfrak{b}$ are chosen among the first i positions.

Therefore, for a fixed operator $\mathfrak{a} \in \mathfrak{B}_k$ with $|\mathfrak{a}|_\succ = i$, among the $\binom{k}{j}$ operators $\mathfrak{b} \in \mathfrak{B}_k$ with $|\mathfrak{b}|_\succ = j$, there are $\binom{i}{j}$ operators such that $(\mathsf{M}_k)_{\mathfrak{a}\mathfrak{b}} = 0$ and $\binom{k}{j} - \binom{i}{j}$ operators such that $(\mathsf{M}_k)_{\mathfrak{a}\mathfrak{b}} = 1$, in both the series and parallel situations. The result immediately follows. □

This motivates to introduce the $(k+1) \times (k+1)$-matrix N_k defined by

$$\mathsf{N}_k := \left[\binom{k}{j} - \binom{i}{j} \right]_{0 \le i,j \le k}$$

Lemma 4.34. *For any $p \in \mathbb{N}$, we have* $\displaystyle\sum_{\mathfrak{a},\mathfrak{b} \in \mathfrak{B}_k} (\mathsf{M}_k^p)_{\mathfrak{a}\mathfrak{b}} = \sum_{0 \le i,j \le k} (\mathsf{N}_k^p)_{ij} \binom{k}{i}$.

Proof. We denote by $\mathbb{1}_n$ the vector with n entries all equal to 1, and by I_n the $(n \times n)$-identity matrix. Consider the vector B_k, the $\big((k+1) \times \mathfrak{B}_k\big)$-matrix L_k and the $\big(\mathfrak{B}_k \times (k+1)\big)$-matrix R_k whose coefficients are given by

$$(\mathsf{B}_k)_i = \binom{k}{i}, \qquad (\mathsf{L}_k)_{i\mathfrak{a}} = \delta_{|\mathfrak{a}|_\succ = i} \binom{k}{i}^{-1} \qquad \text{and} \qquad (\mathsf{R}_k)_{\mathfrak{b}j} = \delta_{|\mathfrak{b}|_\succ = j}.$$

Observe that

$$^{\mathsf{t}}\mathsf{B}_k \cdot \mathsf{L}_k = {}^{\mathsf{t}}\mathbb{1}_{\mathfrak{B}_k}, \qquad \mathsf{R}_k \cdot \mathbb{1}_{k+1} = \mathbb{1}_{\mathfrak{B}_k} \qquad \text{and} \qquad \mathsf{L}_k \cdot \mathsf{R}_k = \mathsf{I}_{k+1}.$$

Moreover, Lemma 4.33 affirms that

$$\mathsf{R}_k \cdot \mathsf{N}_k = \mathsf{M}_k \cdot \mathsf{R}_k.$$

This imply that

$$\mathsf{N}_k^p = \mathsf{L}_k \cdot \mathsf{M}_k^p \cdot \mathsf{R}_k$$

for any $p \in \mathbb{N}$. Indeed, it holds for $p = 0$ since $\mathsf{L}_k \cdot \mathsf{R}_k = \mathsf{I}_{k+1}$, and we obtain by induction that $\mathsf{N}_k^{p+1} = \mathsf{N}_k^p \cdot \mathsf{N}_k = \mathsf{L}_k \cdot \mathsf{M}_k^p \cdot \mathsf{R}_k \cdot \mathsf{N}_k = \mathsf{L}_k \cdot \mathsf{M}_k^p \cdot \mathsf{M}_k \cdot \mathsf{R}_k = \mathsf{L}_k \cdot \mathsf{M}_k^{p+1} \cdot \mathsf{R}_k$. Therefore,

$$\sum_{0 \le i,j \le k} (\mathsf{N}_k^p)_{ij} \binom{k}{i} = {}^{\mathsf{t}}\mathsf{B}_k \cdot \mathsf{N}_k^p \cdot \mathbb{1}_{k+1} = {}^{\mathsf{t}}\mathsf{B}_k \cdot \mathsf{L}_k \cdot \mathsf{M}_k^p \cdot \mathsf{R}_k \cdot \mathbb{1}_{k+1} = {}^{\mathsf{t}}\mathbb{1}_{\mathfrak{B}_k} \cdot \mathsf{M}_k^p \cdot \mathbb{1}_{\mathfrak{B}_k} = \sum_{\mathfrak{a},\mathfrak{b} \in \mathfrak{B}_k} (\mathsf{M}_k^p)_{\mathfrak{a}\mathfrak{b}}. \ \square$$

We thus derive from Proposition 4.31 and Lemma 4.34 the following statement which provides a more efficient way to compute approximations of the generating series $\mathcal{R}_k(t)$.

Corollary 4.35. *The generating series $\mathcal{R}_k(t)$ is given by* $\displaystyle\mathcal{R}_k(t) = 1 + \sum_{p \ge 1} \sum_{0 \le i,j \le k} (\mathsf{N}_k^{p-1})_{ij} \binom{k}{i} t^p$.

For example, the first such matrices are given by

$$\mathsf{N}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathsf{N}_2 = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathsf{N}_3 = \begin{bmatrix} 0 & 3 & 3 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathsf{N}_4 = \begin{bmatrix} 0 & 4 & 6 & 4 & 1 \\ 0 & 3 & 6 & 4 & 1 \\ 0 & 2 & 5 & 4 & 1 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Again, these matrices seem closely related to Eulerian polynomials as illustrated by the following conjecture.

Conjecture 4.36. *The minimal polynomial $\pi_{\mathsf{N}_k}(t)$ and the characteristic polynomial $\chi_{\mathsf{N}_k}(t)$ of the matrix N_k are multiples of the Eulerian polynomial:*

$$\pi_{\mathsf{N}_k}(t) = \chi_{\mathsf{N}_k}(t) = (-1)^k \cdot t \cdot \mathrm{Eul}_k(-t).$$

For example, the minimal and characteristic polynomials of the matrices above are

$$\pi_{\mathsf{N}_1}(t) = t^2, \quad \pi_{\mathsf{N}_2}(t) = t^3 - t^2, \quad \pi_{\mathsf{N}_3}(t) = t^4 - 4t^3 + t^2, \quad \pi_{\mathsf{N}_4}(t) = t^5 - 11t^4 + 11t^3 - t^2,$$
$$\chi_{\mathsf{N}_1}(t) = t^2, \quad \chi_{\mathsf{N}_2}(t) = t^3 - t^2, \quad \chi_{\mathsf{N}_3}(t) = t^4 - 4t^3 + t^2, \quad \chi_{\mathsf{N}_4}(t) = t^5 - 11t^4 + 11t^3 - t^2.$$

5. Combinatorial models and actions

It is well known that the shuffle algebra can be endowed with a structure of dendriform algebra [Lod01] defined for the words xX and yY by

$$xX \prec yY = x\big(X \sqcup\!\sqcup yY\big) \qquad \text{and} \qquad xX \succ yY = y\big(xX \sqcup\!\sqcup Y\big).$$

Replacing the shuffle by the shifted shuffle, one endows similarly the algebra FQSym of permutations with a structure of dendriform algebra. The resulting dendriform algebra is known to be free [Foi15, Von15], and the dendriform subalgebra of FQSym generated by the permutation 1 is known to be the free dendriform algebra on one generator [LR98].

The objective of this section is to show similar actions of the citelangis operads defined in Section 4 on a certain family of permutations and posets. Some of these actions will result in free citelangis algebras, and will thus lead to natural constructions of free citelangis algebras on one generator, providing combinatorial models to index the bases for the citelangis operads.

As the tidy and messy parallel k-citelangis operads are Manin powers of the dendriform and twisted duplicial operads (see Section 2.5 and Propositions 4.2 and 4.9), they naturally act on k-tuples of permutations. Nevertheless, this action is not really interesting here as it does not extend in the series situation. It is much more relevant to define actions of the parallel k-citelangis operads on permutations. Such actions are already known for $k = 1$ (Dend operad) or $k = 2$ (Quad operad), but are unfortunately only defined in these two situations because a permutation has only two ends! In contrast, we will define actions of the series k-citelangis operads for any $k \geq 1$. Even if it is limited to the case $k \leq 2$ and already partially known, we still present the parallel situation in Section 5.1 as it serves as a prototype for the series situation developed in Section 5.2.

We refer the reader to Section 2.8 for definitions and notations on multipermutations and multiposets.

A road map through combinatorial models and actions for parallel/series messy/tidy citelangis operads is given in Table 3.

5.1. Actions of parallel citelangis operads.

We now discuss the action of parallel 2-citelangis operads $\mathsf{TCit}^{\parallel}_{\prec\succ}$ and $\mathsf{MCit}^{\parallel}_{2}$ on certain 2-permutations and 2-posets. We start with some combinatorial considerations on certain 2-permutations.

5.1.1. *Bounded cuts in 2-permutations.*

We introduce first an intruiguing class of 2-permutations. These permutations will be instrumental in studying the action of the tidy parallel 2-citelangis operad, but we also believe that they would deserve further study for their own right.

Definition 5.1. *Let $\sigma \in \mathsf{Perm}_2(n)$ be a 2-permutation of degree n and $\gamma \in [n-1]$. We say that γ is a bounded cut of σ if we can write $\sigma = f\mu\nu l$ where f and l are the first and last letters of σ, while μ and ν are words such that $\mu_i \leq \gamma < \nu_j$ for all $i \in [|\mu|]$ and $j \in [|\nu|]$. We denote by $\mathrm{bcuts}(\sigma)$ the set of bounded cuts of σ. We say that the 2-permutation σ is bounded cuttable if it admits a bounded cut, and bounded uncuttable if it admits no bounded cut.*

For example, 1 is a bounded cut of the 2-permutation 31123424, while the 2-permutation 31421324 is bounded uncuttable. Note that there is no condition on the sizes of μ and ν in Definition 5.1 (they can be empty, or of sizes which are not multiples of 2). We now observe that bounded cuts behave properly with the restrictions of Definition 2.42.

Lemma 5.2. *Consider $L \subseteq [n]$ and $\gamma \in [\min(L), \max(L) - 1]$, and let $\gamma^{|L} := |[\gamma] \cap L|$ denote the number of elements of L between 1 and γ. If γ is a bounded cut of a 2-permutation $\sigma \in \mathsf{Perm}_2(n)$, then $\gamma^{|L}$ is a bounded cut of its restriction $\sigma^{|L}$.*

Proof. Since γ is a bounded cut of σ, we can write $\sigma = f\mu\nu l$ with $\mu_i \leq \gamma < \nu_j$ for all $i \in [|\mu|]$ and $j \in [|\nu|]$. Then $\sigma^{|L} = f^{|L}\mu^{|L}\nu^{|L}l^{|L}$. Moreover $\mu_i^{|L} \leq \gamma^{|L} < \nu_j^{|L}$ for any $i \in [|\mu^{|L}|]$ and $j \in [|\nu^{|L}|]$. Therefore, $\gamma^{|L}$ is a bounded cut of $\sigma^{|L}$. □

Note that the reverse statement is wrong. Consider for instance the 2-permutation $\sigma = 213123$ and the subset $L = \{1, 2\}$. Then 1 is not a bounded cut of σ, but $1^{|L} = 1$ is a bounded cut of $\sigma^{|L} = 2112$.

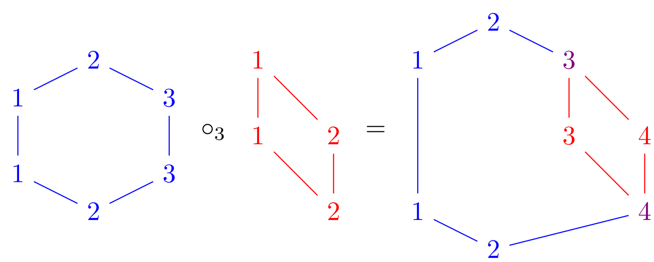
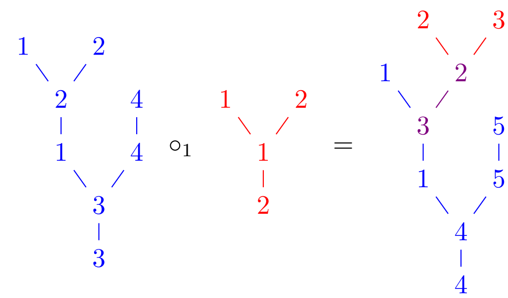
	Parallel (only $k = 2$)	Series (any k)
poset	composition of bounded 2-posets Definition 5.59 and Figure 13 	composition of k-rooted k-posets Definition 5.136 and Figure 19 
	LinExt $\downarrow$ Proposition 5.68	LinExt $\downarrow$ Proposition 5.152
messy	messy parallel composition of 2-permutations Definition 5.65 $31422341 \circ_3 313122 =$ $535341622461 + \cdots + 516223534461$	messy series composition of k-permutations Definition 5.149 $31232144 \circ_2 313122 =$ $514524233166 + \cdots + 534541663122$
	LexMin $\downarrow$ Lemma 5.71	LexMin $\downarrow$ Lemma 5.155
tidy	tidy parallel composition of 2-permutations Proposition 5.27 and Definition 5.70 $31\|422341 \circ_3 313122 = 513534622461$	tidy series composition of k-permutations Proposition 5.103 and Definition 5.154 $312321\|44 \circ_2 313122 = 514521423366$
bases	fully bounded cuttable 2-permutations Definition 5.4 and Propositions 5.69 and 5.72	fully k-rooted cuttable k-permutations Definition 5.76 and Propositions 5.153 and 5.156

TABLE 3. A road map through combinatorial models and actions for parallel/series messy/tidy citelangis operads.

Proposition 5.3. *The following conditions are equivalent for a 2-permutation of degree n:*
(i) its restriction to any interval of $[n]$ of size at least 2 is bounded cuttable,
(ii) its restriction to any subset of $[n]$ of size at least 2 is bounded cuttable.

Proof. Assume that $\sigma \in \mathsf{Perm}_2(n)$ satisfies (i). Let $L \subseteq [n]$ with $|L| \geq 2$. Since $|L| \geq 2$, we have $\min(L) < \max(L)$ so that the restriction $\sigma^{|[\min(L),\max(L)]}$ admits a bounded cut γ with $\min(L) \leq \gamma < \max(L)$. By Remark 2.46, the restriction $\sigma^{|L}$ is just the restriction of $\sigma^{|[\min(L),\max(L)]}$ to $\bar{L} := \{\ell - \min(L) + 1 \mid \ell \in L\}$. By Lemma 5.2, $\sigma^{|L}$ admits a bounded cut $\gamma^{|\bar{L}}$ with $1 \leq \gamma^{|\bar{L}} \leq |L|$. Therefore, σ satisfies (ii). The reverse implication is obvious. □

Definition 5.4. *A 2-permutation is* fully bounded cuttable *if it satisfies the equivalent conditions of Proposition 5.3.*

For example, the 2-permutation 31123424 is bounded cuttable but not fully bounded cuttable (since the restriction to the interval $[2, 4]$ is not bounded cuttable). In contrast, the 2-permutation 5211332454 is fully bounded cuttable.

There is also a characterization of fully bounded cuttable 2-permutations in terms of pattern avoidance. We start by the following technical observation.

Lemma 5.5. *A bounded uncuttable* 2*-permutation of degree at least* 2 *contains a pattern* $b \cdot c \cdot a \cdot b'$ *with* $a \le b, b' \le c$.

Proof. Suppose by contradiction that a 2-permutation $\sigma = \sigma_1 \dots \sigma_{2n}$ contains no such pattern.

Assume first that $\sigma_1 = \sigma_{2n} =: v$. Then for any $u \le v < w$, both values u must appear before both values w, otherwise we would have the pattern $v \cdot w \cdot u \cdot v$. Therefore, both $v-1$ and v (resp. 1, resp. $n-1$) are bounded cuts of σ if $1 < v < n$ (resp. if $v = 1$, resp. if $v = n$).

Assume now that $r := \sigma_1 < \sigma_{2n} =: s$. Let p denote the position of the other s of σ, *i.e.* $p < 2n$ and $\sigma_p = s$. We distinguish two cases:

(i) Assume first that no value of σ appears both before and after the position p. This implies that for $w > s$, both values w appear after the position p, as otherwise we would have a forbidden pattern $w \cdot w \cdot s \cdot s$. Let v be the minimal value that appears after the position p. Note that $v > 1$ as otherwise $r \cdot s \cdot 1 \cdot s$ is a forbidden pattern. We claim that $v-1$ is a bounded cut of σ. Indeed, for any $u < v$, both values u appear before the position p by definition. Moreover, for any $w \ge v$ distinct from s, both values w appear after the position p, as otherwise we would have $v \le w < s$ and thus the forbidden pattern $w \cdot s \cdot v \cdot s$.

(ii) Assume now that there is a value t that appears both before and after the position p. Note that it imposes that $s < t$ as otherwise we would have the forbidden pattern $t \cdot s \cdot t \cdot s$. Let q denote the position of the first value t. We can assume without loss of generality that q is the minimal position of a value that appears both before and after p. Let v be the minimal value that appears after the position q. Note that $v > 1$ as otherwise $r \cdot t \cdot 1 \cdot s$ is a forbidden pattern. We claim that $v-1$ is a bounded cut of σ. Indeed, for any $u < v$, both values u appear before the position q by definition. Moreover, for any $w \ge v$ distinct from s, both values w appear after the position q. Otherwise, by minimality of the position q, the second value w could not be on the right of p. Therefore, either $w > s$ and we have the forbidden pattern $w \cdot w \cdot s \cdot s$, or $v \le w < s < t$ and we have the forbidden pattern $w \cdot t \cdot v \cdot s$.

Finally, the case $\sigma_1 > \sigma_{2n}$ is very similar to the previous one and left to the reader. □

Proposition 5.6. *A* 2*-permutation is fully bounded cuttable if and only if it avoids the pattern* $b \cdot c \cdot a \cdot b'$ *with* $a \le b, b' \le c$.

Proof. There is nothing to prove for the 2-permutation 11. If a 2-permutation σ contains a pattern $b \cdot c \cdot a \cdot b'$ with $a \le b, b' \le c$, then its restriction $\sigma^{|[a,c]}$ is bounded uncuttable, so that σ is not fully bounded cuttable. Conversely, if σ avoids the pattern $b \cdot c \cdot a \cdot b'$ with $a \le b, b' \le c$, then all its restrictions do, so that they are all bounded cuttable by Lemma 5.5. □

We derive in particular the following observation from Proposition 5.6.

Corollary 5.7. *If* $\sigma \in \mathsf{Perm}_2(n)$ *is fully bounded cuttable and* $i \in [n]$*, the factor of* σ *located in between the two occurences of* i *decomposes into* $\mu\nu$ *where* $\mu_p < i < \nu_q$ *for all* $p \in [|\mu|]$ *and* $q \in [|\nu|]$.

Finally, we observe that fully bounded cuttable 2-permutations form a pattern class.

Theorem 5.8. *The set of fully bounded cuttable* 2*-permutations is a* 2*-permutation class: for any fully bounded cuttable* 2*-permutation* $\sigma \in \mathsf{Perm}_2(n)$ *and any* $L \subseteq [n]$*, the restriction* $\sigma^{|L}$ *is fully bounded cuttable.*

Proof. Assume that $\sigma \in \mathsf{Perm}_2(n)$ is fully bounded cuttable and that $L = \{\ell_1, \dots, \ell_{|L|}\} \subseteq [n]$. For any $X \subseteq [|L|]$, the restriction $(\sigma^{|L})^{|X}$ coincides with the restriction $\sigma^{|\{\ell_x \;|\; x \in X\}}$ by Remark 2.57, and is thus bounded cuttable. Therefore, $\sigma^{|L}$ is fully bounded cuttable.

Another approach would be via Proposition 5.6 since patterns are preserved by restriction. □

5.1.2. *Action of* $\mathsf{TCit}^{\parallel}_{\prec\succ}$ *on words and permutations.* We show in this section that FQSym_2 can be endowed with a $\prec\succ$-tidy parallel citelangis structure, and that the resulting $\prec\succ$-tidy parallel citelangis algebra is free. Therefore, the free $\prec\succ$-tidy parallel citelangis subalgebra generated by the 2-permutation 11 provides a combinatorial model for the basis for the $\prec\succ$-tidy parallel citelangis operad.

Action on words. We first observe that the free algebra $\mathcal{A}^{\geq 2}$ can be endowed with a structure of $\prec\succ$-tidy parallel citelangis algebra.

Definition 5.9. *We define the action of the four $\prec\succ$-tidy parallel citelangis operators of $\prec\prec$, $\prec\succ$, $\succ\prec$ and $\succ\succ$ on any two words xXx' and yYy' by*

$$\begin{aligned} xXx' \prec\prec yYy' &= xXyYy'x' \\ xXx' \prec\succ yYy' &= xXx'yYy' \\ xXx' \succ\prec yYy' &= yxXYy'x' \\ xXx' \succ\succ yYy' &= yxXx'Yy'. \end{aligned}$$

In other words, we choose the first and last letters of the result among the first and last letters of xXx' or yYy' depending on the operation, and we concatenate the remaining factors of xXx' and yYy'.

Proposition 5.10. *The free algebra $\mathcal{A}^{\geq 2}$, endowed with the operators of Definition 5.9, defines a $\prec\succ$-tidy parallel citelangis algebra. The concatenation product $\cdot$ of $\mathcal{A}^{\geq 2}$ is given by $\prec\succ$.*

Proof. One immediately checks that our operators on words indeed satisfy the 9 $\prec\succ$-tidy parallel 2-citelangis relations given in Remark 4.14. □

Action on permutations. Replacing the concatenation by the shifted concatenation, one endows similarly the algebra FQSym_2 of 2-permutations with a structure of $\prec\succ$-tidy parallel citelangis algebra. This can be rephrased as follows.

Definition 5.11. *For any operator $\mathfrak{b} \in \mathfrak{B}_2$ and any two 2-permutations μ and ν of degree m and n respectively, $\mu\,\mathfrak{b}\,\nu$ is the 2-permutation $\pi \in \mu \,\bar{\sqcup\!\sqcup}\, \nu$ such that*

- *the first and last entries of π are determined by $\mathfrak{b}$:*

$$\pi_1 = \begin{cases} \mu_1 & \text{if } \mathfrak{b}_1 = \prec, \\ \nu_1 + m & \text{if } \mathfrak{b}_1 = \succ, \end{cases} \qquad \text{and} \qquad \pi_{m+n} = \begin{cases} \mu_m & \text{if } \mathfrak{b}_2 = \prec, \\ \nu_n + m & \text{if } \mathfrak{b}_2 = \succ, \end{cases}$$

- *the remaining entries of π are the concatenation of the remaining entries of μ and the remaining entries of $\nu[m]$.*

For example, for $\mu = 42132413$ and $\nu = 2121$ in FQSym_2 we have $\mu \prec\prec \nu = 421324165653$, $\mu \prec\succ \nu = 421324136565$, $\mu \succ\prec \nu = 642132415653$ and $\mu \succ\succ \nu = 642132413565$. The following statement is immediate from Proposition 5.10.

Proposition 5.12. *The algebra $(\mathsf{FQSym}_2, \bar{\cdot})$, endowed with the operators of Definition 5.11, defines a $\prec\succ$-tidy parallel citelangis algebra. The concatenation product $\bar{\cdot}$ of FQSym_2 is given by $\prec\succ$.*

The goal of this section is to show that the $\prec\succ$-tidy parallel citelangis algebra FQSym_2 is free. To manipulate this $\prec\succ$-tidy parallel citelangis algebra, we consider the evaluations of syntax trees of $\mathbf{Trees}(\mathfrak{B}_2)$ in FQSym_2. See Figure 9 for an illustration.

Definition 5.13. *Denote by $\mathrm{TEval}^{\|}(\mathsf{t}; \sigma_1, \dots, \sigma_p)$ the evaluation of a syntax tree $\mathsf{t} \in \mathbf{Trees}(\mathfrak{B}_2)$ of arity p on p 2-permutations $\sigma_1, \dots, \sigma_p$ of Perm_2 using the $\prec\succ$-tidy parallel citelangis structure of FQSym_2. The tidy parallel permutation evaluation of t is then $\mathrm{TEval}^{\|}(\mathsf{t}) \coloneqq \mathrm{TEval}^{\|}(\mathsf{t}; 11, \dots, 11)$. We extend by linearity t to the elements of $\mathbf{Free}(\mathfrak{B}_2)$ on the one hand, and $\sigma_1, \dots, \sigma_p$ to the elements of FQSym_2 on the other hand.*

Remark 5.14. Let us rephrase algorithmically Definition 5.89. For this, we consider partial syntax trees, *i.e.* trees whose nodes are labeled by operators with 2 letters among $\{\prec, \succ, \perp\}$ and whose leaves are labeled by words. The evaluation $\mathrm{TEval}^{\|}(\mathfrak{s})$ of such a partial syntax tree $\mathfrak{s}$ is defined inductively as follows. If $\mathfrak{s}$ is a leaf labeled by a word w, then $\mathrm{TEval}^{\|}(\mathfrak{s}) = \mathsf{w}$. Otherwise, $\mathrm{TEval}^{\|}(\mathfrak{s})$ is obtained as follows. If the first (resp. second) letter of the root of $\mathfrak{s}$ is $\perp$, then we let $f = \varepsilon$ (resp. $l = \varepsilon$). Otherwise, we let a car traverse the partial syntax tree $\mathfrak{s}$. This car follows and replaces by a $\perp$ the first (resp. second) letter of each signal it traverses, and finally arrives at

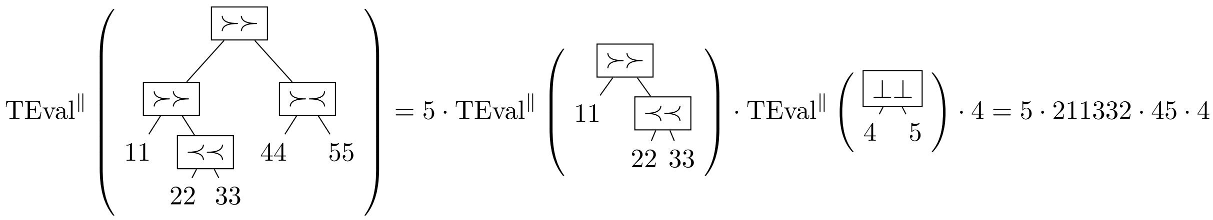


FIGURE 9. Illustration of Remark 5.14.

a leaf where it reads and erases the first (resp. last) letter. Let f (resp. l) be the letter read by the first (resp. second) car at its destination. At this stage, the signal at the root is $\perp\perp$. We are left with the two partial syntax trees $\mathfrak{l}$ and $\mathfrak{r}$ (where some letters of the signals in the nodes and of the words in the leaves have been erased by the cars). The evaluation $\mathrm{TEval}^{\|}(\mathfrak{s})$ is then obtained inductively by

$$\mathrm{TEval}^{\|}(\mathfrak{s}) = f \cdot \mathrm{TEval}^{\|}(\mathfrak{l}) \cdot \mathrm{TEval}^{\|}(\mathfrak{r}) \cdot l.$$

Finally, for a syntax tree $\mathfrak{t}$ of arity p and 2-permutations $\sigma_1 \in \mathsf{Perm}_2(n_1), \dots, \sigma_p \in \mathsf{Perm}_2(n_p)$, the evaluation $\mathrm{TEval}^{\|}(\mathfrak{t}; \sigma_1, \dots, \sigma_p)$ is the evaluation of the partial syntax tree $\mathfrak{s}$ obtained from $\mathfrak{t}$ by putting the permutation $\sigma_i[n_1 + \cdots + n_{i-1}]$ at the i-th leaf for all $i \in [p]$. In particular, for $\mathrm{TEval}^{\|}(\mathfrak{t}) = \mathrm{TEval}^{\|}(\mathfrak{t}; 11, \dots, 11)$, the i-th leaf is labeled by the word ii. See Figure 9.

As FQSym_2 is a $\prec\succ$-tidy parallel citelangis algebra, the tidy parallel permutation evaluation is preserved by the $\prec\succ$-tidy parallel citelangis relations of Remark 4.14. Thus, $\mathrm{TEval}^{\|}(\mathfrak{s}; F_1, \dots, F_p) = \mathrm{TEval}^{\|}(\mathfrak{t}; F_1, \dots, F_p)$ for any $\mathfrak{s}, \mathfrak{t} \in \mathbf{Free}(\mathfrak{B}_2)(p)$ which are equivalent modulo the $\prec\succ$-tidy parallel citelangis relations, and for any $F_1, \dots, F_p \in \mathsf{FQSym}_2$. The objective of this section is to show the reciprocal statement. The proof is based on bounded cuts in 2-permutations introduced in Definition 5.1.

Tidy parallel permutation evaluations and bounded cuts. Our next two lemmas state that the bounded cuts of a 2-permutation ρ precisely correspond to its decompositions of the form $\rho = \sigma \, \mathfrak{b} \, \tau$. Their proofs immediately follow from Definition 5.11 and are thus left to the reader.

Lemma 5.15. *For any* 2*-permutations $\sigma \in \mathsf{Perm}_2(m)$ and $\tau \in \mathsf{Perm}_2(n)$, and any operator $\mathfrak{b} \in \mathfrak{B}_2$, the degree m of σ is a bounded cut of $\sigma \, \mathfrak{b} \, \tau$.*

Lemma 5.16. *For any* 2*-permutation $\rho \in \mathsf{Perm}_2(\ell)$ and any bounded cut $\gamma \in \mathrm{bcuts}(\rho)$, there is a unique $\mathfrak{b} \in \mathfrak{B}_2$ (defined by $\mathfrak{b}_i := \prec$ if $\rho_i \le \gamma$ and $\mathfrak{b}_i := \succ$ if $\rho_i > \gamma$) such that $\rho = \rho^{|[\gamma]} \, \mathfrak{b} \, \rho^{|[\ell] \smallsetminus [\gamma]}$.*

Remark 5.17. Lemma 5.16 gives an inductive algorithm to compute all decompositions of a given 2-permutation ρ as an evaluation of the form $\rho = \mathrm{TEval}^{\|}(\mathfrak{t}; \sigma_1, \dots, \sigma_p)$. Namely, ρ admits

- the trivial evaluation $\rho = \mathrm{TEval}^{\|}(\mathbb{1}; \rho)$, where $\mathbb{1}$ is the unit syntax tree with no node and a single leaf, and
- the evaluation $\rho = \mathrm{TEval}^{\|}(\mathfrak{t}; \sigma_1, \dots, \sigma_l, \tau_1, \dots, \tau_r)$, for any bounded cut $\gamma \in \mathrm{bcuts}(\rho)$ and any evaluations $\rho^{|[\gamma]} = \mathrm{TEval}^{\|}(\mathfrak{l}; \sigma_1, \dots, \sigma_l)$ and $\rho^{|[\ell] \smallsetminus [\gamma]} = \mathrm{TEval}^{\|}(\mathfrak{r}; \tau_1, \dots, \tau_r)$, where $\mathfrak{t}$ is the syntax tree with root $\mathfrak{b} \in \mathfrak{B}_2$ defined by Lemma 5.16 and with subtrees $\mathfrak{l}$ and $\mathfrak{r}$.

This algorithm implies the existence of decompositions of the form $\rho = \mathrm{TEval}^{\|}(\mathfrak{t}; \sigma_1, \dots, \sigma_p)$ where $\sigma_1, \dots, \sigma_p$ are bounded uncuttable. In fact, we can even impose the position of the first bounded cut.

Corollary 5.18. *For any* 2*-permutation $\rho \in \mathsf{Perm}_2$ and any bounded cut $\gamma \in \mathrm{bcuts}(\rho)$, there exists a syntax tree $\mathfrak{t}$ of arity p with left subtree of arity l and bounded uncuttable* 2*-permutations $\sigma_1 \in \mathsf{Perm}_2(n_1), \dots, \sigma_p \in \mathsf{Perm}_2(n_p)$ such that $\rho = \mathrm{TEval}^{\|}(\mathfrak{t}; \sigma_1, \dots, \sigma_p)$ and $\gamma = n_1 + \cdots + n_l$.*

We now characterize the 2-permutations ρ that admit a decomposition of the form $\rho = \mathrm{TEval}^{\|}(\mathfrak{t})$.

Proposition 5.19. *The tidy parallel permutation evaluations of the syntax trees of* $\mathbf{Trees}(\mathfrak{B}_2)$ *are precisely the fully bounded cuttable* 2*-permutations.*

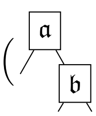
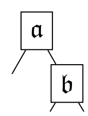
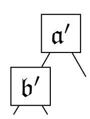
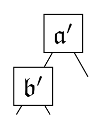
Proof. Consider first a 2-permutation $\rho = \mathrm{TEval}^{\|}(\mathfrak{t})$ with $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_2)(\ell)$. We prove by induction on ℓ that ρ is fully bounded cuttable. If $\ell = 1$, there is nothing to prove. Assume that $\ell \geq 2$ and let $1 \leq a < b \leq \ell$. Let $\mathfrak{l}$ and $\mathfrak{r}$ denote the left and right subtrees of $\mathfrak{t}$, and let γ be the arity of $\mathfrak{l}$, so that $\rho^{|[\gamma]} = \mathrm{TEval}^{\|}(\mathfrak{l})$ and $\rho^{|[\ell] \smallsetminus [\gamma]} = \mathrm{TEval}^{\|}(\mathfrak{r})$. We distinguish three cases:

- Assume that $b \leq \gamma$. Since $\rho^{|[\ell]} = \mathrm{TEval}^{\|}(\mathfrak{l})$ is fully bounded cuttable by induction hypothesis and bounded cuts are preserved by restriction by Lemma 5.2, we obtain that $\rho^{|[a,b]} = (\rho^{|[\gamma]})^{|[a,b]}$ is bounded cuttable.
- Assume that $\gamma \leq a$. The argument is similar since $\rho^{|[a,b]} = (\rho^{|[\ell] \smallsetminus [\gamma]})^{|[a-\gamma, b-\gamma]}$.
- Assume finally that $a < \gamma < b$. By Lemma 5.15, γ is a bounded cut of ρ. Therefore, $\gamma - a$ is a bounded cut of $\rho^{|[a,b]}$ by Lemma 5.2.

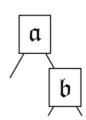
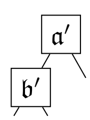
Conversely, consider now a fully bounded cuttable 2-permutation $\rho \in \mathsf{Perm}_2(\ell)$. Similarly to Remark 5.17, we prove by induction on ℓ that ρ is the tidy parallel permutation evaluation of a syntax tree. If $\ell = 1$, then $\rho = \mathrm{TEval}^{\|}(\mathbb{1})$. If $\ell \geq 2$, then ρ admits at least one bounded cut γ by assumption. Moreover, $\rho^{|[\gamma]}$ and $\rho^{|[\ell] \smallsetminus [\gamma]}$ are both fully bounded cuttable by Theorem 5.8. By induction, we obtain that $\rho^{|[\gamma]} = \mathrm{TEval}^{\|}(\mathfrak{l})$ and $\rho^{|[\ell] \smallsetminus [\gamma]} = \mathrm{TEval}^{\|}(\mathfrak{r})$. Then $\rho = \mathrm{TEval}^{\|}(\mathfrak{t})$, where $\mathfrak{t}$ is the syntax tree with root $\mathfrak{b} \in \mathfrak{B}_2$ defined by Lemma 5.16 and with subtrees $\mathfrak{l}$ and $\mathfrak{r}$. $\square$

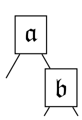
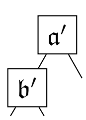
Freeness. Our objective is now to prove that the decompositions of Corollary 5.18 for a given 2-permutation ρ are all equivalent up to the $\prec\!\succ$-tidy parallel citelangis relations of Remark 4.14. Our first step is to understand the evalutations of a quadratic syntax tree on three permutations. We start from a simple observation, which again immediately follows from Definition 5.11. Recall that a syntax tree is $\prec\!\succ$-tidy parallel when all its first (resp. last) traffic signals not contained in its first (resp. last) parallel routes point to the left (resp. right).

Lemma 5.20. *For any 2-permutations $\rho \in \mathsf{Perm}_2(\ell)$, $\sigma \in \mathsf{Perm}_2(m)$ and $\tau \in \mathsf{Perm}_2(n)$, and any operators $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}', \mathfrak{b}' \in \mathfrak{B}_2$, we have*

- *$\ell{+}m$ is a bounded cut of* $\mathrm{TEval}^{\|}\Big($  $; \rho, \sigma, \tau\Big)$ *if and only if*  *is $\prec\!\succ$-tidy parallel,*
- *ℓ is a bounded cut of* $\mathrm{TEval}^{\|}\Big($  $; \rho, \sigma, \tau\Big)$ *if and only if*  *is $\prec\!\succ$-tidy parallel.*

Lemma 5.21. *For any 2-permutations $\rho, \sigma, \tau \in \mathsf{Perm}_2$, and any operators $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}', \mathfrak{b}' \in \mathfrak{B}_2$, if*

$\mathrm{TEval}^{\|}\Big($  $; \rho, \sigma, \tau\Big) = \mathrm{TEval}^{\|}\Big($  $; \rho, \sigma, \tau\Big),$

then  $=$  *is a $\prec\!\succ$-tidy parallel citelangis relation.*

Proof. Let π denote the 2-permutation obtained by these evaluations, and let ℓ, m and n denote the degrees of ρ, σ and τ respectively. By Lemma 5.15, we obtain that ℓ and $\ell + m$ are bounded cuts of π. By Lemma 5.20, we therefore derive that the two syntax trees are $\prec\!\succ$-tidy parallel. Moreover, the parallel destination vector of the two syntax trees are both given by the first and last letters of π where we replace all letters between 1 and ℓ by 1, all letters between $\ell{+}1$ and $\ell{+}m$ by 2, and all letters between $\ell + m + 1$ and $\ell + m + n$ by 3. Since they are $\prec\!\succ$-tidy parallel and have the same parallel destination vector, the two syntax trees form a $\prec\!\succ$-tidy parallel citelangis relation. $\square$

We now prove that, up to the $\prec\!\succ$-tidy parallel citelangis relations, any 2-permutation can be obtained in a unique way as the evaluation of a syntax tree on bounded uncuttable 2-permutations.

Proposition 5.22. *For any syntax trees $\mathfrak{t}, \mathfrak{t}' \in \mathbf{Trees}(\mathfrak{B}_2)$ of arity p and p' respectively, and any bounded uncuttable 2-permutations $\sigma_1, \dots, \sigma_p$, $\sigma'_1, \dots, \sigma'_{p'} \in \mathsf{Perm}_2$, if*

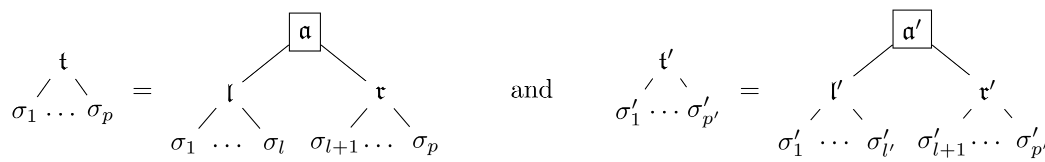
$$\mathrm{TEval}^{\|}(\mathfrak{t}; \sigma_1, \dots, \sigma_p) = \mathrm{TEval}^{\|}(\mathfrak{t}'; \sigma'_1, \dots, \sigma'_{p'}),$$

then $p = p'$, $\mathfrak{t} = \mathfrak{t}'$ modulo the $\prec\!\succ$-tidy parallel citelangis relations and $\sigma_i = \sigma'_i$ for all $i \in [p]$.

Proof. Let $\pi = \mathrm{TEval}^{\|}(\mathfrak{t}; \sigma_1, \dots, \sigma_p) = \mathrm{TEval}^{\|}(\mathfrak{t}'; \sigma'_1, \dots, \sigma'_{p'})$ and let n denote its degree.

We prove the result by induction on p. If $p = 1$, then π is bounded uncuttable, so that $p' = 1$, $\mathfrak{t} = \mathfrak{t}'$ is the only syntax tree of arity 1, and $\sigma_1 = \sigma'_1 = \pi$.

Assume now that $p > 1$. Let $\mathfrak{a}$ be the root of $\mathfrak{t}$, let $\mathfrak{l}$ and $\mathfrak{r}$ be its left and right subtrees, let l be the arity of $\mathfrak{l}$ and let γ be the corresponding bounded cut of π. We thus have $\mathrm{TEval}^{\|}(\mathfrak{l}; \sigma_1, \dots, \sigma_l) = \pi^{|[\gamma]}$ and $\mathrm{TEval}^{\|}(\mathfrak{r}; \sigma_{l+1}, \dots, \sigma_p) = \pi^{|[n] \smallsetminus [\gamma]}$. Define $\mathfrak{a}', \mathfrak{l}', \mathfrak{r}', l'$ and γ' similarly for $\mathfrak{t}'$. These notations are illustrated below:

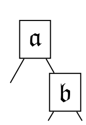
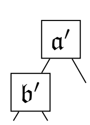


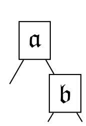
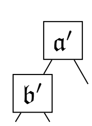
Assume first that $\gamma = \gamma'$. Then $\mathfrak{a} = \mathfrak{a}'$, $\mathrm{TEval}^{\|}(\mathfrak{l}; \sigma_1, \dots, \sigma_l) = \pi^{|[\gamma]} = \mathrm{TEval}^{\|}(\mathfrak{l}'; \sigma'_1, \dots, \sigma'_{l'})$ and $\mathrm{TEval}^{\|}(\mathfrak{r}; \sigma_{l+1}, \dots, \sigma_p) = \pi^{|[n] \smallsetminus [\gamma]} = \mathrm{TEval}^{\|}(\mathfrak{r}'; \sigma'_{l'+1}, \dots, \sigma'_{p'})$ by Lemma 5.16. By induction hypothesis, the first equality ensures that $l = l'$, that $\mathfrak{l} = \mathfrak{l}'$ modulo the $\prec\!\succ$-tidy parallel citelangis relations and that $\sigma_i = \sigma'_i$ for all $i \in [l]$, while the second equality ensures that $p - l = p' - l'$, that $\mathfrak{r} = \mathfrak{r}'$ modulo the $\prec\!\succ$-tidy parallel citelangis relations and that $\sigma_i = \sigma'_i$ for all $i \in [p] \smallsetminus [l]$. We thus conclude that $p = p'$, that $\mathfrak{t} = \mathfrak{t}'$ modulo the $\prec\!\succ$-tidy parallel citelangis relations, and that $\sigma_i = \sigma'_i$ for all $i \in [p]$.

Assume now without loss of generality that $\gamma < \gamma'$. Consider the 2-permutations $\rho := \pi^{|[\gamma]}$, $\sigma := \pi^{|[\gamma'] \smallsetminus [\gamma]}$ and $\tau := \pi^{|[n] \smallsetminus [\gamma']}$. Since γ' is a bounded cut of π larger than γ, Lemma 5.2 ensures that $\gamma' - \gamma$ is a bounded cut of $\pi^{|[n] \smallsetminus [\gamma]}$. By Corollary 5.18 and induction hypothesis, there exists a syntax tree $\mathfrak{s}$ of arity $p - l$ with root $\mathfrak{b}$, left subtree $\mathfrak{u}$ of arity u and right subtree $\mathfrak{v}$, such that $\mathfrak{l}' = \mathfrak{s}'$ modulo the $\prec\!\succ$-tidy parallel citelangis relations and $\mathrm{TEval}^{\|}(\mathfrak{s}; \sigma_{l+1}, \dots, \sigma_p) = \pi^{|[n] \smallsetminus [\gamma]}$, $\mathrm{TEval}^{\|}(\mathfrak{u}; \sigma_{l+1}, \dots, \sigma_{l+u}) = \sigma$ and $\mathrm{TEval}^{\|}(\mathfrak{v}; \sigma_{l+u+1}, \dots, \sigma_p) = \tau$. Similarly, since γ is a bounded cut of π smaller than γ', Lemma 5.2 ensures that γ is a bounded cut of $\pi^{|[\gamma']}$. By Corollary 5.18 and induction hypothesis, there exists a syntax tree $\mathfrak{s}'$ of arity l', with root $\mathfrak{b}'$, left subtree $\mathfrak{u}'$ of arity u' and right subtree $\mathfrak{v}'$, such that $\mathfrak{l}' = \mathfrak{s}'$ modulo the $\prec\!\succ$-tidy parallel citelangis relations and $\mathrm{TEval}^{\|}(\mathfrak{s}'; \sigma'_1, \dots, \sigma'_{l'}) = \pi^{|[\gamma']}$, $\mathrm{TEval}^{\|}(\mathfrak{u}'; \sigma'_1, \dots, \sigma'_{u'}) = \rho$ and $\mathrm{TEval}^{\|}(\mathfrak{v}'; \sigma_{u'+1}, \dots, \sigma_{l'}) = \sigma$. Since

$$\begin{aligned} \mathrm{TEval}^{\|}(\mathfrak{l}; \sigma_1, \dots, \sigma_l) &= \rho = \mathrm{TEval}^{\|}(\mathfrak{u}'; \sigma'_1, \dots, \sigma'_{u'}), \\ \mathrm{TEval}^{\|}(\mathfrak{u}; \sigma_{l+1}, \dots, \sigma_{l+u}) &= \sigma = \mathrm{TEval}^{\|}(\mathfrak{v}'; \sigma_{u'+1}, \dots, \sigma_{l'}), \\ \text{and} \quad \mathrm{TEval}^{\|}(\mathfrak{v}; \sigma_{l+u+1}, \dots, \sigma_p) &= \tau = \mathrm{TEval}^{\|}(\mathfrak{r}'; \sigma'_{l'+1}, \dots, \sigma'_{p'}), \end{aligned}$$

we obtain by three applications of the induction hypothesis that $p = p'$ and $\sigma_i = \sigma'_i$ for all $i \in [p]$. Moreover, we have

$\mathrm{TEval}^{\|}\Big($  $; \rho, \sigma, \tau\Big) = \pi = \mathrm{TEval}^{\|}\Big($  $; \rho, \sigma, \tau\Big).$

By Lemma 5.21, we conclude that  $=$  is a $\prec\!\succ$-tidy parallel citelangis relation, and thus that $\mathfrak{t} = \mathfrak{t}'$ up to $\prec\!\succ$-tidy parallel citelangis relations. ☐

Remark 5.23. From Remark 5.17 and Proposition 5.22, we derive an inductive algorithm to compute the decomposition of a given 2-permutation ρ as an evaluation of the form $\rho = \mathrm{TEval}^{\Vert}(\mathfrak{t}; \sigma_1, \dots, \sigma_p)$, where $\mathfrak{t}$ is in normal form in the $\prec\succ$-tidy parallel citelangis rewriting system of Section 4.3.2 and $\sigma_1, \dots, \sigma_p$ are bounded uncuttable 2-permutations. Namely,

- if ρ is bounded uncuttable, then $\rho = \mathrm{TEval}^{\Vert}(\mathbb{1}; \rho)$, where $\mathbb{1}$ is the unit syntax tree with no node and a single leaf, and
- otherwise, $\rho = \mathrm{TEval}^{\Vert}(\mathfrak{t}; \sigma_1, \dots, \sigma_l, \tau_1, \dots, \tau_r)$, where γ be the rightmost bounded cut of ρ, $\rho^{|[\gamma]} = \mathrm{TEval}^{\Vert}(\mathfrak{l}; \sigma_1, \dots, \sigma_l)$ and $\rho^{|[\ell] \smallsetminus [\gamma]} = \mathrm{TEval}^{\Vert}(\mathfrak{r}; \tau_1, \dots, \tau_r)$ are such that $\mathfrak{l}, \mathfrak{r}$ are in normal form and $\sigma_1, \dots, \sigma_l, \tau_1, \dots, \tau_r$ are bounded uncuttable, and $\mathfrak{t}$ is the syntax tree with root $\mathfrak{b} \in \mathfrak{B}_2$ defined by Lemma 5.16 and with subtrees $\mathfrak{l}$ and $\mathfrak{r}$.

Proposition 5.22 proves the main result of this section.

Theorem 5.24. *The $\prec\succ$-tidy parallel citelangis algebra FQSym_2 is free on bounded uncuttable 2-permutations.*

<u>A complete combinatorial model for $\mathsf{TCit}^{\Vert}_{\prec\succ}$</u>. By Theorem 5.24, the $\prec\succ$-tidy parallel citelangis operad $\mathsf{TCit}^{\Vert}_{\prec\succ}$ can be fully understood from the $\prec\succ$-tidy parallel citelangis subalgebra of FQSym_2 generated by the 2-permutation 11. We close this section with a completely explicit combinatorial model for this algebra. We first obtain from Proposition 5.19 a combinatorial model for the operations of $\mathsf{TCit}^{\Vert}_{\prec\succ}$.

Proposition 5.25. *The tidy parallel permutation evaluation $\mathfrak{t} \mapsto \mathrm{TEval}^{\Vert}(\mathfrak{t})$ is a graded bijection from the $\prec\succ$-tidy parallel equivalence classes of syntax trees of $\mathbf{Trees}(\mathfrak{B}_2)$ to the fully bounded cuttable 2-permutations.*

Proof. This map $\mathfrak{t} \mapsto \mathrm{TEval}^{\Vert}(\mathfrak{t})$ is surjective on fully bounded cuttable 2-permutations by Proposition 5.19. It is compatible with the $\prec\succ$-tidy parallel citelangis relations by Proposition 5.12. Finally, it is bijective since the $\prec\succ$-tidy parallel citelangis algebra FQSym_2 is free on uncuttable 2-permutations by Theorem 5.24. $\square$

Therefore, the fully bounded cuttable 2-permutations can be thought of as a basis of the $\prec\succ$-tidy parallel citelangis operad $\mathsf{TCit}^{\Vert}_{\prec\succ}$. Through the bijection of Proposition 5.19, we can thus define the compositions of the $\prec\succ$-tidy parallel citelangis operad $\mathsf{TCit}^{\Vert}_{\prec\succ}$ directly on fully bounded cuttable 2-permutations.

Definition 5.26. *For any integers $i \le m$ and n, and any two fully bounded cuttable 2-permutations $\sigma = \mathrm{TEval}^{\Vert}(\mathfrak{s}) \in \mathsf{Perm}_2(m)$ and $\tau = \mathrm{TEval}^{\Vert}(\mathfrak{t}) \in \mathsf{Perm}_2(n)$, we define*

$$\sigma \circ_i \tau := \mathrm{TEval}^{\Vert}(\mathfrak{s} \circ_i \mathfrak{t}) \in \mathsf{Perm}_2(m+n-1).$$

The following statement provides a direct combinatorial description of the action of the compositions $\circ_i$ of the $\prec\succ$-tidy parallel citelangis operad $\mathsf{TCit}^{\Vert}_{\prec\succ}$ on fully bounded cuttable 2-permutations.

Proposition 5.27. *Let $\sigma \in \mathsf{Perm}_2(m)$ and $\tau \in \mathsf{Perm}_2(n)$ be two fully bounded cuttable 2-permutations and $i \in [m]$. Write $\sigma = \lambda\, i\, \mu\, \nu\, i\, \omega$, where $\mu_p < i < \nu_q$ for all $p \in [|\mu|]$ and $q \in [|\nu|]$ according to Corollary 5.7, and write $\tau = f\, \theta\, l$ where f and l are its first and last letters. Then*

$$\begin{aligned} \sigma \circ_i \tau &= \lambda[i,n]\, f[i-1]\, \mu[i,n]\, \theta[i-1]\, \nu[i,n] \quad l[i-1]\, \omega[i,n] \\ &= \lambda[i,n]\, f[i-1]\, \mu \qquad \theta[i-1]\, \nu[n-1]\, l[i-1]\, \omega[i,n]. \end{aligned}$$

Proof. Consider arbitrary syntax trees $\mathfrak{s}$ and $\mathfrak{t}$ such that $\sigma = \mathrm{TEval}^{\Vert}(\mathfrak{s})$ and $\tau = \mathrm{TEval}^{\Vert}(\mathfrak{t})$. By Remark 5.14, $\mathrm{TEval}^{\Vert}(\mathfrak{s} \circ_i \mathfrak{t})$ is obtained inductively by letting 2 cars traverse $\mathfrak{s} \circ_i \mathfrak{t}$ in parallel and recording the position of their arrival. The cars that arrive at position i in $\mathfrak{s}$ thus continue their journey through $\mathfrak{t}$. Therefore, the first and last values i in σ are replaced by the first and last values of τ, and the remaining values of τ are placed at the only possible position in $\sigma \circ_i \tau$. $\square$

Here are some examples of compositions on fully bounded cuttable 2-permutations:

$$61|1432325546 \circ_1 211233 = 8211233654547768,$$
$$611432|325546 \circ_2 211233 = 8116532234547768,$$
$$611432|325546 \circ_3 211233 = 8116423345527768,$$
$$61143232|5546 \circ_4 211233 = 8115323244567768,$$
$$611432325|546 \circ_5 211233 = 8114323265567748,$$
$$61143232554|6 \circ_6 211233 = 7114323255466788,$$

where we have materialized by a vertical bar the separation $\mu|\nu$.

Remark 5.28. Motivated by Proposition 5.27, we will extend in Section 5.1.6 the compositions of Definition 5.26 from fully bounded cuttable 2-permutations to all 2-permutations.

5.1.3. *Action of* $\mathsf{MCit}_2^{\parallel}$ *on words and multipermutations.* As observed by M. Aguiar and J.-L. Loday in [AL04], FQSym_2 can also be endowed with a messy parallel 2-citelangis algebra structure (*i.e.* quadri-algebra structure). In this section, we recall the construction and we revisit the proof that the resulting messy parallel 2-citelangis algebra is free [Foi15, Von15].

Action on words. The shuffle subalgebra $\mathsf{Shuffle}^{\geq 2}$ can be endowed with a structure of parallel 2-citelangis algebra (or quadri-algebra [AL04]) defined as follows.

Definition 5.29. *We define the action of the four messy parallel* 2*-citelangis operators* $\prec\prec$, $\prec\succ$, $\succ\prec$ *and* $\succ\succ$ *on any two words* xXx' *and* yYy' *by*

$$\begin{aligned} xXx' \prec\prec yYy' &= x\big(X \sqcup\!\sqcup yYy'\big)x' \\ xXx' \prec\succ yYy' &= x\big(Xx' \sqcup\!\sqcup yY\big)y' \\ xXx' \succ\prec yYy' &= y\big(xX \sqcup\!\sqcup Yy'\big)x' \\ xXx' \succ\succ yYy' &= y\big(xXx' \sqcup\!\sqcup Y\big)y'. \end{aligned}$$

In other words, we choose the first and last letters of the results among the first and last letters of xXx' *or* yYy' *depending on the operation, and we shuffle the remaining factors of* xXx' *and* yYy'.

Proposition 5.30. *The shuffle algebra* $\mathsf{Shuffle}^{\geq 2}$*, endowed with the operators of Definition 5.29, defines a messy parallel* 2*-citelangis algebra (or quadri-algebra). The shuffle product* $\sqcup\!\sqcup$ *of* $\mathsf{Shuffle}^{\geq 2}$ *is given by* $\rtimes\!\!\ltimes\rtimes\!\!\ltimes = \prec\prec + \prec\succ + \succ\prec + \succ\succ$.

Proof. One immediately checks that our operators on words indeed satisfy the 9 messy parallel 2-citelangis relations given in Example 4.5. □

Action on permutations. Replacing the shuffle by the shifted shuffle, one endows similarly the algebra FQSym_2 of 2-permutations with a structure of messy parallel 2-citelangis algebra. This was already observed by M. Aguiar and J.-L. Loday in [AL04]. This can be rephrased as follows.

Definition 5.31. *For any operator* $\mathfrak{b} \in \mathfrak{B}_2$ *and any two* 2*-permutations* μ *and* ν *of degree* m *and* n *respectively,* $\mu\,\mathfrak{b}\,\nu$ *is the sum of all* 2*-permutations* $\pi \in \mu \,\bar{\sqcup\!\sqcup}\, \nu$ *such that*

$$\pi_1 = \begin{cases} \mu_1 & \text{if } \mathfrak{b}_1 = \prec, \\ \nu_1 + m & \text{if } \mathfrak{b}_1 = \succ, \end{cases} \qquad \text{and} \qquad \pi_{m+n} = \begin{cases} \mu_m & \text{if } \mathfrak{b}_2 = \prec, \\ \nu_n + m & \text{if } \mathfrak{b}_2 = \succ. \end{cases}$$

For example, for $\mu = 321312$ and $\nu = 213231$ in FQSym_2 we have $\mu \prec\prec \nu = 3(2131 \sqcup\!\sqcup 546564)2$, $\mu \prec\succ \nu = 3(21312 \sqcup\!\sqcup 54656)4$, $\mu \succ\prec \nu = 5(32131 \sqcup\!\sqcup 46564)2$ and $\mu \succ\succ \nu = 5(321312 \sqcup\!\sqcup 4656)4$. The next statement is immediate from Proposition 5.30.

Proposition 5.32 ([AL04])**.** *The algebra* $(\mathsf{FQSym}_2, \bar{\sqcup\!\sqcup})$*, endowed with the operators of Definition 5.31, defines a messy parallel* 2*-citelangis algebra (or quadri-algebra). The shifted shuffle product* $\bar{\sqcup\!\sqcup}$ *of* FQSym_2 *is given by* $\rtimes\!\!\ltimes\rtimes\!\!\ltimes$.

Similarly to Definition 5.13, we consider the evaluations of syntax trees of $\mathbf{Trees}(\mathfrak{B}_2)$ in FQSym_2 to manipulate this messy parallel 2-citelangis algebra. See Figure 12.

Definition 5.33. *Denote by* $\mathrm{MEval}^{\|}(\mathfrak{t}; \sigma_1, \dots, \sigma_p)$ *the evaluation of a syntax tree* $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_2)$ *of arity* p *on* p *2-permutations* $\sigma_1, \dots, \sigma_p$ *of* Perm_2 *using the messy parallel 2-citelangis structure of* FQSym_2*. The* messy parallel permutation evaluation *of* $\mathfrak{t}$ *is then* $\mathrm{MEval}^{\|}(\mathfrak{t}) := \mathrm{MEval}^{\|}(\mathfrak{t}; 11, \dots, 11)$*. We extend by linearity* $\mathfrak{t}$ *to the elements of* $\mathbf{Free}(\mathfrak{B}_2)$ *on the one hand, and* $\sigma_1, \dots, \sigma_p$ *to the elements of* FQSym_2 *on the other hand.*

Remark 5.34. Let us rephrase algorithmically Definition 5.33. For this, we consider partial syntax trees as in Remark 5.14, where nodes are labeled by operators with 2 letters among $\{\prec, \succ, \perp\}$ and leaves are labeled by words. The messy parallel permutation evaluation $\mathrm{MEval}^{\|}(\mathfrak{s})$ of such a partial syntax tree $\mathfrak{s}$ is defined inductively by

- $\mathrm{MEval}^{\|}(\mathfrak{s}) = \mathsf{w}$ if $\mathfrak{s}$ is a leaf labeled by the word w,
- $\mathrm{MEval}^{\|}(\mathfrak{s}) = f \cdot \big(\mathrm{MEval}^{\|}(\mathfrak{l}) \sqcup \mathrm{MEval}^{\|}(\mathfrak{r})\big) \cdot l$ otherwise, where f and l are the letters read by two cars in parallel on $\mathfrak{s}$, and $\mathfrak{l}$ and $\mathfrak{r}$ are the two partial syntax trees left after these two cars traversed $\mathfrak{s}$ while replacing by $\perp$ the letters they use in the signal in the nodes and erasing the letters they read in the words in the leaves as described in Remark 5.14.

Finally, for a syntax tree $\mathfrak{t}$ of arity p and 2-permutations $\sigma_1 \in \mathsf{Perm}_2(n_1), \dots, \sigma_p \in \mathsf{Perm}_2(n_p)$, the evaluation $\mathrm{MEval}^{\|}(\mathfrak{t}; \sigma_1, \dots, \sigma_p)$ is the evaluation of the partial syntax tree $\mathfrak{s}$ obtained from $\mathfrak{t}$ by putting the permutation $\sigma_i[n_1 + \dots + n_{i-1}]$ at the i-th leaf for all $i \in [p]$. In particular, for $\mathrm{MEval}^{\|}(\mathfrak{t}) = \mathrm{MEval}^{\|}(\mathfrak{t}; 11, \dots, 11)$, the i-th root is labeled by the word ii. See Figure 12.

Freeness. It turns out that the tidy and messy parallel permutation evaluations are related by triangularity for the lexicographic order.

Definition 5.35. *For an homogeneous element* $F \in \mathsf{FQSym}_2$*, we denote by* $\mathrm{LexMin}(F)$ *the lexicographic minimal 2-permutation with a non-zero coefficient in* F*.*

Lemma 5.36. *For any* $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_2)(p)$ *and* $\sigma_1, \dots, \sigma_p \in \mathsf{Perm}_2$*, we have*

$$\mathrm{TEval}^{\|}(\mathfrak{t}; \sigma_1, \dots, \sigma_p) = \mathrm{LexMin}\left(\mathrm{MEval}^{\|}(\mathfrak{t}; \sigma_1, \dots, \sigma_p)\right).$$

In particular, $\mathrm{TEval}^{\|}(\mathfrak{t}) = \mathrm{LexMin}\left(\mathrm{MEval}^{\|}(\mathfrak{t})\right)$*.*

Proof. The proof works by induction using the descriptions of $\mathrm{TEval}^{\|}$ in Remark 5.14 and of $\mathrm{MEval}^{\|}$ in Remark 5.34. Indeed, for any partial syntax tree $\mathfrak{s}$, we have

- if $\mathfrak{s}$ is a leaf labeled by the word w, then $\mathrm{TEval}^{\|}(\mathfrak{s}) = \mathsf{w} = \mathrm{MEval}^{\|}(\mathfrak{s})$,
- otherwise, if and $\mathfrak{l}$ and $\mathfrak{r}$ are the two partial syntax trees left after 2 cars traversed $\mathfrak{s}$ in parallel while erasing the first and last letters of the signals in the nodes and of first and last letters of the words in the leaves as described in Remark 5.14, then

$$\begin{aligned}\mathrm{TEval}^{\|}(\mathfrak{s}) &= f \cdot \mathrm{TEval}^{\|}(\mathfrak{l}) \cdot \mathrm{TEval}^{\|}(\mathfrak{r}) \cdot l = f \cdot \mathrm{LexMin}\left(\mathrm{MEval}^{\|}(\mathfrak{l})\right) \cdot \mathrm{LexMin}\left(\mathrm{MEval}^{\|}(\mathfrak{r})\right) \cdot l \\ &= \mathrm{LexMin}\left(f \cdot \left(\mathrm{MEval}^{\|}(\mathfrak{l}) \sqcup \mathrm{MEval}^{\|}(\mathfrak{r})\right) \cdot l\right) = \mathrm{LexMin}\left(\mathrm{MEval}^{\|}(\mathfrak{s})\right).\end{aligned}$$

The result then follows by application on the syntax tree $\mathfrak{s}$ obtained from $\mathfrak{t}$ by putting the permutation $\sigma_i[n_1 + \dots + n_{i-1}]$ at the i-th leaf for all $i \in [p]$. □

Remark 5.37. Since $\mathrm{TEval}^{\|}$ is increasing for the lexicographic order, Lemma 5.36 extends to FQSym_2. Namely, for any $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_2)(p)$ and homogeneous elements $F_1, \dots, F_p \in \mathsf{FQSym}_2$,

$$\mathrm{LexMin}\left(\mathrm{MEval}^{\|}(\mathfrak{t}; F_1, \dots, F_p)\right) = \mathrm{TEval}^{\|}\left(\mathfrak{t}; \mathrm{LexMin}(F_1), \dots, \mathrm{LexMin}(F_p)\right).$$

Lemma 5.36 enables us to revisit the following result of [Foi15, Von15].

Theorem 5.38 ([Foi15, Von15])**.** *The messy parallel 2-citelangis algebra* FQSym_2 *(aka. quadri-algebra) is free on bounded uncuttable 2-permutations.*

Proof. Consider the $\prec\succ$-tidy and the messy parallel 2-citelangis rewriting systems defined in Section 4.3.2. As seen in Section 4.3.3, these two rewriting systems have the same normal forms. Consider two such normal forms $\mathfrak{t}, \mathfrak{t}'$ of arity p and p' respectively, and some bounded uncuttable 2-permutations $\sigma_1, \dots, \sigma_p, \sigma'_1, \dots, \sigma'_{p'}$, such that $\mathrm{MEval}^{\|}(\mathfrak{t}; \sigma_1, \dots, \sigma_p) = \mathrm{MEval}^{\|}(\mathfrak{t}'; \sigma'_1, \dots, \sigma'_{p'})$. It then follows from Lemma 5.36 that $\mathrm{TEval}^{\|}(\mathfrak{t}; \sigma_1, \dots, \sigma_p) = \mathrm{TEval}^{\|}(\mathfrak{t}'; \sigma'_1, \dots, \sigma'_{p'})$. Proposition 5.22 then implies that $p = p'$, that $\sigma_i = \sigma'_i$ for all $i \in [p]$, and that $\mathfrak{t}$ and $\mathfrak{t}'$ are tidy parallel 2-citelangis equivalent. As we assumed that they were in normal form, we obtain that $\mathfrak{t} = \mathfrak{t}'$. The result follows. □

Remark 5.39. By Theorem 5.38, the messy parallel 2-citelangis operad (or Quad operad) can be fully understood from the messy parallel 2-citelangis subalgebra of FQSym_2 generated by the 2-permutation 11. However, while the fully bounded cuttable 2-permutations still provide a combinatorial model for the basis of this messy parallel 2-citelangis algebra similarly to Proposition 5.25, the compositions $\circ_i$ of the messy parallel 2-citelangis operad $\mathsf{MCit}_k^{\|}$ on fully bounded cuttable 2-permutations are more intricate than Proposition 5.27.

5.1.4. *Operations of* $\mathsf{MCit}_2^{\|}$ *on bounded* 2*-posets.* In this section, we observe that any messy parallel permutation evaluation is the sum of all linear extensions of a well-chosen 2-poset. This observation allows us to encode the messy parallel permutation evaluation by an alternative combinatorial model and to study directly on this model the action of the messy parallel 2-citelangis operad $\mathsf{MCit}_2^{\|}$. It also motivates the introduction of the parallel 2-poset operad that will be studied in Section 5.1.5.

<u>Operations on bounded 2-posets and parallel poset evaluations</u>. We first define some operations on the following 2-posets.

Definition 5.40. *A* 2*-poset* $\le_M$ *is* bounded *if it admits a unique minimal element* $\min(\le_M)$ *and a unique maximal element* $\max(\le_M)$*. We then denote by* $\le_{M_*}$ *the poset induced by* $\le_M$ *on* $M_* := M \smallsetminus \{\min(\le_M)\}$ *and by* $\le_{M^*}$ *the poset induced by* $\le_M$ *on* $M^* := M \smallsetminus \{\max(\le_M)\}$*.*

Definition 5.41. *Consider an operator* $\mathfrak{b} \in \mathfrak{B}_2 := \{\prec\prec, \prec\succ, \succ\prec, \succ\succ\}$ *and two bounded* 2*-posets* $\le_M$ *and* $\le_N$ *of degrees* m *and* n *respectively. We define a bounded* 2*-poset* $\le_P := \le_M \mathfrak{b} \le_N$ *by:*

- $P := M \sqcup N[m]$ *where* $N[m]$ *is the* 2*-set* $(m+1)^{\{2\}} \dots (m+n)^{\{2\}}$ *obtained from* N *by shifting each element by the degree* m *of* M *as in Definition 2.54,*
- *in* $\le_P$*, the elements of* M *(respectively of* $N[m]$*) are ordered by* $\le_M$ *(resp. by* $\le_{N[m]}$*), and the only other cover relations are* $\{\min(\le_M), \min(\le_{N[m]})\}$ *whose order is given by the first traffic signal of* $\mathfrak{b}$*, and* $\{\max(\le_M), \max(\le_{N[m]})\}$ *whose order is given by the second traffic signal of* $\mathfrak{b}$*. Formally, the comparison* $x \le_P y$ *holds for* $x, y \in P$ *if and only if one of the following statements holds:*
  - $x \in M$*,* $y \in M$ *and* $x \le_M y$*, or*
  - $x \in N[m]$*,* $y \in N[m]$ *and* $x \le_{N[m]} y$*, or*
  - $x = \min(\le_M)$*,* $y \in N[m]$ *and* $\mathfrak{b}_1 = \prec$*, or*
  - $x = \min(\le_{N[m]})$*,* $y \in M$ *and* $\mathfrak{b}_1 = \succ$*, or*
  - $x \in N[m]$*,* $y = \max(\le_M)$ *and* $\mathfrak{b}_2 = \prec$*, or*
  - $x \in M$*,* $y = \max(\le_{N[m]})$ *and* $\mathfrak{b}_2 = \succ$*.*

Remark 5.42. Definition 5.41 can be conveniently rephrased in terms of ordered sums and disjoint unions of posets presented in Definition 2.58. Indeed,

$$\begin{aligned}
\le_M \prec\prec \le_N &= \{\min(\le_M)\} + \big(\le_{M_*^*} \sqcup \le_{N[m]}\big) + \{\max(\le_M)\},\\
\le_M \prec\succ \le_N &= \{\min(\le_M)\} + \big(\le_{M_*} \sqcup \le_{N[m]^*}\big) + \{\max(\le_{N[m]})\},\\
\le_M \succ\prec \le_N &= \{\min(\le_{N[m]})\} + \big(\le_{M^*} \sqcup \le_{N[m]_*}\big) + \{\max(\le_M)\},\\
\le_M \succ\succ \le_N &= \{\min(\le_{N[m]})\} + \big(\le_M \sqcup \le_{N[m]_*^*}\big) + \{\max(\le_{N[m]})\},
\end{aligned}$$

where $m = |M|$.

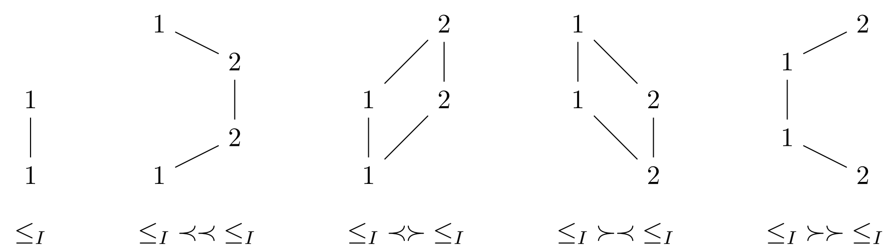


FIGURE 10. Four parallel operations on bounded 2-posets. See Definition 5.41.

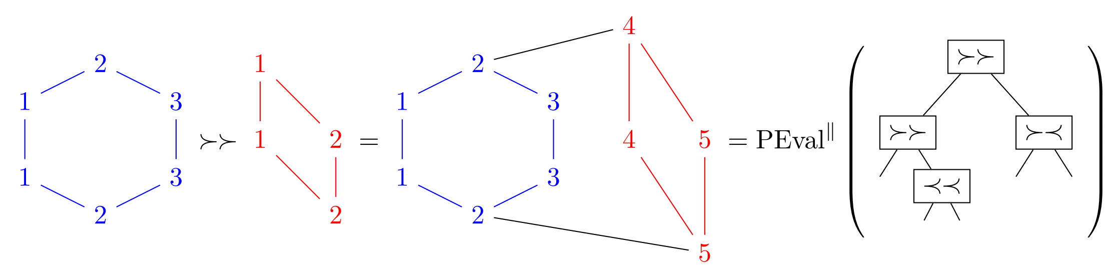


FIGURE 11. Another example of parallel operation on bounded 2-posets. See Definition 5.41.

For example, let $\leq_I$ denote the 2-chain on $1^{\{2\}}$ represented in Figure 10 (left). Then the four bounded 2-posets $\leq_I \ \mathfrak{b} \ \leq_I$, for $\mathfrak{b} \in \mathfrak{B}_2$, are represented in Figure 10 (left). See Figure 11 for another example. The next statement clearly follows from Remark 5.42.

Lemma 5.43. *For any operator $\mathfrak{b} \in \mathfrak{B}_2$ and any two bounded 2-posets $\leq_M$ and $\leq_N$ of degrees m and n respectively, $\leq_M \ \mathfrak{b} \ \leq_N$ is a bounded 2-poset of degree $m+n$.*

This statement allows to define the evaluation of a syntax tree on posets. See Figure 11.

Definition 5.44. *Denote by $\mathrm{PEval}^{\|}(\mathfrak{t}; \leq_{M_1}, \dots, \leq_{M_p})$ the evaluation of a syntax tree $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_2)$ of arity p on p bounded 2-posets $\leq_{M_1}, \dots, \leq_{M_p}$ using the operators of Definition 5.41. The parallel poset evaluation of $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_2)$ is then $\mathrm{PEval}^{\|}(\mathfrak{t}) := \mathrm{PEval}^{\|}(\mathfrak{t}; \leq_I, \dots, \leq_I)$, where $\leq_I$ is the 2-chain on $1^{\{2\}}$.*

Remark 5.45. It is not difficult to check that, if $\leq_M$ and $\leq_N$ are lattices, then $\leq_M \mathfrak{b} \leq_N$ is a lattice for any operator $\mathfrak{b} \in \mathfrak{B}_2$. Therefore, for any $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_2)$, the parallel poset evaluation $\mathrm{PEval}^{\|}(\mathfrak{t})$ is a lattice 2-poset. We will characterize the parallel poset evaluations later in Proposition 5.53.

Bounded cuts in 2-posets. We now aim at characterizing parallel poset evaluations. As in Section 5.1.2, this understanding goes through bounded cuts in bounded 2-posets.

Definition 5.46. *Let $\leq_M$ be a bounded 2-poset of degree n and $\gamma \in [n-1]$. We say that γ is a bounded cut of $\leq_M$ if we can decompose M into $M = \{\min(\leq_M), \max(\leq_M)\} \sqcup L \sqcup R$ such that, for all $\ell \in L$ and $r \in R$, we have $\ell \leq \gamma < r$ and ℓ and r are incomparable for $\leq_M$. We denote by $\mathrm{bcuts}(\leq_M)$ the set of bounded cuts of $\leq_M$. We say that the bounded 2-poset $\leq_M$ is bounded cuttable if it admits a bounded cut, and bounded uncuttable if it admits no bounded cut.*

The following statements are similar to Lemma 5.2 and Proposition 5.3.

Lemma 5.47. *Consider $L \subseteq [m]$ and $\gamma \in [\min(L), \max(L)-1]$, and let $\gamma^{|L} := |[\gamma] \cap L|$ denote the number of elements of L between 1 and γ. If γ is a bounded cut of a bounded 2-poset $\leq_M$ of degree m, then $\gamma^{|L}$ is a bounded cut of its restriction $\leq_{M^{|L}}$.*

Proposition 5.48. *The following conditions are equivalent for a bounded 2-poset of degree n:*

(i) its restriction to any interval of $[n]$ of size at least 2 is bounded cuttable,

(ii) its restriction to any subset of $[n]$ of size at least 2 is bounded cuttable.

Definition 5.49. *A bounded 2-poset is* fully bounded cuttable *if it satisfies the equivalent conditions of Proposition 5.48.*

The following statements are similar to Lemmas 5.15 and 5.16.

Lemma 5.50. *For any bounded 2-posets $\leq_M$ and $\leq_N$, and any operator $\mathfrak{b} \in \mathfrak{B}_2$, the degree m of $\leq_M$ is a bounded cut of $\leq_M \ \mathfrak{b} \leq_N$.*

Lemma 5.51. *For any bounded 2-poset $\leq_P$ of degree p and any bounded cut $\gamma \in \mathrm{bcuts}(\leq_P)$, there is a unique $\mathfrak{b} \in \mathfrak{B}_2$ (defined by $\mathfrak{b}_1 := \prec$ if $\min(\leq_P) \leq \gamma$ while $\mathfrak{b}_1 := \succ$ if $\min(\leq_P) > \gamma$, and $\mathfrak{b}_2 := \prec$ if $\max(\leq_P) \leq \gamma$ while $\mathfrak{b}_2 := \succ$ if $\max(\leq_P) > \gamma$) such that $\leq_P = \leq_{P|[\gamma]} \mathfrak{b} \leq_{P|[p] \smallsetminus [\gamma]}$.*

Remark 5.52. Similarly to Remark 5.17, observe that Lemma 5.51 gives an inductive algorithm to compute all decompositions of a given bounded 2-poset $\leq_P$ as an evaluation of the form $\leq_P = \mathrm{PEval}^{\parallel}(\mathfrak{t}; \leq_{M_1}, \dots, \leq_{M_p})$. Namely, $\leq_P$ admits

- the trivial evaluation $\leq_0 = \mathrm{PEval}^{\parallel}(\mathbb{1}; \leq_0)$, where $\mathbb{1}$ is the unit syntax tree with no node and a single leaf, and
- the evaluation $\leq_P = \mathrm{PEval}^{\parallel}(\mathfrak{t}; \leq_{M_1}, \dots, \leq_{M_l}, \leq_{N_1}, \dots, \leq_{N_r})$, for any $\gamma \in \mathrm{bcuts}(\leq_P)$ and any $\leq_{P|[\gamma]} = \mathrm{PEval}^{\parallel}(\mathfrak{l}; \leq_{M_1}, \dots, \leq_{M_l})$ and $\leq_{P|[\ell] \smallsetminus [\gamma]} = \mathrm{PEval}^{\parallel}(\mathfrak{r}; \leq_{N_1}, \dots, \leq_{N_r})$, where $\mathfrak{t}$ is the syntax tree with root $\mathfrak{b} \in \mathfrak{B}_2$ defined by Lemma 5.51 and with subtrees $\mathfrak{l}$ and $\mathfrak{r}$.

This algorithm implies the existence of decompositions of the form $\leq_P = \mathrm{PEval}^{\parallel}(\mathfrak{t}; \leq_{M_1}, \dots, \leq_{M_p})$ where $\leq_{M_1}, \dots, \leq_{M_p}$ are bounded uncuttable.

Note that not all bounded 2-posets are obtained by evaluating syntax trees on $\mathfrak{B}_2$. For instance, the evaluation the syntax trees of arity 2 only produces the four bounded 2-posets of Figure 10, thus only two of the six linear bounded 2-posets of degree 2. We now characterize the bounded 2-posets which are parallel poset evaluations of syntax trees.

Proposition 5.53. *The parallel poset evaluations of the syntax trees of* $\mathbf{Trees}(\mathfrak{B}_2)$ *are precisely the fully bounded cuttable 2-posets.*

Proof. The proof is exactly the same as Proposition 5.19. Consider first a parallel poset evaluation $\leq_P = \mathrm{PEval}^{\parallel}(\mathfrak{t})$ with $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_2)(\ell)$. We prove by induction on ℓ that $\leq_P$ is fully bounded cuttable. If $\ell = 1$, there is nothing to prove. Assume that $\ell \geq 2$ and let $1 \leq a < b \leq \ell$. Let $\mathfrak{l}$ and $\mathfrak{r}$ denote the left and right subtrees of $\mathfrak{t}$, and let γ be the arity of $\mathfrak{l}$, so that $\leq_{P|[\gamma]} = \mathrm{PEval}^{\parallel}(\mathfrak{l})$ and $\leq_{M|[\ell] \smallsetminus [\gamma]} = \mathrm{PEval}^{\parallel}(\mathfrak{r})$. We distinguish three cases:

- Assume that $b \leq \gamma$. Since $\leq_{M|[\ell]} = \mathrm{PEval}^{\parallel}(\mathfrak{l})$ is fully bounded cuttable by induction hypothesis and bounded cuts are preserved by restriction by Lemma 5.47, we obtain that $\leq_{P|[a,b]} = (\leq_{P|[\gamma]})^{|[a,b]}$ is bounded cuttable.
- Assume that $\gamma \leq a$. The argument is similar since $\leq_{P|[a,b]} = (\leq_{P|[\ell] \smallsetminus [\gamma]})^{|[a-\gamma, b-\gamma]}$.
- Assume finally that $a < \gamma < b$. By Lemma 5.50, γ is a bounded cut of $\leq_P$. Therefore, $\gamma - a$ is a bounded cut of $\leq_{P|[a,b]}$ by Lemma 5.47.

Conversely, consider now a fully bounded cuttable 2-poset $\leq_P$ of degree ℓ. Similarly to Remark 5.52, we prove by induction on ℓ that $\leq_P$ is the parallel poset evaluation of a syntax tree. If $\ell = 1$, then $\leq_P = \mathrm{PEval}^{\parallel}(\mathbb{1})$. If $\ell \geq 2$, then $\leq_P$ admits at least one bounded cut γ by assumption. Moreover, $\leq_{P|[\gamma]}$ and $\leq_{P|[\ell] \smallsetminus [\gamma]}$ are both fully bounded cuttable by Lemma 5.47. By induction, we obtain that $\leq_{P|[\gamma]} = \mathrm{PEval}^{\parallel}(\mathfrak{l})$ and $\leq_{P|[\ell] \smallsetminus [\gamma]} = \mathrm{PEval}^{\parallel}(\mathfrak{r})$. Then $\leq_P = \mathrm{PEval}^{\parallel}(\mathfrak{t})$, where $\mathfrak{t}$ is the syntax tree with root $\mathfrak{b} \in \mathfrak{B}_2$ defined by Lemma 5.16 and with subtrees $\mathfrak{l}$ and $\mathfrak{r}$. □

Connections between $\mathrm{TEval}^{\parallel}$, $\mathrm{MEval}^{\parallel}$ and $\mathrm{PEval}^{\parallel}$. We have seen in Lemma 5.36 that $\mathrm{TEval}^{\parallel}(\mathfrak{t}) = \mathrm{LexMin}\big(\mathrm{MEval}^{\parallel}(\mathfrak{t})\big)$ for any syntax tree $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_2)$. We now compare with the parallel poset evaluation $\mathrm{PEval}^{\parallel}(\mathfrak{t})$.

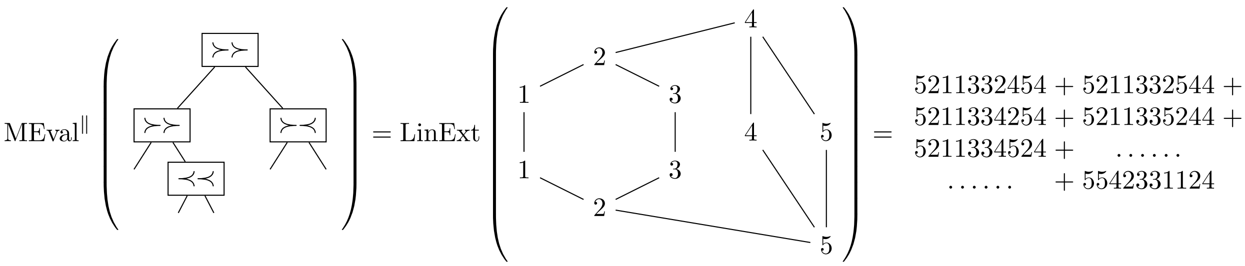


FIGURE 12. Illustration of Lemma 5.54.

For a multiposet $\leq_M$, we let

$$\mathrm{LinExt}(\leq_M) := \sum_{\mu \in \mathcal{L}(\leq_M)} \mu$$

be the sum of all linear extensions of $\leq_M$. We obtain the following statement, illustrated in Figure 12.

Lemma 5.54. *For any syntax tree* $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_2)$*, we have* $\mathrm{MEval}^{\|}(\mathfrak{t}) = \mathrm{LinExt}\big(\mathrm{PEval}^{\|}(\mathfrak{t})\big)$.

Proof. Remark 5.42 and Lemma 2.64 prove that

$$\mathrm{LinExt}(\leq_M)\ \mathfrak{b}\ \mathrm{LinExt}(\leq_N) = \mathrm{LinExt}(\leq_M \mathfrak{b} \leq_N).$$

for any operator $\mathfrak{b} \in \mathfrak{B}_2$ and any bounded 2-posets $\leq_M$ and $\leq_N$. The statement then follows by induction on syntax trees. □

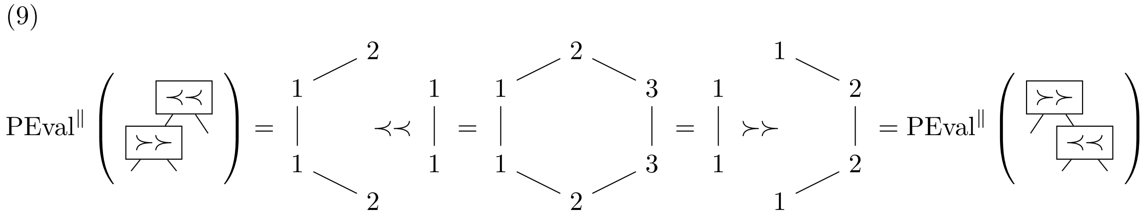
Conversely, we can obtain $\mathrm{PEval}^{\|}(\mathfrak{t})$ from $\mathrm{MEval}^{\|}(\mathfrak{t})$ by intersecting all linear orders corresponding to the 2-permutations that appear in $\mathrm{MEval}^{\|}(\mathfrak{t})$.

We now describe the connection between $\mathrm{TEval}^{\|}(\mathfrak{t})$ and $\mathrm{PEval}^{\|}(\mathfrak{t})$. By Lemmas 5.36 and 5.54, the tidy parallel permutation evaluation $\mathrm{TEval}^{\|}(\mathfrak{t})$ is the minimal linear extension of $\mathrm{PEval}^{\|}(\mathfrak{t})$. It can thus be obtained from $\mathrm{PEval}^{\|}(\mathfrak{t})$ by repeatedly reading and deleting the minimal source.

Conversely, there are several syntax trees $\mathfrak{t}$ whose parallel poset evaluation $\mathrm{PEval}^{\|}(\mathfrak{t})$ has the same minimal linear extension. Given a bounded cuttable 2-permutation σ, we can find all 2-posets $\mathrm{PEval}^{\|}(\mathfrak{t})$ whose minimal linear extension is σ by first using Remark 5.14 to compute all possible syntax trees $\mathfrak{t}$ such that $\sigma = \mathrm{TEval}^{\|}(\mathfrak{t})$, and then applying Definitions 5.41 and 5.44 to get their parallel poset evaluations $\mathrm{PEval}^{\|}(\mathfrak{t})$.

Relations among parallel poset evaluations. We now study which syntax trees evaluate to the same bounded 2-poset. Observe first the following quadratic relation:

(9)



Warning! The operators of Definition 5.41 on bounded 2-posets do not verify the other messy parallel 2-citelangis relations, and thus do not define a messy parallel 2-citelangis algebra. In fact, we will now show that Equation (9) is the only relation that leads to the same parallel poset evaluations.

Lemma 5.55. *For any operators* $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}_2$ *and any bounded 2-posets* $\leq_M, \leq_N, \leq_{M'}, \leq_{N'}$*, we have* $\leq_M \mathfrak{b} \leq_N = \leq_{M'} \mathfrak{b}' \leq_{N'}$ *if and only if*

- *either* $\mathfrak{b} = \mathfrak{b}'$, $\leq_M = \leq_{M'}$ *and* $\leq_N = \leq_{N'}$,

- *or* $\mathfrak{b} = \prec\prec$ *and* $\mathfrak{b}' = \succ\succ$*, and there exists a bounded* 2*-poset* $\le_O$ *such that* $\le_M = \le_{M'} \succ\succ \le_O$ *and* $\le_{N'} = \le_O \prec\prec \le_N$.

Proof. The "if" direction is immediate as the second case was already observed in Equation (9).

The "only if" direction relies on the fact that for any $\mathfrak{b} \in \mathfrak{B}_2$ and any two bounded 2-posets $\le_M$ and $\le_N$, we can easily identify the positions of the minimums and maximums of $\le_M$ and $\le_N$ in the 2-poset $\le_P := \le_M \mathfrak{b} \le_N$. For this, let $a := \min(\le_P)$ and $t := \max(\le_P)$, let b and c denote the minimal and maximal values covering a in $\le_P$ and let r and s denote the minimal and maximal values covered by t in $\le_P$. Then the 2-posets $\le_M$ and $\le_N$ are (up to the label shift) the sub-2-posets of $\le_P$ induced by $\tilde{M}$ and $\tilde{N}$ depending on $\mathfrak{b}$ as follows:

- if $\mathfrak{b} = \prec\prec$ then $\tilde{M} := [a,t] \smallsetminus [c,s]$ and $\tilde{N} := [c,s] = [a,s] \smallsetminus \{a\} = [c,t] \smallsetminus \{t\}$,
- if $\mathfrak{b} = \prec\succ$ then $\tilde{M} := [a,r]$ and $\tilde{N} := [c,t]$,
- if $\mathfrak{b} = \succ\prec$ then $\tilde{M} := [b,t]$ and $\tilde{N} := [a,s]$,
- if $\mathfrak{b} = \succ\succ$ then $\tilde{M} := [b,r] = [a,r] \smallsetminus \{a\} = [b,t] \smallsetminus \{t\}$ and $\tilde{N} := [a,t] \smallsetminus [b,r]$,

where $[x,y]$ denotes the interval between x and y in $\le_P$. We now argue the "only if" direction of the statement by case analysis:

- Case $\mathfrak{b} = \mathfrak{b}'$: We immediately obtain that $\le_M = \le_{M'}$ and $\le_N = \le_{N'}$ since the description above enables us to reconstruct $\le_M$ and $\le_N$ from $\le_M \mathfrak{b} \le_N$ when we know $\mathfrak{b}$.
- Case $\mathfrak{b} = \prec\succ$ and $\mathfrak{b}' = \succ\prec$: Impossible since $\min(\le_M \prec\succ \le_N) < \max(\le_M \prec\succ \le_N)$ while $\min(\le_{M'} \succ\prec \le_{N'}) > \max(\le_{M'} \succ\prec \le_{N'})$ (where $<$ and $>$ denote the natural order on the integer values).
- Case $\mathfrak{b} = \prec\succ$ and $\mathfrak{b}' = \prec\prec$: Impossible since the above discussion would imply $\tilde{N} = [c,t]$ and $\tilde{N}' = [c,t] \smallsetminus \{t\}$ (up to the label shift), contradicting the fact that both $\tilde{N}$ and $\tilde{N}'$ should be even (because we work with 2-posets).
- Case $\mathfrak{b} = \prec\prec$ and $\mathfrak{b}' = \succ\succ$: Let $\le_O$ denote the bounded sub-2-posets of $\le_P$ induced by $\tilde{O} := [a,t] \smallsetminus ([b,r] \cup [c,s])$. Since $\le_N$ is the sub-2-poset of $\le_P$ induced by $\tilde{N} := [c,s]$ and $\le_{M'}$ is the sub-2-poset of $\le_P$ induced by $\tilde{N}' := [b,r]$, we obtain that $\le_M = \le_{M'} \succ\succ \le_O$ and $\le_{N'} = \le_O \prec\prec \le_N$.
- All other cases are symmetric.

An alternative argument would be to show that for any bounded 2-posets $\le_M$ and $\le_N$, the 2-posets $\le_M \prec\succ \le_N$ and $\le_M \succ\prec \le_N$ have a unique cut and to apply Lemmas 5.50 and 5.51 to prove that the only remaining option is that $\mathfrak{b} = \prec\prec$ and $\mathfrak{b}' = \succ\succ$. ☐

Remark 5.56. Lemma 5.55 fails for multiposets, as we used a parity argument on 2-posets.

Finally, the next statement is similar to Proposition 5.22.

Proposition 5.57. *For any syntax trees* $\mathfrak{t}, \mathfrak{t}' \in \mathbf{Trees}(\mathfrak{B}_2)$ *of arity* p *and* p' *respectively, and any bounded uncuttable* 2*-posets* $\le_{M_1}, \dots, \le_{M_p}, \le_{M'_1}, \dots, \le_{M'_{p'}}$*, if*

$$\mathrm{TEval}^{\parallel}(\mathfrak{t}; \le_{M_1}, \dots, \le_{M_p}) = \mathrm{TEval}^{\parallel}(\mathfrak{t}'; \le_{M'_1}, \dots, \le_{M'_{p'}}),$$

then $p = p'$*,* $\mathfrak{t} = \mathfrak{t}'$ *modulo rewritings using Equation* (9) *and* $\le_{M_i} = \le_{M'_i}$ *for all* $i \in [p]$*.*

Proof. Immediate by induction from Lemma 5.55. ☐

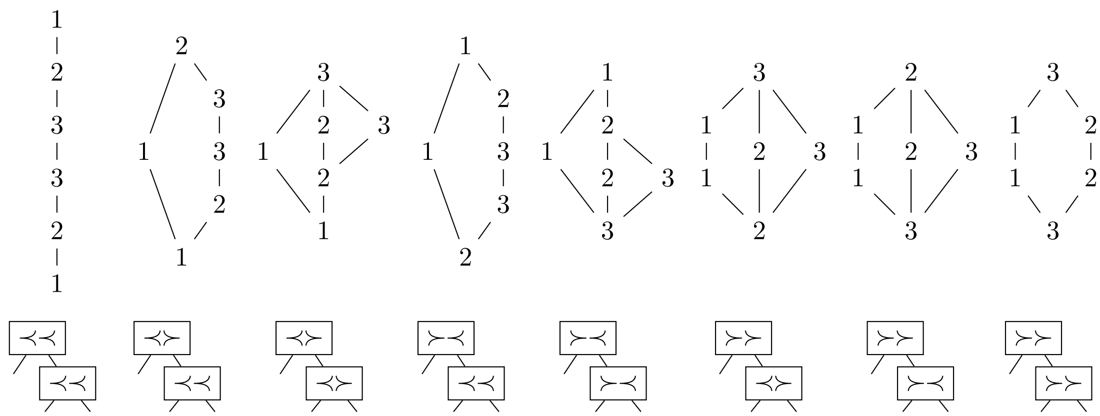
<u>Parallel poset evaluations of parallel 2-citelangis normal forms</u>. As the operators of Definition 5.41 on bounded 2-posets do not verify all the messy parallel 2-citelangis relations, the parallel poset evaluations of all syntax trees result to two many different 2-posets. To obtain an alternative combinatorial model for the messy parallel 2-citelangis operad, we thus need to restrict our attention to parallel poset evaluations of normal forms of the messy parallel 2-citelangis rewriting system described in Section 4.3.2.

Definition 5.58. *A* parallel normal 2*-poset is a* 2*-poset that can be obtained as the parallel poset evaluation of a normal form of the messy parallel* 2*-citelangis rewriting system.*

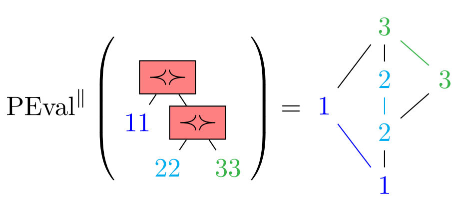
In particular, by Remark 5.45 and Proposition 5.53, a parallel normal 2-poset is a fully bounded cuttable lattice. Besides these conditions, there seems to be no simple characterization of the parallel normal 2-posets.

There is however a simple algorithm to decide whether a given fully bounded cuttable 2-poset $\leq_P$ is parallel normal. Namely, compute its minimal linear extension σ, then the normal form $\mathfrak{t}$ of the $\prec\!\succ$-tidy parallel citelangis rewriting system such that $\sigma = \mathrm{TEval}^{\parallel}(\mathfrak{t})$ by Remark 5.23, and check whether $\leq_P = \mathrm{PEval}^{\parallel}(\mathfrak{t})$.

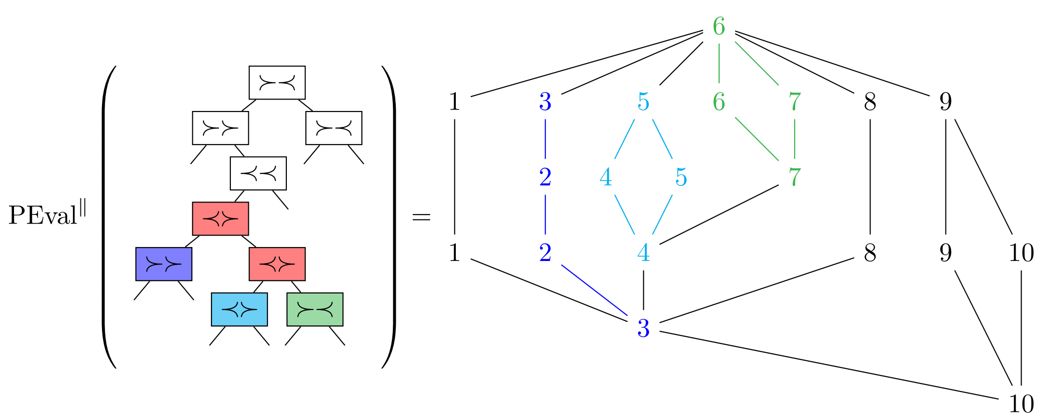
Finally, one can also consider the list of all parallel poset evaluations of the 8 quadratic parallel 2-signaletic right combs distinct from that of Equation (9). These posets are given by



were we have represented below each 2-poset the parallel 2-signaletic comb from which it was evaluated. A fully bounded cuttable 2-poset is parallel normal if it does not contain any of these forbidden 2-posets as patterns. The meaning of patterns in bounded 2-posets should be clear with the poset compositions defined in Definition 5.59. Here, we prefer to illustrate intuitively the notion on an example:



is a pattern in



5.1.5. *Parallel poset operad.* As observed above, the operators $\mathfrak{B}_2$ on bounded 2-posets defined in Definition 5.41 do not verify all messy parallel 2-citelangis relations. However, they satisfy a subset of these relations which in turn defines an operad on posets. This operad can be seen as a suboperad of an operad on all bounded 2-posets defined by the following composition rules, illustrated in Figure 13.

Definition 5.59. *Let $\le_M$ and $\le_N$ be two bounded 2-posets of degrees m and n respectively, and let $i \in [m]$. Let i_1 and i_2 denote the two copies of i in M such that $i_1 <_M i_2$. We define the composition $\le_P := \le_M \circ_i \le_N$ by:*

- $P := 1^{\{2\}} \dots (m+n-1)^{\{2\}}$,
- *$\le_P$ is obtained by inserting $\le_N$ in $\le_M$ by placing $\min(\le_N)$ at i_1 and $\max(\le_N)$ at i_2, and performing the appropriate shift. More precisely, using the notations of Definitions 2.37 and 2.38, the comparison $x \le_P y$ holds for $x, y \in P$ if and only if one of the following statements holds:*
  - $x \in M[i,n]$, $y \in M[i,n]$ *and* $\overline{x} \le_M \overline{y}$, *or*
  - $x \in N[i-1]$, $y \in N[i-1]$ *and* $\overline{x} \le_N \overline{y}$, *or*
  - $x \in M[i,n]$, $y \in N[i-1]$ *and* $\overline{x} \le_M i_1$, *or*
  - $x \in N[i-1]$, $y \in M[i,n]$ *and* $i_2 \le_M \overline{y}$.

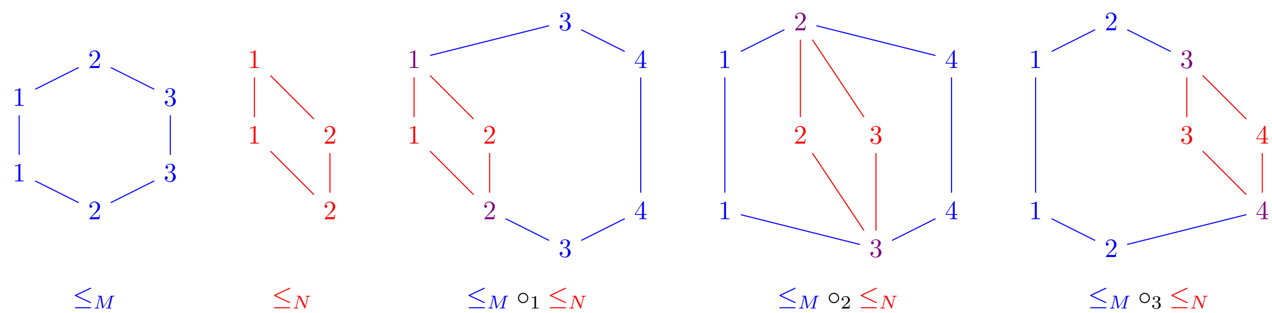


FIGURE 13. Examples of compositions in the parallel 2-poset operad. See Definition 5.59.

The following statement is left to the reader.

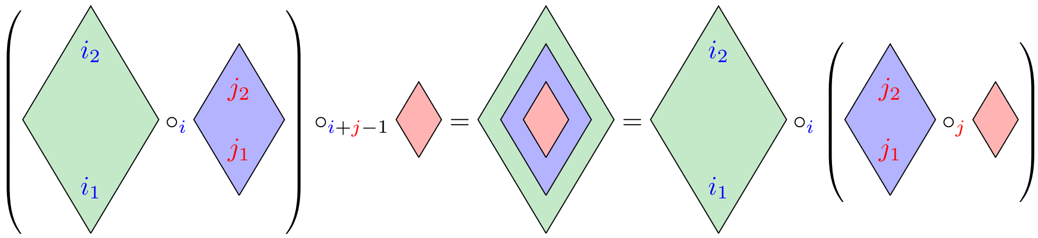
Lemma 5.60. *For any two bounded 2-posets $\le_M$ and $\le_N$ of degrees m and n respectively, and any $i \in [m]$, the composition $\le_M \circ_i \le_N$ is a bounded 2-poset of degree $m+n-1$.*

Proposition 5.61. *The compositions $\circ_i$ of Definition 5.59 define an operad structure $\mathsf{BP}_2^{\parallel}$ on bounded 2-posets.*

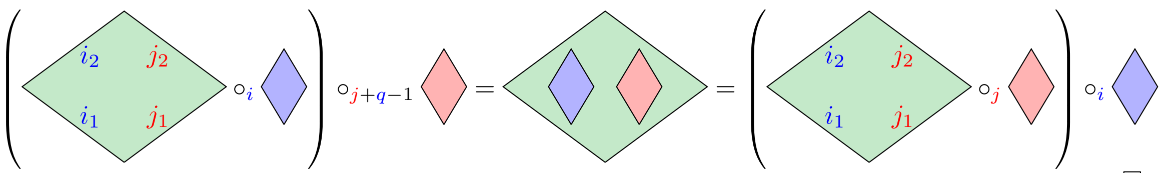
Proof. We start with the series composition (see Section 2.1). Let $\le_P$, $\le_Q$ and $\le_R$ be three bounded 2-posets of respective arity p, q and r and $i \in [p]$, $j \in [q]$. Then applying the definition, one sees that $(\le_P \circ_i \le_Q) \circ_{j+q-1} \le_R$ and $(\le_P \circ_j \le_R) \circ_i \le_Q$ are both equal to $\le_S$ defined by the following:

- Set $S := 1^{\{2\}} \dots (p+q+r-2)^{\{2\}}$, and denote by $P' := P[i, q+r-1]$, by $Q' := Q[j,r][i]$ and by $R' := R[i+j-1]$. For x in Q', we still denote $\overline{x}$ the corresponding unshifted element in Q' (which should be denoted $\overline{\overline{x}}$, according to Definition 2.38). Finally, let i_1, i_2 and j_1, j_2 denote the respective coies of i and j in P and Q.
- $x \le_S y$ holds for $x, y \in S$ if and only if one of the following statements holds:
  - $x \in T'$, $y \in T'$ and $\overline{x} \le_T \overline{y}$ for $T \in \{P, Q, R\}$, or
  - $x \in P'$, $y \in Q' \cup R'$ and $\overline{x} <_P i_1$, or
  - $x \in Q' \cup R'$, $y \in P'$ and $i_2 <_P \overline{y}$, or
  - $x \in Q'$, $y \in R'$ and $\overline{x} <_Q j_1$, or
  - $x \in R'$, $y \in Q'$ and $j_2 <_Q \overline{y}$.

This is illustrated in the following picture:



A similar computation can be done for parallel composition (see Section 2.1). We leave the details to the reader, but provide the corresponding illustration:



□

The following statements connect this operad $\mathsf{BP}_2^{\parallel}$ with the parallel poset evaluation of Definition 5.44.

Lemma 5.62. *For any operator $\mathfrak{b} \in \mathfrak{B}_2$ and any bounded 2-posets $\leq_M$ and $\leq_N$, we have*

$$\leq_M \mathfrak{b} \leq_N = \mathrm{PEval}^{\parallel}(\mathfrak{b}) \circ (\leq_M, \leq_N),$$

where the posets $\mathrm{PEval}^{\parallel}(\mathfrak{b}) := \leq_I \mathfrak{b} \leq_I$ are illustrated in Figure 10.

Proof. By Definition 5.59, the composition $\mathrm{PEval}^{\parallel}(\mathfrak{b}) \circ (\leq_M, \leq_N)$ places $\min(\leq_M)$ at 1_1, $\max(\leq_M)$ at 1_2, $\min(\leq_N)$ at 2_1, and $\max(\leq_N)$ at 2_2 in the 2-poset $\mathrm{PEval}^{\parallel}(\mathfrak{b})$. This coincides with the result of the operation $\leq_M \mathfrak{b} \leq_N$ described in Definition 5.41 and Remark 5.42. □

Lemma 5.62 enables us to interpret the operations on bounded 2-posets of the previous section as a suboperad of $\mathsf{BP}_2^{\parallel}$. We say that two syntax trees $\mathfrak{t}, \mathfrak{t}'$ on $\mathfrak{B}_2$ with the same arity are ***parallel poset equivalent*** and we write $\mathfrak{t} \simeq^{\parallel} \mathfrak{t}'$ if they have the same parallel poset evaluation.

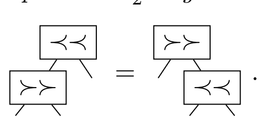
Theorem 5.63. *The parallel poset equivalence is compatible with the grafting of syntax trees: for any syntax trees $\mathfrak{t} \simeq^{\parallel} \mathfrak{t}'$ of arity p and $\mathfrak{s} \simeq^{\parallel} \mathfrak{s}'$ of arity q and $i \in [p]$, we have $\mathfrak{t} \circ_i \mathfrak{s} \simeq^{\parallel} \mathfrak{t}' \circ_i \mathfrak{s}'$. Therefore, there exists a* parallel 2-poset operad

$$\mathsf{Pos}_2^{\parallel} := \mathrm{PEval}^{\parallel}\big(\mathbf{Trees}(\mathfrak{B}_2)\big).$$

Proof. By induction on Lemma 5.62, the map $\mathrm{PEval}^{\parallel}$ coincides with the unique operad morphism from the free operad $\mathbf{Free}(\mathfrak{B}_2)$ to $\mathsf{BP}_2^{\parallel}$ that sends a generator $\mathfrak{b} \in \mathfrak{B}_2$ to $\mathrm{PEval}^{\parallel}(\mathfrak{b})$. Therefore, $\mathsf{Pos}_2^{\parallel}$ is the suboperad of $\mathsf{BP}_2^{\parallel}$ generated by $\big\{\mathrm{PEval}^{\parallel}(\mathfrak{b}) \mid \mathfrak{b} \in \mathfrak{B}_2\big\}$. □

By construction, the parallel 2-poset operad $\mathsf{Pos}_2^{\parallel}$ satisfies the relation of Equation (9). Proposition 5.57 implies that it is the only relation in $\mathsf{Pos}_2^{\parallel}$.

Theorem 5.64. *The parallel 2-poset operad $\mathsf{Pos}_2^{\parallel}$ is generated by $\mathfrak{B}_2$ with the unique relation*



In fact, the parallel 2-poset operad $\mathsf{Pos}_2^{\parallel}(\mathfrak{t})$ is actually isomorphic to the series 2-poset operad $\mathsf{Pos}_2^{\ddagger}(\mathfrak{t})$ that will be defined for arbitrary $k \geq 1$ in Section 5.2.5. We refer in particular to Section 5.2.5 for enumerative properties of $\mathsf{Pos}_2^{\parallel}(\mathfrak{t})$.

5.1.6. *Parallel 2-Zinbiel operads.* Recall that the dendriform operad is the suboperad of the nonsymmetric Zinbiel operad generated by $\prec := 12$ and $\succ := 21$, see Section 2.4.3. It is therefore natural to look for an operad $\mathsf{MZin}_2^{\parallel}$ on 2-permutations which contains the operad $\mathsf{MCit}_2^{\parallel}$ as a suboperad generated in degree 2 and which moreover closes the following left square of operad morphisms. We will also find a tidy version $\mathsf{TZin}_{\prec\succ}^{\parallel}$, closing the right square where the dashed arrows are not operad morphisms, but bijections of normal forms.

$$\begin{array}{ccccc}
\mathsf{BP}_2^{\parallel} & \xrightarrow{\text{LinExt}} & \mathsf{MZin}_2^{\parallel} & \xleftarrow{\text{LexMin}} \dashrightarrow & \mathsf{TZin}_{\prec\succ}^{\parallel} \\
\uparrow & & \uparrow & & \uparrow \\
\mathsf{Pos}_2^{\parallel} & \xrightarrow{\text{LinExt}} & \mathsf{MCit}_2^{\parallel} & \xleftarrow{\text{LexMin}} \dashrightarrow & \mathsf{TCit}_{\prec\succ}^{\parallel}
\end{array}$$

<u>Messy parallel 2-Zinbiel operad</u>. We start with the messy version.

Definition 5.65. *Let $\sigma \in \mathsf{Perm}_2(m)$ and $\tau \in \mathsf{Perm}_2(n)$ be two 2-permutations and $i \in [m]$. Write $\sigma = \lambda\, i\, \mu\, i\, \omega$ and $\tau = f\, \theta\, l$ where f and l are its first and last letters. Then the i-th composition of σ and μ is defined by*

$$\sigma \circ_i \tau = \lambda[i,n]\, f[i-1]\, (\mu[i,n] \sqcup \theta[i-1])\, l[i-1]\, \omega[i,n].$$

We extends this definition by linearity.

Here are some examples of compositions on 2-permutations:

$$\begin{aligned}
31422341 \circ_1 313122 &= 531312644562 + 531316244562 + 531316424562 + \dots + 536143415622 \\
&\quad + \cdots (126 \text{ terms}) \cdots + 536445163122 + 536445613122, \\
31422341 \circ_2 313122 &= 516424233561, \\
31422341 \circ_3 313122 &= 535341622461 + 535314622461 + 535316422461 + \dots + 513562342461 \\
&\quad + \cdots (70 \text{ terms}) \cdots + 516232534461 + 516223534461, \\
31422341 \circ_4 313122 &= 316464522351 + 316464252351 + 316464225351 + \dots + 316246425351 \\
&\quad + \cdots (35 \text{ terms}) \cdots + 316224364551 + 316223464551.
\end{aligned}$$

The following two lemmas relate the compositions of 2-permutations and the compositions of bounded 2-posets. They play the same role for the composition as Lemma 5.54 played for the operations.

Lemma 5.66. *For any two 2-permutations $\sigma \in \mathsf{Perm}_2(m)$ and $\tau \in \mathsf{Perm}_2(n)$ and $i \in [m]$,*

$$\sigma \circ_i \tau = \mathrm{LinExt}\left(\leq_\sigma \circ_i \leq_\tau\right),$$

where $\leq_\mu$ denotes the total bounded 2-poset associated with the 2-permutation μ, and the composition $\circ_i$ on the left is the composition on 2-permutations of Definition 5.65, while the composition $\circ_i$ on the right is the composition on bounded 2-posets of Definition 5.59.

Proof. This follows from Lemma 2.64 and the decomposition of the poset $\leq_\sigma \circ_i \leq_\tau$ as basic manipulations on posets. Details are left to the reader. □

Lemma 5.67. *For any two bounded 2-posets $\leq_M \in \mathsf{BP}_2^{\parallel}(m)$ and $\leq_N \in \mathsf{BP}_2^{\parallel}(n)$ and $i \in [m]$,*

$$\mathrm{LinExt}(\leq_M) \circ_i \mathrm{LinExt}(\leq_N) = \mathrm{LinExt}(\leq_M \circ_i \leq_N),$$

where the composition $\circ_i$ on the left is the composition on 2-permutations of Definition 5.65, while the composition $\circ_i$ on the right is the composition on 2-posets of Definition 5.59.

Proof. Fix three integers m, n, and $i \in [m]$. Remark that if ν is a permutation appearing in the composition $\sigma \circ_i \tau$ for some $\sigma \in \mathsf{Perm}_2(m)$ and $\tau \in \mathsf{Perm}_2(n)$, one can recover uniquely these two permutations from ν. As a consequence, $\mathrm{LinExt}(\leq_M) \circ_i \mathrm{LinExt}(\leq_N)$ cannot have multiplicities, so that we can argue by double inclusion. We conclude by Lemma 2.64. □

As a consequence, Definition 5.65 actually defines an operad composition.

Proposition 5.68. *The family* $(\mathbb{K}\mathsf{Perm}_2(n))_{n>0}$ *endowed with the composition rules of Definition 5.65 defines an operad* $\mathsf{MZin}_2^{\parallel}$*, called the* messy parallel 2-Zinbiel operad*. Moreover,* LinExt *is a surjective operad morphism from* $\mathsf{BP}_2^{\parallel}$ *to* $\mathsf{MZin}_2^{\parallel}$*.*

Proof. Thanks to Lemmas 5.66 and 5.67, we have

$$\sigma \circ_i (\tau \circ_j \mu) = \mathrm{LinExt}\left(\leq_\sigma \circ_i (\leq_\tau \circ_j \leq_\mu)\right).$$

Using similar equalities for the other compositions, we prove the operad axioms. The morphism property is just Lemma 5.66. The surjectivity follows from $\mathrm{LinExt}(\leq_\sigma) = \sigma$. □

Proposition 5.69. *The operad* $\mathsf{MCit}_2^{\parallel}$ *is the suboperad of* $\mathsf{MZin}_2^{\parallel}$ *generated by the four elements* 1221*,* $1122 + 1212$*,* $2121 + 2211$ *and* 2112*.*

Proof. The generators are given by the linear extensions of the bounded 2-posets of Figure 10:

$$\begin{array}{lcl} \mathrm{LinExt}(\leq_I \prec\prec \leq_I) = 1221, & & \mathrm{LinExt}(\leq_I \prec\succ \leq_I) = 1122 + 1212, \\ \mathrm{LinExt}(\leq_I \succ\prec \leq_I) = 2121 + 2211 & \text{and} & \mathrm{LinExt}(\leq_I \succ\succ \leq_I) = 2112. \end{array}$$ □

Tidy parallel 2-Zinbiel operad. The goal of this section is to generalize to any 2-permutations the composition formula of Proposition 5.27 for fully bounded cuttable 2-permutations, as suggested by Remark 5.28.

Definition 5.70. *Let* $\sigma \in \mathsf{Perm}_2(m)$ *and* $\tau \in \mathsf{Perm}_2(n)$ *be two 2-permutations and* $i \in [m]$*. Write* $\sigma = \lambda\, i\, \mu\, \nu\, i\, \omega$*, where* μ *is the maximal factor such that* $\mu_p < i$ *for all* $p \in [|\mu|]$*. Therefore either* ν *is empty or* $i < \nu_1$*. Write moreover* $\tau = f\,\theta\,l$ *where* f *and* l *are its first and last letters. Then the* i*-th composition of* σ *and* μ *is defined by*

$$\begin{aligned} \sigma \circ_i \tau &:= \lambda[i,n]\, f[i-1]\, \mu[i,n]\, \theta[i-1]\, \nu[i,n]\, l[i-1]\, \omega[i,n] \\ &= \lambda[i,n]\, f[i-1]\, \mu \qquad \theta[i-1]\, \nu[i,n]\, l[i-1]\, \omega[i,n]. \end{aligned}$$

Note that this is the same composition formula as in Proposition 5.27. Here are some examples of compositions on 2-permutations:

$$131|422341 \circ_1 313122 = 531312644562,$$
$$3142|2341 \circ_2 313122 = 516424233561,$$
$$31|422341 \circ_3 313122 = 513534622461,$$
$$314223|41 \circ_4 313122 = 316223464551,$$

where we have materialized by a vertical bar the separation $\mu|\nu$.

Similar to Lemma 5.36, there is a relation between messy and tidy compositions.

Lemma 5.71. *For any two 2-permutations* $\sigma \in \mathsf{Perm}_2(m)$ *and* $\tau \in \mathsf{Perm}_2(n)$ *and* $i \in [m]$*,*

$$\sigma \circ_i \tau = \mathrm{LexMin}(\sigma \circ_i \tau),$$

where the composition $\circ_i$ *on the left is the* $\prec\succ$*-tidy composition of 2-permutations of Definition 5.70, while the composition* $\circ_i$ *on the right is the messy composition of 2-permutations of Definition 5.65.*

As a consequence, the composition rules of Definition 5.70 define an operad.

Proposition 5.72. *The family* $(\mathsf{Perm}_2(n))_{n>0}$ *endowed with the composition rules of Definition 5.70 defines a non-symmetric set operad* $\mathsf{TZin}_{\prec\succ}^{\parallel}$*, called the* tidy parallel Zinbiel operad*. Moreover* $\mathsf{TCit}_{\prec\succ}^{\parallel}$ *is the suboperad of* $\mathsf{TZin}_{\prec\succ}^{\parallel}$ *given by fully bounded cuttable 2-permutations.*

5.2. **Actions of series citelangis operads.** We now discuss the action of series k-citelangis operads $\mathsf{TCit}_k^{\ddagger}$ and $\mathsf{MCit}_k^{\ddagger}$ on certain k-permutations and k-posets. Our presentation closely follows the prototype given by the actions of the parallel k-citelangis operads presented in Section 5.1. We start with some combinatorial considerations on certain k-permutations.

5.2.1. *k-rooted cuts in k-permutations.* We introduce first an intriguing class of k-permutations. These permutations will be instrumental in studying the action of the tidy series k-citelangis operad, but we also believe that they would deserve further study for their own right.

Definition 5.73. *Let $\sigma \in \mathsf{Perm}_k(n)$ be a k-permutation of degree n and $\gamma \in [n-1]$. We say that γ is a k-rooted cut of σ if we can write $\sigma = \mu\nu\omega$ where μ, ν and ω are words such that $|\mu| = k$ and $\nu_i \le \gamma < \omega_j$ for all $i \in [|\nu|]$ and $j \in [|\omega|]$. We denote by $\mathrm{rcuts}_k(\sigma)$ the set of k-rooted cuts of σ. We say that the k-permutation σ is k-rooted cuttable if it admits a k-rooted cut, and k-rooted uncuttable if it admits no k-rooted cut.*

For example, 3 is a 2-rooted cut of the 2-permutation 31213244, while the 2-permutation 31421324 is 2-rooted uncuttable. Note that there is no condition on the sizes of ν and ω in Definition 5.73 (they can be empty, or of sizes which are not multiples of k). In particular, it is convenient to observe that we could change the condition $|\mu| = k$ by $|\mu| \le k$. Indeed, if $|\mu| < k$, one can always recover the situation where $|\mu| = k$ by transferring the $k - |\mu|$ letters at the beginning of ν to the end of μ. We now observe that k-rooted cuts behave properly with the restrictions of Definition 2.42.

Lemma 5.74. *Consider $L \subseteq [n]$ and $\gamma \in [\min(L), \max(L) - 1]$, and let $\gamma^{|L} := |[\gamma] \cap L|$ denote the number of elements of L between 1 and γ. If γ is a k-rooted cut of a k-permutation $\sigma \in \mathsf{Perm}_k(n)$, then $\gamma^{|L}$ is a k-rooted cut of its restriction $\sigma^{|L}$.*

Proof. Since γ is a k-rooted cut of σ, we can write $\sigma = \mu\nu\omega$ with $|\mu| = k$ and $\nu_i \le \gamma < \omega_j$ for all $i \in [|\nu|]$ and $j \in [|\omega|]$. Then $\sigma^{|L} = \mu^{|L}\nu^{|L}\omega^{|L}$. Moreover $|\mu^{|L}| \le k$ and $\nu_i^{|L} \le \gamma^{|L} < \omega_j^{|L}$ for any $i \in [|\nu^{|L}|]$ and $j \in [|\omega^{|L}|]$. Therefore, $\gamma^{|L}$ is a k-rooted cut of $\sigma^{|L}$. □

Note that the reverse statement is wrong. Consider for instance the 2-permutation $\sigma = 213123$ and the subset $L = \{1, 2\}$. Then 1 is not a 2-rooted cut of σ, but $1^{|L} = 1$ is a 2-rooted cut of $\sigma^{|L} = 2112$.

Proposition 5.75. *The following conditions are equivalent for a k-permutation of degree n:*

(i) its restriction to any interval of $[n]$ of size at least 2 is k-rooted cuttable,
(ii) its restriction to any subset of $[n]$ of size at least 2 is k-rooted cuttable.

Proof. Assume that $\sigma \in \mathsf{Perm}_k(n)$ satisfies (i). Let $L \subseteq [n]$ with $|L| \ge 2$. Since $|L| \ge 2$, we have $\min(L) < \max(L)$ so that the restriction $\sigma^{|[\min(L),\max(L)]}$ admits a k-rooted cut γ with $\min(L) \le \gamma < \max(L)$. By Remark 2.46, the restriction $\sigma^{|L}$ is just the restriction of $\sigma^{|[\min(L),\max(L)]}$ to $\bar{L} := \{\ell - \min(L) + 1 \mid \ell \in L\}$. By Lemma 5.74, $\sigma^{|L}$ admits a k-rooted cut $\gamma^{|\bar{L}}$ with $1 \le \gamma^{|\bar{L}} \le |L|$. Therefore, σ satisfies (ii). The reverse implication is obvious. □

Definition 5.76. *A k-permutation is fully k-rooted cuttable if it satisfies the equivalent conditions of Proposition 5.75.*

For example, the 2-permutation 31213244 is 2-rooted cuttable but not fully 2-rooted cuttable (since the restriction to the interval $[1, 3]$ is not 2-rooted cuttable). In contrast, the 2-permutation 363121244556 is fully 2-rooted cuttable.

When $k = 1$, we drop 1- in 1-rooted, and just say fully rooted cuttable permutations. They coincide with the usual Catalan permutations avoiding the pattern 231. Recall that a 231 pattern in a permutation is a subword $b{\cdot}c{\cdot}a$ of three (non-necessarily consecutive) letters such that $a < b < c$. Let us start with the following simple observation.

Lemma 5.77. *A rooted uncuttable permutation of degree at least 2 contains a pattern 231.*

Proof. Let σ be a rooted uncuttable permutation and let b be its first letter. Since b is not a rooted cut of σ, there is $a < b < c$ such that c appears before a in σ. Therefore, $b \cdot c \cdot a$ is a 231 pattern in σ. □

Proposition 5.78. *A permutation is fully rooted cuttable if and only if it avoids the pattern 231.*

Proof. There is nothing to prove for the permutation 1. If a permutation σ contains a 231-pattern $b \cdot c \cdot a$, then its restriction $\sigma^{|[a,c]}$ is uncuttable, so that σ is not fully rooted cuttable. Conversely, if σ avoids the pattern 231, then all its restrictions do, so that they are all rooted cuttable by Lemma 5.77. □

Similar statements hold when $k = 2$.

Lemma 5.79. *A* 2*-rooted uncuttable* 2*-permutation of degree at least* 2 *contains a pattern* $b \cdot b' \cdot c \cdot a$ *with* $a \le b, b' \le c$.

Proof. Suppose by contradiction that a 2-permutation $\sigma = \sigma_1 \dots \sigma_{2n}$ contains no such pattern.

Assume first that $\sigma_1 = \sigma_2 =: v$. Then for any $u \le v < w$, both values u must appear before both values w, otherwise we would have the pattern $v \cdot v \cdot w \cdot u$. Therefore, both $v - 1$ and v (resp. 1, resp. $n - 1$) are 2-rooted cuts of σ if $1 < v < n$ (resp. if $v = 1$, resp. if $v = n$).

Assume now that $\sigma_1 \ne \sigma_2$, and let $r := \min(\sigma_1, \sigma_2)$ and $s := \max(\sigma_1, \sigma_2)$. Let p denote the position of the second s of σ, *i.e.* $p > 2$ and $\sigma_p = s$. We distinguish two cases:

(i) Assume first that no value of σ appears both before and after the position p. This implies that for $w > s$, both values w appear after the position p, as otherwise we would have a forbidden pattern $s \cdot w \cdot w \cdot s$. Let v be the minimal value that appears after the position p. Note that $v > 1$ as otherwise $\{r, s\} \cdot s \cdot 1$ is a forbidden pattern. We claim that $v - 1$ is a 2-rooted cut of σ. Indeed, for any $u < v$, both values u appear before the position p by definition. Moreover, for any $w \ge v$ distinct from s, both values w appear after the position p, as otherwise we would have $v \le w < s$ and thus the forbidden pattern $s \cdot w \cdot s \cdot v$.

(ii) Assume now that there is a value t that appears both before and after the position p. Note that it imposes that $s < t$ as otherwise we would have the forbidden pattern $s \cdot t \cdot s \cdot t$. Let q denote the position of the first value t. We can assume without loss of generality that q is the minimal position of a value that appears both before and after p. Let v be the minimal value that appears after the position q. Note that $v > 1$ as otherwise $\{r, s\} \cdot t \cdot 1$ is a forbidden pattern. We claim that $v - 1$ is a 2-rooted cut of σ. Indeed, for any $u < v$, both values u appear before the position q by definition. Moreover, for any $w \ge v$ distinct from s, both values w appear after the position q. Otherwise, by minimality of the position q, the second value w could not be on the right of p. Therefore, either $w > s$ and we have the forbidden pattern $s \cdot w \cdot w \cdot s$, or $v \le w < s < t$ and we have the forbidden pattern $s \cdot w \cdot t \cdot v$. □

Proposition 5.80. *A* 2*-permutation is fully* 2*-rooted cuttable if and only if it avoids the pattern* $b \cdot b' \cdot c \cdot a$ *with* $a \le b, b' \le c$.

Proof. There is nothing to prove for the 2-permutation 11. If a 2-permutation σ contains a pattern $b \cdot b' \cdot c \cdot a$ with $a \le b, b' \le c$, then its restriction $\sigma^{|[a,c]}$ is 2-rooted uncuttable, so that σ is not fully 2-rooted cuttable. Conversely, if σ avoids the pattern $b \cdot b' \cdot c \cdot a$ with $a \le b, b' \le c$, then all its restrictions do, so that they are all 2-rooted cuttable by Lemma 5.79. □

In contrast, for $k > 2$, there is no clear characterization of fully k-rooted cuttable k-permutations in terms of pattern avoidance. The following statement provides a necessary and a sufficient pattern avoiding conditions, although these two conditions do not match. The necessary condition (1) was considered for permutations in [Pil18].

Proposition 5.81. *Let* σ *be a* k*-permutation.*

(1) If σ *if fully* k*-rooted cuttable, it contains no pattern* $b_1 \cdots b_k \cdot c \cdot a$ *with* $a \le b_i \le c$ *for* $i \in [k]$*,*
(2) If σ *is not fully* k*-rooted cuttable, it contains a pattern* $a_1 \cdots a_k \cdot b \cdot a$ *with* $a < b$ *and* $a_i \le b$ *for* $i \in [k]$*, and a pattern* $b_1 \cdots b_k \cdot b \cdot a$ *with* $a < b$ *and* $a \le b_i$ *for* $i \in [k]$*.*

Proof. For (1), assume by contradiction that σ contains a pattern $b_1 \cdots b_k \cdot c \cdot a$ with $a \le b_i \le c$ for $i \in [k]$. Then in its restriction $\sigma^{|[a,c]}$ to the interval $[a, c]$, the letters a and c appear after the first k letters. It follows that $\sigma^{|[a,c]}$ is k-rooted uncuttable, so that σ is not fully k-rooted cuttable.

For (2), assume that σ is not fully k-rooted cuttable. Let $\alpha < \beta$ be such that $\sigma^{|[\alpha,\beta]}$ is not cuttable. Denote by $a_1, \dots, a_k$ the first k letters of σ in $[\alpha, \beta]$ and by b the first occurrence of β

in σ. Since β is not a k-rooted cut of $\sigma^{|[\alpha,\beta]}$, there is a letter a after b in σ with $a < b$. We have thus found a pattern $a_1 \cdots a_k \cdot b \cdot a$ with $a < b$ and $a_i \le b$ for $i \in [k]$. The proof for the other pattern is symmetric considering the last occurrence of α in σ. □

Remark 5.82. In Proposition 5.81, observe that

- the necessary condition (1) is not sufficient: for instance, the 3-permutation 113213223 is 3-rooted uncuttable while it contains no pattern $b_1 \cdot b_2 \cdot b_3 \cdot c \cdot a$ with $a \le b_i \le c$ for $i \in [3]$.
- the sufficient condition (2) is not necessary: for instance, the permutation 1432 is fully 1-rooted cuttable (it avoids 231) but contains a pattern $a_1 \cdot b \cdot a$ with $a < b$ and $a_1 \le b$ (consider 143) and a pattern $b_1 \cdot b \cdot a$ with $a < b$ and $a \le b_1$ (consider 432).

We derive in particular the following observation from Proposition 5.81 (1).

Corollary 5.83. *If $\sigma \in \mathsf{Perm}_k(n)$ is fully k-rooted cuttable and $i \in [n]$, the suffix of σ located after the last occurence of i decomposes into $\mu\nu$ where $\mu_p < i < \nu_q$ for all $p \in [|\mu|]$ and $q \in [|\nu|]$.*

Finally, we observe that fully k-rooted cuttable k-permutations form a pattern class.

Theorem 5.84. *The set of fully k-rooted cuttable k-permutations is a k-permutation class: for any fully k-rooted cuttable k-permutation $\sigma \in \mathsf{Perm}_k(n)$ and any $L \subseteq [n]$, the restriction $\sigma^{|L}$ is fully k-rooted cuttable.*

Proof. Assume that $\sigma \in \mathsf{Perm}_k(n)$ is fully k-rooted cuttable and that $L = \{\ell_1, \dots, \ell_{|L|}\} \subseteq [n]$. For any $X \subseteq [|L|]$, the restriction $(\sigma^{|L})^{|X}$ coincides with the restriction $\sigma^{|\{\ell_x \mid x \in X\}}$ by Remark 2.57, and is thus k-rooted cuttable. Therefore, $\sigma^{|L}$ is fully k-rooted cuttable. □

5.2.2. *Action of $\mathsf{TCit}_k^{\ddagger}$ on words and permutations.* We show in this section that FQSym_k can be endowed with a tidy series k-citelangis structure, and that the resulting tidy series k-citelangis algebra is free. Therefore, the free tidy series k-citelangis subalgebra generated by the k-permutation $1^{\{k\}}$ provides a combinatorial model for the basis for the tidy series k-citelangis operad.

Action on words. We first show that the free algebra $\mathcal{A}^{\ge k}$ can be endowed with a structure of tidy series k-citelangis algebra.

Definition 5.85. *For a tidy series k-citelangis operator $\mathfrak{b} \in \{\prec, \succ\}^*$ and two words X and Y such that $|\mathfrak{b}| \le \min(|X|, |Y|)$, we define inductively*

$$X \;\mathfrak{b}\; Y = \begin{cases} XY & \text{if } \mathfrak{b} = \varepsilon, \\ x(\underline{X} \;\underline{\mathfrak{b}}\; Y) & \text{if } \mathfrak{b} = \prec\underline{\mathfrak{b}} \text{ and } X = x\underline{X}, \\ y(X \;\underline{\mathfrak{b}}\; \underline{Y}) & \text{if } \mathfrak{b} = \succ\underline{\mathfrak{b}} \text{ and } Y = y\underline{Y}. \end{cases}$$

In other words, we choose the first letters of the result among the first letters of X or Y depending on the operation $\mathfrak{b}$, and we then concatenate the remaining suffixes of X and Y.

For example, when $k = 1$, the two operators are given for the words xX and yY by

$$xX \prec yY = xXyY \qquad \text{and} \qquad xX \succ yY = yxXY.$$

In particular, the concatenation is given by $\cdot = \prec$. When $k = 2$, the four operators are given for the words x_1x_2X and y_1y_2Y by

$$\begin{aligned} x_1x_2X \prec\prec y_1y_2Y &= x_1x_2Xy_1y_2Y, \\ x_1x_2X \prec\succ y_1y_2Y &= x_1y_1x_2Xy_2Y, \\ x_1x_2X \succ\prec y_1y_2Y &= y_1x_1x_2Xy_2Y, \\ x_1x_2X \succ\succ y_1y_2Y &= y_1y_2x_1x_2XY. \end{aligned}$$

Again, the concatenation is given by $\cdot = \prec\prec$.

Proposition 5.86. *The free algebra $\mathcal{A}^{\ge k}$, endowed with the operators of Definition 5.85, defines a tidy series k-citelangis algebra. The concatenation product $\cdot$ of $\mathcal{A}^{\ge k}$ is given by $\prec^k$.*

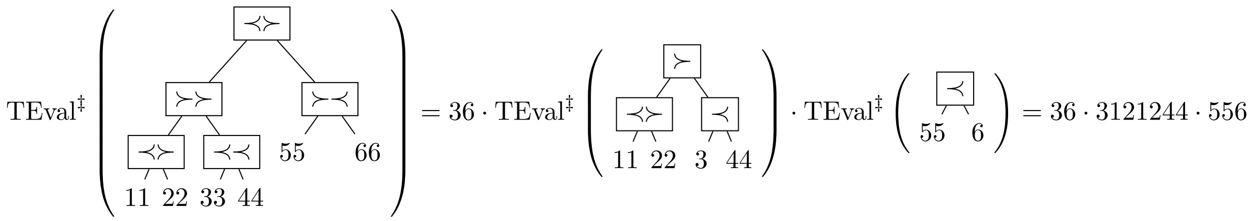


FIGURE 14. Illustration of Remark 5.90.

Proof. Consider a destination vector $\mathsf{p} \in [3]^k$, and the corresponding tidy series k-citelangis operators $\mathfrak{a}_\mathsf{p}, \mathfrak{b}_\mathsf{p}, \mathfrak{c}_\mathsf{p}, \mathfrak{d}_\mathsf{p} \in \{\prec, \succ\}^k$ defined in Proposition 4.20. Then for any words X, Y, Z, the words that appear in

$$X \, \mathfrak{a}_\mathsf{p} \, (Y \, \mathfrak{b}_\mathsf{p} \, Z) \qquad \text{and} \qquad (X \, \mathfrak{d}_\mathsf{p} \, Y) \, \mathfrak{c}_\mathsf{p} \, Z$$

is the word in $X \sqcup Y \sqcup Z$ such that

- for $i \in [k]$, its i-th letter was taken from the word X if $\mathsf{p}_i = 1$, from the word Y if $\mathsf{p}_i = 2$ and from the word Z if $\mathsf{p}_i = 3$, and
- its remaining letters are taken first from X, then from Y and finally from Z.

Therefore our operators on words indeed satisfy the tidy series k-citelangis relation ($\mathsf{TCit}^\ddagger\ \mathsf{p}$). □

Action on permutations. Replacing the concatenation by the shifted concatenation, one endows similarly the algebra FQSym_k of k-permutations with a structure of tidy series k-citelangis algebra. This can be rephrased as follows.

Definition 5.87. *For any operator $\mathfrak{b} \in \mathfrak{B}_k$ and any two k-permutations μ and ν of degree m and n respectively, $\mu \, \mathfrak{b} \, \nu$ is the k-permutation $\pi \in \mu \,\bar{\sqcup\!\sqcup}\, \nu$ such that*

- *for all $i \in [k]$, we have $\pi_i \le m$ if $\mathfrak{b}_i = \prec$ while $\pi_i > m$ if $\mathfrak{b}_i = \succ$,*
- *the remaining entries of π are the concatenation of the remaining entries of μ and the remaining entries of $\nu[m]$.*

For example, for $\mu = 321312132$ and $\nu = 221211$ in FQSym_3 we have $\mu \prec\succ\succ \nu = 3552131212324544$ and $\mu \succ\prec\prec \nu = 5321312132554544$. The following statement is immediate from Proposition 5.86.

Proposition 5.88. *The algebra $(\mathsf{FQSym}_k, \bar{\cdot})$, endowed with the operators of Definition 5.87, defines a tidy series k-citelangis algebra. The concatenation product $\bar{\cdot}$ of FQSym_k is given by $\prec^k$.*

The goal of this section is to show that the tidy series k-citelangis algebra FQSym_k is free. To manipulate this tidy series k-citelangis algebra, we consider the evaluations of syntax trees of $\mathbf{Trees}(\mathfrak{B}_k)$ in FQSym_k. See Figure 14 for an illustration.

Definition 5.89. *Denote by $\mathrm{TEval}^\ddagger(\mathfrak{t}; \sigma_1, \dots, \sigma_p)$ the evaluation of a syntax tree $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_k)$ of arity p on p k-permutations $\sigma_1, \dots, \sigma_p$ of Perm_k using the tidy series k-citelangis structure of FQSym_k. The* tidy series permutation evaluation *of $\mathfrak{t}$ is then $\mathrm{TEval}^\ddagger(\mathfrak{t}) := \mathrm{TEval}^\ddagger(\mathfrak{t}; 1^{\{k\}}, \dots, 1^{\{k\}})$. We extend by linearity $\mathfrak{t}$ to the elements of $\mathbf{Free}(\mathfrak{B}_k)$ on the one hand, and $\sigma_1, \dots, \sigma_p$ to the elements of FQSym_k on the other hand.*

Remark 5.90. Let us rephrase algorithmically Definition 5.89. For this, we consider partial syntax trees, *i.e.* trees whose nodes are labeled by operators with at most k letters $\{\prec, \succ\}$ and whose leaves are labeled by words. The evaluation $\mathrm{TEval}^\ddagger(\mathfrak{s})$ of such a partial syntax tree $\mathfrak{s}$ is defined inductively as follows. If $\mathfrak{s}$ is a leaf labeled by a word w, then $\mathrm{TEval}^\ddagger(\mathfrak{s}) = \mathsf{w}$. Otherwise, if $\mathfrak{s}$ has a root with ℓ signals, $\mathrm{TEval}^\ddagger(\mathfrak{s})$ is obtained as follows. Let ℓ cars traverse the partial syntax tree $\mathfrak{s}$ in series. The j-th car follows and erases the first letter of each signal it traverses, and finally arrives at a leaf where it reads and erases the first letter w_j. Let $\mathsf{w} = w_1 \dots w_\ell$ denote the word formed by the letters read by the ℓ cars at their destination. At this stage, all letters of the signal at the root have been erased. We are left with the two partial syntax trees $\mathfrak{l}$ and $\mathfrak{r}$

(where some letters of the signals in the nodes and of the words in the leaves have been erased by the ℓ cars). The evaluation $\mathrm{TEval}^{\ddagger}(\mathfrak{s})$ is then obtained inductively by

$$\mathrm{TEval}^{\ddagger}(\mathfrak{s}) = \mathsf{w} \cdot \mathrm{TEval}^{\ddagger}(\mathfrak{l}) \cdot \mathrm{TEval}^{\ddagger}(\mathfrak{r}).$$

Finally, for a syntax tree $\mathfrak{t}$ of arity p and k-permutations $\sigma_1 \in \mathsf{Perm}_k(n_1), \dots, \sigma_p \in \mathsf{Perm}_k(n_p)$, the evaluation $\mathrm{TEval}^{\ddagger}(\mathfrak{t}; \sigma_1, \dots, \sigma_p)$ is the evaluation of the partial syntax tree $\mathfrak{s}$ obtained from $\mathfrak{t}$ by putting the permutation $\sigma_i[n_1 + \cdots + n_{i-1}]$ at the i-th leaf for all $i \in [p]$. In particular, for $\mathrm{TEval}^{\ddagger}(\mathfrak{t}) = \mathrm{TEval}^{\ddagger}(\mathfrak{t}; 1^{\{k\}}, \dots, 1^{\{k\}})$, the i-th leaf is labeled by the word $i^{\{k\}}$. See Figure 14.

As FQSym_k is a tidy series k-citelangis algebra, the tidy series permutation evaluation is preserved by the tidy series k-citelangis relations of Proposition 4.20. Thus, $\mathrm{TEval}^{\ddagger}(\mathfrak{s}; F_1, \dots, F_p) = \mathrm{TEval}^{\ddagger}(\mathfrak{t}; F_1, \dots, F_p)$ for any $\mathfrak{s}, \mathfrak{t} \in \mathbf{Free}(\mathfrak{B}_k)(p)$ which are equivalent modulo the tidy series k-citelangis relations, and for any $F_1, \dots, F_p \in \mathsf{FQSym}_k$. The objective of this section is to show the reciprocal statement. The proof is based on k-rooted cuts in k-permutations introduced in Definition 5.73.

<u>Tidy series permutation evaluations and k-rooted cuts</u>. Our next two lemmas state that the k-rooted cuts of a k-permutation ρ precisely correspond to its decompositions of the form $\rho = \sigma \,\mathfrak{b}\, \tau$. Their proofs immediately follow from Definition 5.87 and are thus left to the reader.

Lemma 5.91. *For any k-permutations $\sigma \in \mathsf{Perm}_k(m)$ and $\tau \in \mathsf{Perm}_k(n)$, and any operator $\mathfrak{b} \in \mathfrak{B}_k$, the degree m of σ is a k-rooted cut of $\sigma \,\mathfrak{b}\, \tau$.*

Lemma 5.92. *For any k-permutation $\rho \in \mathsf{Perm}_k(\ell)$ and any k-rooted cut $\gamma \in \mathrm{rcuts}_k(\rho)$, there is a unique $\mathfrak{b} \in \mathfrak{B}_k$ (defined by $\mathfrak{b}_i := \prec$ if $\rho_i \le \gamma$ and $\mathfrak{b}_i := \succ$ if $\rho_i > \gamma$) such that $\rho = \rho^{|[\gamma]} \,\mathfrak{b}\, \rho^{|[\ell] \smallsetminus [\gamma]}$.*

Remark 5.93. Lemma 5.92 gives an inductive algorithm to compute all decompositions of a given k-permutation ρ as an evaluation of the form $\rho = \mathrm{TEval}^{\ddagger}(\mathfrak{t}; \sigma_1, \dots, \sigma_p)$. Namely, ρ admits

- the trivial evaluation $\rho = \mathrm{TEval}^{\ddagger}(\mathbb{1}; \rho)$, where $\mathbb{1}$ is the unit syntax tree with no node and a single leaf, and
- the evaluation $\rho = \mathrm{TEval}^{\ddagger}(\mathfrak{t}; \sigma_1, \dots, \sigma_l, \tau_1, \dots, \tau_r)$, for any k-rooted cut $\gamma \in \mathrm{rcuts}_k(\rho)$ and any evaluations $\rho^{|[\gamma]} = \mathrm{TEval}^{\ddagger}(\mathfrak{l}; \sigma_1, \dots, \sigma_l)$ and $\rho^{|[\ell] \smallsetminus [\gamma]} = \mathrm{TEval}^{\ddagger}(\mathfrak{r}; \tau_1, \dots, \tau_r)$, where $\mathfrak{t}$ is the syntax tree with root $\mathfrak{b} \in \mathfrak{B}_k$ defined by Lemma 5.92 and with subtrees $\mathfrak{l}$ and $\mathfrak{r}$.

This algorithm implies the existence of decompositions of the form $\rho = \mathrm{TEval}^{\ddagger}(\mathfrak{t}; \sigma_1, \dots, \sigma_p)$ where $\sigma_1, \dots, \sigma_p$ are k-rooted uncuttable. In fact, we can even impose the position of the first k-rooted cut.

Corollary 5.94. *For any k-permutation $\rho \in \mathsf{Perm}_k$ and any k-rooted cut $\gamma \in \mathrm{rcuts}_k(\rho)$, there exists a syntax tree $\mathfrak{t}$ of arity p with left subtree of arity l and k-rooted uncuttable k-permutations $\sigma_1 \in \mathsf{Perm}_k(n_1), \dots, \sigma_p \in \mathsf{Perm}_k(n_p)$ such that $\rho = \mathrm{TEval}^{\ddagger}(\mathfrak{t}; \sigma_1, \dots, \sigma_p)$ and $\gamma = n_1 + \cdots + n_l$.*

We now characterize the k-permutations ρ that admit a decomposition of the form $\rho = \mathrm{TEval}^{\ddagger}(\mathfrak{t})$.

Proposition 5.95. *The tidy series permutation evaluations of the syntax trees of $\mathbf{Trees}(\mathfrak{B}_k)$ are precisely the fully k-rooted cuttable k-permutations.*

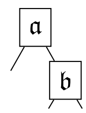
Proof. Consider first a k-permutation $\rho = \mathrm{TEval}^{\ddagger}(\mathfrak{t})$ with $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_k)(\ell)$. We prove by induction on ℓ that ρ is fully k-rooted cuttable. If $\ell = 1$, there is nothing to prove. Assume that $\ell \ge 2$ and let $1 \le a < b \le \ell$. Let $\mathfrak{l}$ and $\mathfrak{r}$ denote the left and right subtrees of $\mathfrak{t}$, and let γ be the arity of $\mathfrak{l}$, so that $\rho^{|[\gamma]} = \mathrm{TEval}^{\ddagger}(\mathfrak{l})$ and $\rho^{|[\ell] \smallsetminus [\gamma]} = \mathrm{TEval}^{\ddagger}(\mathfrak{r})$. We distinguish three cases:

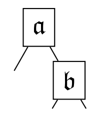
- Assume that $b \le \gamma$. Since $\rho^{|[\ell]} = \mathrm{TEval}^{\ddagger}(\mathfrak{l})$ is fully k-rooted cuttable by induction hypothesis and k-rooted cuts are preserved by restriction by Lemma 5.74, we obtain that $\rho^{|[a,b]} = (\rho^{|[\gamma]})^{|[a,b]}$ is k-rooted cuttable.
- Assume that $\gamma \le a$. The argument is similar since $\rho^{|[a,b]} = (\rho^{|[\ell] \smallsetminus [\gamma]})^{|[a-\gamma, b-\gamma]}$.
- Assume finally that $a < \gamma < b$. By Lemma 5.91, γ is a k-rooted cut of ρ. Therefore, $\gamma - a$ is a k-rooted cut of $\rho^{|[a,b]}$ by Lemma 5.74.

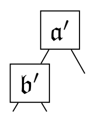
Conversely, consider now a fully k-rooted cuttable k-permutation $\rho \in \mathsf{Perm}_k(\ell)$. Similarly to Remark 5.93, we prove by induction on ℓ that ρ is the tidy series permutation evaluation of a syntax tree. If $\ell = 1$, then $\rho = \mathrm{TEval}^{\ddagger}(\mathbb{1})$. If $\ell \geq 2$, then ρ admits at least one k-rooted cut γ by assumption. Moreover, $\rho^{|[\gamma]}$ and $\rho^{|[\ell]\smallsetminus[\gamma]}$ are both fully k-rooted cuttable by Theorem 5.84. By induction, we obtain that $\rho^{|[\gamma]} = \mathrm{TEval}^{\ddagger}(\mathfrak{l})$ and $\rho^{|[\ell]\smallsetminus[\gamma]} = \mathrm{TEval}^{\ddagger}(\mathfrak{r})$. Then $\rho = \mathrm{TEval}^{\ddagger}(\mathfrak{t})$, where $\mathfrak{t}$ is the syntax tree with root $\mathfrak{b} \in \mathfrak{B}_k$ defined by Lemma 5.92 and with subtrees $\mathfrak{l}$ and $\mathfrak{r}$. □

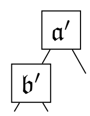
<u>Freeness</u>. Our objective is now to prove that the decompositions of Corollary 5.94 for a given k-permutation ρ are all equivalent up to the tidy series k-citelangis relations of Proposition 4.20. Our first step is to understand the evalutations of a quadratic syntax tree on three permutations. We start from a simple observation, which again immediately follows from Definition 5.87. Recall that a syntax tree is tidy series when all the traffic signals not contained in its series routes point to the left.

Lemma 5.96. *For any k-permutations $\rho \in \mathsf{Perm}_k(\ell)$, $\sigma \in \mathsf{Perm}_k(m)$ and $\tau \in \mathsf{Perm}_k(n)$, and any operators $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}', \mathfrak{b}' \in \mathfrak{B}_k$, we have*

- *$\ell + m$ is a k-rooted cut of* $\mathrm{TEval}^{\ddagger}\Big($ 

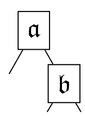
$;\rho,\sigma,\tau\Big)$ *if and only if* 

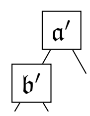
is tidy series,
- *ℓ is a k-rooted cut of* $\mathrm{TEval}^{\ddagger}\Big($ 

$;\rho,\sigma,\tau\Big)$ *if and only if* 

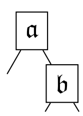
is tidy series.

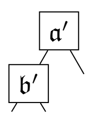
Lemma 5.97. *For any k-permutations $\rho, \sigma, \tau \in \mathsf{Perm}_k$, and any operators $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}', \mathfrak{b}' \in \mathfrak{B}_k$, if*

$\mathrm{TEval}^{\ddagger}\Big($ 

$;\rho,\sigma,\tau\Big) = \mathrm{TEval}^{\ddagger}\Big($ 

$;\rho,\sigma,\tau\Big),$

then 

$=$ 

is a tidy series k-citelangis relation.

Proof. Let π denote the k-permutation obtained by these evaluations, and let ℓ, m and n denote the degrees of ρ, σ and τ respectively. By Lemma 5.91, we obtain that ℓ and $\ell + m$ are k-rooted cuts of π. By Lemma 5.96, we therefore derive that the two syntax trees are tidy series. Moreover, the series destination vector of the two syntax trees are both given by the first k letters of π where we replace all letters between 1 and ℓ by 1, all letters between $\ell + 1$ and $\ell + m$ by 2, and all letters between $\ell + m + 1$ and $\ell + m + n$ by 3. Since they are tidy series and have the same series destination vector, the two syntax trees form a tidy series k-citelangis relation. □

We now prove that, up to the tidy series k-citelangis relations, any k-permutation can be obtained in a unique way as the evaluation of a syntax tree on k-rooted uncuttable k-permutations.

Proposition 5.98. *For any syntax trees $\mathfrak{t}, \mathfrak{t}' \in \mathbf{Trees}(\mathfrak{B}_k)$ of arity p and p' respectively, and any k-rooted uncuttable k-permutations $\sigma_1, \ldots, \sigma_p$, $\sigma'_1, \ldots, \sigma'_{p'} \in \mathsf{Perm}_k$, if*

$$\mathrm{TEval}^{\ddagger}(\mathfrak{t}; \sigma_1, \ldots, \sigma_p) = \mathrm{TEval}^{\ddagger}(\mathfrak{t}'; \sigma'_1, \ldots, \sigma'_{p'}),$$

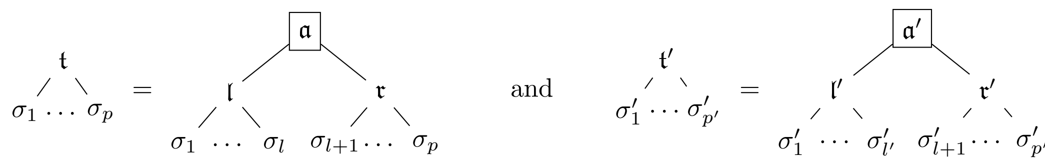
then $p = p'$, $\mathfrak{t} = \mathfrak{t}'$ modulo the tidy series k-citelangis relations and $\sigma_i = \sigma'_i$ for all $i \in [p]$.

Proof. Let $\pi = \mathrm{TEval}^{\ddagger}(\mathfrak{t}; \sigma_1, \ldots, \sigma_p) = \mathrm{TEval}^{\ddagger}(\mathfrak{t}'; \sigma'_1, \ldots, \sigma'_{p'})$ and let n denote its degree.

We prove the result by induction on p. If $p = 1$, then π is k-rooted uncuttable, so that $p' = 1$, $\mathfrak{t} = \mathfrak{t}'$ is the only syntax tree of arity 1, and $\sigma_1 = \sigma'_1 = \pi$.

Assume now that $p > 1$. Let $\mathfrak{a}$ be the root of $\mathfrak{t}$, let $\mathfrak{l}$ and $\mathfrak{r}$ be its left and right subtrees, let l be the arity of $\mathfrak{l}$ and let γ be the corresponding k-rooted cut of π. We thus have $\mathrm{TEval}^{\ddagger}(\mathfrak{l}; \sigma_1, \ldots, \sigma_l) = \pi^{|[\gamma]}$

and $\mathrm{TEval}^{\ddagger}(\mathfrak{r}; \sigma_{l+1}, \dots, \sigma_p) = \pi^{|[n] \smallsetminus [\gamma]}$. Define $\mathfrak{a}', \mathfrak{l}', \mathfrak{r}', l'$ and γ' similarly for $\mathfrak{t}'$. These notations are illustrated below:

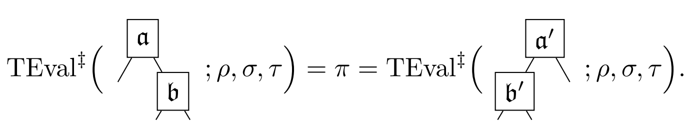


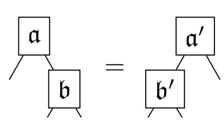
Assume first that $\gamma = \gamma'$. Then $\mathfrak{a} = \mathfrak{a}'$, $\mathrm{TEval}^{\ddagger}(\mathfrak{l}; \sigma_1, \dots, \sigma_l) = \pi^{|[\gamma]} = \mathrm{TEval}^{\ddagger}(\mathfrak{l}'; \sigma'_1, \dots, \sigma'_{l'})$ and $\mathrm{TEval}^{\ddagger}(\mathfrak{r}; \sigma_{l+1}, \dots, \sigma_p) = \pi^{|[n] \smallsetminus [\gamma]} = \mathrm{TEval}^{\ddagger}(\mathfrak{r}'; \sigma'_{l'+1}, \dots, \sigma'_{p'})$ by Lemma 5.92. By induction hypothesis, the first equality ensures that $l = l'$, that $\mathfrak{l} = \mathfrak{l}'$ modulo the tidy series k-citelangis relations and that $\sigma_i = \sigma'_i$ for all $i \in [l]$, while the second equality ensures that $p - l = p' - l'$, that $\mathfrak{r} = \mathfrak{r}'$ modulo the tidy series k-citelangis relations and that $\sigma_i = \sigma'_i$ for all $i \in [p] \smallsetminus [l]$. We thus conclude that $p = p'$, that $\mathfrak{t} = \mathfrak{t}'$ modulo the tidy series k-citelangis relations, and that $\sigma_i = \sigma'_i$ for all $i \in [p]$.

Assume now without loss of generality that $\gamma < \gamma'$. Consider the k-permutations $\rho := \pi^{|[\gamma]}$, $\sigma := \pi^{|[\gamma'] \smallsetminus [\gamma]}$ and $\tau := \pi^{|[n] \smallsetminus [\gamma']}$. Since γ' is a k-rooted cut of π larger than γ, Lemma 5.74 ensures that $\gamma' - \gamma$ is a k-rooted cut of $\pi^{|[n] \smallsetminus [\gamma]}$. By Corollary 5.94 and induction hypothesis, there exists a syntax tree $\mathfrak{s}$ of arity $p - l$ with root $\mathfrak{b}$, left subtree $\mathfrak{u}$ of arity u and right subtree $\mathfrak{v}$, such that $\mathfrak{l}' = \mathfrak{s}'$ modulo the tidy series k-citelangis relations and $\mathrm{TEval}^{\ddagger}(\mathfrak{s}; \sigma_{l+1}, \dots, \sigma_p) = \pi^{|[n] \smallsetminus [\gamma]}$, $\mathrm{TEval}^{\ddagger}(\mathfrak{u}; \sigma_{l+1}, \dots, \sigma_{l+u}) = \sigma$ and $\mathrm{TEval}^{\ddagger}(\mathfrak{v}; \sigma_{l+u+1}, \dots, \sigma_p) = \tau$. Similarly, since γ is a k-rooted cut of π smaller than γ', Lemma 5.74 ensures that γ is a k-rooted cut of $\pi^{|[\gamma']}$. By Corollary 5.94 and induction hypothesis, there exists a syntax tree $\mathfrak{s}'$ of arity l', with root $\mathfrak{b}'$, left subtree $\mathfrak{u}'$ of arity u' and right subtree $\mathfrak{v}'$, such that $\mathfrak{l}' = \mathfrak{s}'$ modulo the tidy series k-citelangis relations and $\mathrm{TEval}^{\ddagger}(\mathfrak{s}'; \sigma'_1, \dots, \sigma'_{l'}) = \pi^{|[\gamma']}$, $\mathrm{TEval}^{\ddagger}(\mathfrak{u}'; \sigma'_1, \dots, \sigma'_{u'}) = \rho$ and $\mathrm{TEval}^{\ddagger}(\mathfrak{v}'; \sigma_{u'+1}, \dots, \sigma_{l'}) = \sigma$. Since

$$\begin{aligned} \mathrm{TEval}^{\ddagger}(\mathfrak{l}; \sigma_1, \dots, \sigma_l) &= \rho = \mathrm{TEval}^{\ddagger}(\mathfrak{u}'; \sigma'_1, \dots, \sigma'_{u'}), \\ \mathrm{TEval}^{\ddagger}(\mathfrak{u}; \sigma_{l+1}, \dots, \sigma_{l+u}) &= \sigma = \mathrm{TEval}^{\ddagger}(\mathfrak{v}'; \sigma_{u'+1}, \dots, \sigma_{l'}), \\ \text{and} \quad \mathrm{TEval}^{\ddagger}(\mathfrak{v}; \sigma_{l+u+1}, \dots, \sigma_p) &= \tau = \mathrm{TEval}^{\ddagger}(\mathfrak{r}'; \sigma'_{l'+1}, \dots, \sigma'_{p'}), \end{aligned}$$

we obtain by three applications of the induction hypothesis that $p = p'$ and $\sigma_i = \sigma'_i$ for all $i \in [p]$. Moreover, we have



By Lemma 5.97, we conclude that  is a tidy series k-citelangis relation, and thus that $\mathfrak{t} = \mathfrak{t}'$ up to tidy series k-citelangis relations. □

Remark 5.99. From Remark 5.93 and Proposition 5.98, we derive an inductive algorithm to compute the decomposition of a given k-permutation ρ as an evaluation of the form $\rho = \mathrm{TEval}^{\ddagger}(\mathfrak{t}; \sigma_1, \dots, \sigma_p)$, where $\mathfrak{t}$ is in normal form in the tidy series k-citelangis rewriting system of Section 4.3.2 and $\sigma_1, \dots, \sigma_p$ are k-rooted uncuttable k-permutations. Namely,

- if ρ is k-rooted uncuttable, then $\rho = \mathrm{TEval}^{\ddagger}(\mathbb{1}; \rho)$, where $\mathbb{1}$ is the unit syntax tree with no node and a single leaf, and
- otherwise, $\rho = \mathrm{TEval}^{\ddagger}(\mathfrak{t}; \sigma_1, \dots, \sigma_l, \tau_1, \dots, \tau_r)$, where γ be the rightmost k-rooted cut of ρ, $\rho^{|[\gamma]} = \mathrm{TEval}^{\ddagger}(\mathfrak{l}; \sigma_1, \dots, \sigma_l)$ and $\rho^{|[\ell] \smallsetminus [\gamma]} = \mathrm{TEval}^{\ddagger}(\mathfrak{r}; \tau_1, \dots, \tau_r)$ are such that $\mathfrak{l}, \mathfrak{r}$ are in normal form and $\sigma_1, \dots, \sigma_l, \tau_1, \dots, \tau_r$ are k-rooted uncuttable, and $\mathfrak{t}$ is the syntax tree with root $\mathfrak{b} \in \mathfrak{B}_k$ defined by Lemma 5.92 and with subtrees $\mathfrak{l}$ and $\mathfrak{r}$.

Proposition 5.98 proves the main result of this section.

Theorem 5.100. *The tidy series k-citelangis algebra FQSym_k is free on k-rooted uncuttable k-permutations.*

A complete combinatorial model for $\mathsf{TCit}_k^{\ddagger}$. By Theorem 5.100, the tidy series k-citelangis operad $\mathsf{TCit}_k^{\ddagger}$ can be fully understood from the tidy series k-citelangis subalgebra of FQSym_k generated by the k-permutation $1^{\{k\}}$. We close this section with a completely explicit combinatorial model for this algebra. We first obtain from Proposition 5.95 a combinatorial model for the operations of $\mathsf{TCit}_k^{\ddagger}$.

Proposition 5.101. *The tidy series permutation evaluation $\mathfrak{t} \mapsto \mathrm{TEval}^{\ddagger}(\mathfrak{t})$ is a graded bijection from the tidy series equivalence classes of syntax trees of $\mathbf{Trees}(\mathfrak{B}_k)$ to the fully k-rooted cuttable k-permutations.*

Proof. This map $\mathfrak{t} \mapsto \mathrm{TEval}^{\ddagger}(\mathfrak{t})$ is surjective on fully k-rooted cuttable k-permutations by Proposition 5.95. It is compatible with the tidy series k-citelangis relations by Proposition 5.88. Finally, it is bijective since the tidy series k-citelangis algebra FQSym_k is free on uncuttable k-permutations by Theorem 5.100. □

Therefore, the fully k-rooted cuttable k-permutations can be thought of as a basis of the tidy series k-citelangis operad $\mathsf{TCit}_k^{\ddagger}$. Through the bijection of Proposition 5.95, we can thus define the compositions of the tidy series k-citelangis operad $\mathsf{TCit}_k^{\ddagger}$ directly on fully k-rooted cuttable k-permutations.

Definition 5.102. *For any integers $i \le m$ and n, and any two fully k-rooted cuttable k-permutations $\sigma = \mathrm{TEval}^{\ddagger}(\mathfrak{s}) \in \mathsf{Perm}_k(m)$ and $\tau = \mathrm{TEval}^{\ddagger}(\mathfrak{t}) \in \mathsf{Perm}_k(n)$, we define*

$$\sigma \circ_i \tau := \mathrm{TEval}^{\ddagger}(\mathfrak{s} \circ_i \mathfrak{t}) \in \mathsf{Perm}_k(m+n-1).$$

The following statement provides a direct combinatorial description of the action of the compositions $\circ_i$ of the tidy series k-citelangis operad $\mathsf{TCit}_k^{\ddagger}$ on fully k-rooted cuttable k-permutations.

Proposition 5.103. *Let $\sigma \in \mathsf{Perm}_k(m)$ and $\tau \in \mathsf{Perm}_k(n)$ be two fully k-rooted cuttable k-permutations and $i \in [m]$. Write $\sigma = \omega_0\, i\, \omega_1\, i\, \omega_2 \cdots \omega_{k-1}\, i\, \omega_k$, where $\omega_k = \mu\, \nu$ with $\mu_p < i < \nu_q$ for all $p \in [|\mu|]$ and $q \in [|\nu|]$ according to Corollary 5.83, and write $\tau = \tau_1 \cdots \tau_k\, \theta$. Then*

$$\begin{aligned}\sigma \circ_i \tau &= \omega_0[i,n]\, \tau_1[i-1]\, \omega_1[i,n]\, \tau_2[i-1]\, \omega_2[i,n] \cdots \omega_{k-1}[i,n]\, \tau_k[i-1]\, \mu[i,n]\, \theta[i-1]\, \nu[i,n] \\ &= \omega_0[i,n]\, \tau_1[i-1]\, \omega_1[i,n]\, \tau_2[i-1]\, \omega_2[i,n] \cdots \omega_{k-1}[i,n]\, \tau_k[i-1]\, \mu \qquad \theta[i-1]\, \nu[n-1].\end{aligned}$$

Proof. Consider arbitrary syntax trees $\mathfrak{s}$ and $\mathfrak{t}$ such that $\sigma = \mathrm{TEval}^{\ddagger}(\mathfrak{s})$ and $\tau = \mathrm{TEval}^{\ddagger}(\mathfrak{t})$. By Remark 5.90, $\mathrm{TEval}^{\ddagger}(\mathfrak{s} \circ_i \mathfrak{t})$ is obtained inductively by letting k cars traverse $\mathfrak{s} \circ_i \mathfrak{t}$ in series and recording the position of their arrival. The cars that arrive at position i in $\mathfrak{s}$ thus continue their journey through $\mathfrak{t}$. Therefore, the k values i in σ are replaced by the first k values of τ, and the remaining values of τ are placed at the only possible position in $\sigma \circ_i \tau$. □

Here are some examples of compositions on fully 2-rooted cuttable 2-permutations:

$$\begin{aligned}
66141|4322355 \circ_1 232113 &= 88263211365445\,77,\\
6614143\,22|355 \circ_2 232113 &= 8816165343224577,\\
6614143223|55 \circ_3 232113 &= 8816164225433577,\\
6614143223|55 \circ_4 232113 &= 8815163223544677,\\
661414322355| \circ_5 232113 &= 8814143223676557,\\
661414322355| \circ_6 232113 &= 7814143223557668,
\end{aligned}$$

where we have materialized by a vertical bar the separation $\mu|\nu$.

Remark 5.104. Motivated by Proposition 5.103, we will extend in Section 5.2.7 the compositions of Definition 5.102 from fully k-rooted cuttable k-permutations to all k-permutations.

5.2.3. *Action of* $\mathsf{MCit}_k^{\ddagger}$ *on words and permutations.* The objective of this section is to show that FQSym_k can also be endowed with a messy series k-citelangis algebra structure, generalizing the dendriform algebra structure on FQSym_1. Moreover, the resulting messy series k-citelangis algebra is free. Therefore, the free messy series k-citelangis subalgebra generated by the k-permutation $1^{\{k\}}$ provides a combinatorial model for the basis for the messy series k-citelangis operad.

Action on words. We first show that the shuffle algebra $\mathsf{Shuffle}^{\geq k}$ can be endowed with a structure of messy series k-citelangis algebra.

Definition 5.105. *For a messy series k-citelangis operator $\mathfrak{b} \in \{\prec, \succ\}^*$ and two words X and Y such that $|\mathfrak{b}| \leq \min(|X|, |Y|)$, we define*

$$X \; \mathfrak{b} \; Y = \begin{cases} X \sqcup\!\sqcup Y & \text{if } \mathfrak{b} = \varepsilon, \\ x(\underline{X} \; \underline{\mathfrak{b}} \; Y) & \text{if } \mathfrak{b} = \prec \underline{\mathfrak{b}} \text{ and } X = x\underline{X}, \\ y(X \; \underline{\mathfrak{b}} \; \underline{Y}) & \text{if } \mathfrak{b} = \succ \underline{\mathfrak{b}} \text{ and } Y = y\underline{Y}. \end{cases}$$

In other words, we consider the shuffle of X and Y, except that the i-th letter of $X \, \mathfrak{b} \, Y$ is forced to belong to X (resp. to Y) if the i-th letter of $\mathfrak{b}$ is $\prec$ (resp. is $\succ$).

For example, when $k = 1$, the two operators are given for the words xX and yY by

$$xX \prec yY = x\big(X \sqcup\!\sqcup yY\big) \qquad \text{and} \qquad xX \succ yY = y\big(xX \sqcup\!\sqcup Y\big).$$

In particular, the shuffle product is given by $\sqcup\!\sqcup = \rtimes\!\!\!\ltimes := \prec + \succ$. When $k = 2$, the four operators are given for the words x_1x_2X and y_1y_2Y by

$$\begin{aligned} x_1x_2X \prec\prec y_1y_2Y &= x_1x_2\big(X \sqcup\!\sqcup y_1y_2Y\big), \\ x_1x_2X \prec\succ y_1y_2Y &= x_1y_1\big(x_2X \sqcup\!\sqcup y_2Y\big), \\ x_1x_2X \succ\prec y_1y_2Y &= y_1x_1\big(x_2X \sqcup\!\sqcup y_2Y\big), \\ x_1x_2X \succ\succ y_1y_2Y &= y_1y_2\big(x_1x_2X \sqcup\!\sqcup Y\big). \end{aligned}$$

Again, the shuffle product is given by $\sqcup\!\sqcup = \rtimes\!\!\!\ltimes\rtimes\!\!\!\ltimes := \prec\prec + \prec\succ + \succ\prec + \succ\succ$.

Proposition 5.106 ([Pil18])**.** *The shuffle algebra $\mathsf{Shuffle}^{\geq k}$, endowed with the operators of Definition 5.105, defines a messy series k-citelangis algebra. The shuffle product $\sqcup\!\sqcup$ of $\mathsf{Shuffle}^{\geq k}$ is given by $\rtimes\!\!\!\ltimes^k := \sum_{\mathfrak{b} \in \mathfrak{B}_k} \mathfrak{b}$.*

Proof. Consider a destination vector $\mathsf{p} \in [3]^k$, and the corresponding messy series k-citelangis operators $\mathfrak{a}_\mathsf{p}, \mathfrak{b}_\mathsf{p}, \mathfrak{c}_\mathsf{p}, \mathfrak{d}_\mathsf{p} \in \{\prec, \rtimes\!\!\!\ltimes, \succ\}^k$ defined in Proposition 4.16. Then for any words X, Y, Z, the words that appear in

$$X \; \mathfrak{a}_\mathsf{p} \; (Y \; \mathfrak{b}_\mathsf{p} \; Z) \qquad \text{and} \qquad (X \; \mathfrak{d}_\mathsf{p} \; Y) \; \mathfrak{c}_\mathsf{p} \; Z$$

are precisely the words in $X \sqcup\!\sqcup Y \sqcup\!\sqcup Z$ such that for $i \in [k]$, their i-th letter was taken from the word X if $\mathsf{p}_i = 1$, from the word Y if $\mathsf{p}_i = 2$ and from the word Z if $\mathsf{p}_i = 3$. Therefore our operators on words indeed satisfy the messy series k-citelangis relation ($\mathsf{MCit}^{\ddagger}\,\mathsf{p}$). □

Action on permutations. Replacing the shuffle by the shifted shuffle, one endows similarly the algebra FQSym_k of k-permutations with a structure of messy series k-citelangis algebra. This can be rephrased as follows.

Definition 5.107. *For any operator $\mathfrak{b} \in \mathfrak{B}_k$ and any two k-permutations μ and ν of degree m and n respectively, $\mu \, \mathfrak{b} \, \nu$ is the sum of all k-permutations $\pi \in \mu \,\bar{\sqcup\!\sqcup}\, \nu$ such that for all $i \in [k]$, we have $\pi_i \leq m$ if $\mathfrak{b}_i = \prec$ while $\pi_i > m$ if $\mathfrak{b}_i = \succ$.*

For example, for $\mu = 321312132$ and $\nu = 221211$ we have $\mu \prec\succ\succ \nu = 355(21312132 \sqcup\!\sqcup 4544)$ and $\mu \succ\prec\prec \nu = 532(1312132 \sqcup\!\sqcup 54544)$. The next statement is immediate from Proposition 5.106.

Proposition 5.108 ([Pil18])**.** *The algebra $(\mathsf{FQSym}_k, \bar{\sqcup\!\sqcup})$, endowed with the operators of Definition 5.107, defines a messy series k-citelangis algebra. The shifted shuffle product $\bar{\sqcup\!\sqcup}$ of FQSym_k is given by $\rtimes\!\!\!\ltimes^k$.*

Similarly to Definition 5.89, we consider the evaluations of syntax trees of $\mathbf{Trees}(\mathfrak{B}_k)$ in FQSym_k to manipulate this messy series k-citelangis algebra. See Figure 17.

Definition 5.109. *Denote by* $\mathrm{MEval}^{\ddagger}(\mathfrak{t}; \sigma_1, \dots, \sigma_p)$ *the evaluation of a syntax tree* $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_k)$ *of arity* p *on* p k*-permutations* $\sigma_1, \dots, \sigma_p$ *of* Perm_k *using the messy series* k*-citelangis structure of* FQSym_k*. The* messy series permutation evaluation *of* $\mathfrak{t}$ *is then* $\mathrm{MEval}^{\ddagger}(\mathfrak{t}) := \mathrm{MEval}^{\ddagger}(\mathfrak{t}; 1^{\{k\}}, \dots, 1^{\{k\}})$*. We extend by linearity* $\mathfrak{t}$ *to the elements of* $\mathbf{Free}(\mathfrak{B}_k)$ *on the one hand, and* $\sigma_1, \dots, \sigma_p$ *to the elements of* FQSym_k *on the other hand.*

Remark 5.110. Let us rephrase algorithmically Definition 5.109. For this, we consider partial syntax trees as in Remark 5.90, where nodes are labeled by operators with at most k letters $\{\prec, \succ\}$ and leaves are labeled by words. The messy series permutation evaluation $\mathrm{MEval}^{\ddagger}(\mathfrak{s})$ of such a partial syntax tree $\mathfrak{s}$ is defined inductively by

- $\mathrm{MEval}^{\ddagger}(\mathfrak{s}) = \mathsf{w}$ if $\mathfrak{s}$ is a leaf labeled by the word w,
- $\mathrm{MEval}^{\ddagger}(\mathfrak{s}) = \mathsf{w} \cdot \big(\mathrm{MEval}^{\ddagger}(\mathfrak{l}) \sqcup\!\sqcup \mathrm{MEval}^{\ddagger}(\mathfrak{r})\big)$ if $\mathfrak{s}$ has a root with ℓ signals, and $\mathfrak{l}$ and $\mathfrak{r}$ are the two partial syntax trees left after ℓ cars traversed $\mathfrak{s}$ in series while erasing the first letters of the signals in the nodes and of the words in the leaves as described in Remark 5.90.

Finally, for a syntax tree $\mathfrak{t}$ of arity p and k-permutations $\sigma_1 \in \mathsf{Perm}_k(n_1), \dots, \sigma_p \in \mathsf{Perm}_k(n_p)$, the evaluation $\mathrm{MEval}^{\ddagger}(\mathfrak{t}; \sigma_1, \dots, \sigma_p)$ is the evaluation of the partial syntax tree $\mathfrak{s}$ obtained from $\mathfrak{t}$ by putting the permutation $\sigma_i[n_1 + \dots + n_{i-1}]$ at the i-th leaf for all $i \in [p]$. In particular, for $\mathrm{TEval}^{\ddagger}(\mathfrak{t}) = \mathrm{TEval}^{\ddagger}(\mathfrak{t}; 1^{\{k\}}, \dots, 1^{\{k\}})$, the i-th root is labeled by the word $i^{\{k\}}$. See Figure 17.

Freeness. It turns out that the tidy and messy series permutation evaluations are related by triangularity for the lexicographic order.

Definition 5.111. *For an homogeneous element* $F \in \mathsf{FQSym}_k$*, we denote by* $\mathrm{LexMin}(F)$ *the lexicographic minimal* k*-permutation with a non-zero coefficient in* F*.*

Lemma 5.112. *For any* $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_k)(p)$ *and* $\sigma_1, \dots, \sigma_p \in \mathsf{Perm}_k$*, we have*

$$\mathrm{TEval}^{\ddagger}(\mathfrak{t}; \sigma_1, \dots, \sigma_p) = \mathrm{LexMin}\left(\mathrm{MEval}^{\ddagger}(\mathfrak{t}; \sigma_1, \dots, \sigma_p)\right).$$

In particular, $\mathrm{TEval}^{\ddagger}(\mathfrak{t}) = \mathrm{LexMin}\left(\mathrm{MEval}^{\ddagger}(\mathfrak{t})\right)$*.*

Proof. The proof works by induction using the descriptions of $\mathrm{TEval}^{\ddagger}$ in Remark 5.90 and of $\mathrm{MEval}^{\ddagger}$ in Remark 5.110. Indeed, for any partial syntax tree $\mathfrak{s}$, we have

- if $\mathfrak{s}$ is a leaf labeled by the word w, then $\mathrm{TEval}^{\ddagger}(\mathfrak{s}) = \mathsf{w} = \mathrm{MEval}^{\ddagger}(\mathfrak{s})$,
- if $\mathfrak{s}$ has a root with ℓ signals, and $\mathfrak{l}$ and $\mathfrak{r}$ are the two partial syntax trees left after ℓ cars traversed $\mathfrak{s}$ in series while erasing the first letters of the signals in the nodes and of the words in the leaves as described in Remark 5.90, then

$$\begin{aligned}\mathrm{TEval}^{\ddagger}(\mathfrak{s}) &= \mathsf{w} \cdot \mathrm{TEval}^{\ddagger}(\mathfrak{l}) \cdot \mathrm{TEval}^{\ddagger}(\mathfrak{r}) = \mathsf{w} \cdot \mathrm{LexMin}\left(\mathrm{MEval}^{\ddagger}(\mathfrak{l})\right) \cdot \mathrm{LexMin}\left(\mathrm{MEval}^{\ddagger}(\mathfrak{r})\right) \\ &= \mathrm{LexMin}\left(\mathsf{w} \cdot \left(\mathrm{MEval}^{\ddagger}(\mathfrak{l}) \sqcup\!\sqcup \mathrm{MEval}^{\ddagger}(\mathfrak{r})\right)\right) = \mathrm{LexMin}\left(\mathrm{MEval}^{\ddagger}(\mathfrak{s})\right).\end{aligned}$$

The result then follows by application on the syntax tree $\mathfrak{s}$ obtained from $\mathfrak{t}$ by putting the permutation $\sigma_i[n_1 + \dots + n_{i-1}]$ at the i-th leaf for all $i \in [p]$. □

Remark 5.113. Since $\mathrm{TEval}^{\ddagger}$ is increasing for the lexicographic order, Lemma 5.112 extends to FQSym_k. Namely, for any $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_k)(p)$ and homogeneous elements $F_1, \dots, F_p \in \mathsf{FQSym}_k$,

$$\mathrm{LexMin}\left(\mathrm{MEval}^{\ddagger}(\mathfrak{t}; F_1, \dots, F_p)\right) = \mathrm{TEval}^{\ddagger}\left(\mathfrak{t}; \mathrm{LexMin}(F_1), \dots, \mathrm{LexMin}(F_p)\right).$$

We derive from Lemma 5.112 the main result of this section.

Theorem 5.114. *The messy series* k*-citelangis algebra* FQSym_k *is free on* k*-rooted uncuttable* k*-permutations.*

Proof. Consider the tidy and the messy series k-citelangis rewriting systems defined in Section 4.3.2. As seen in Section 4.3.3, these two rewriting systems have the same normal forms. Consider two such normal forms $\mathfrak{t}, \mathfrak{t}'$ of arity p and p' respectively, and some k-rooted uncuttable

k-permutations $\sigma_1, \dots, \sigma_p, \sigma'_1, \dots, \sigma'_{p'}$, such that $\mathrm{MEval}^{\ddagger}(\mathfrak{t}; \sigma_1, \dots, \sigma_p) = \mathrm{MEval}^{\ddagger}(\mathfrak{t}'; \sigma'_1, \dots, \sigma'_{p'})$. It then follows from Lemma 5.112 that $\mathrm{TEval}^{\ddagger}(\mathfrak{t}; \sigma_1, \dots, \sigma_p) = \mathrm{TEval}^{\ddagger}(\mathfrak{t}'; \sigma'_1, \dots, \sigma'_{p'})$. Proposition 5.98 then implies that $p = p'$, that $\sigma_i = \sigma'_i$ for all $i \in [p]$, and that $\mathfrak{t}$ and $\mathfrak{t}'$ are tidy series k-citelangis equivalent. As we assumed that they were in normal form, we obtain that $\mathfrak{t} = \mathfrak{t}'$. The result follows. □

Remark 5.115. By Theorem 5.114, the messy series k-citelangis operad can be fully understood from the messy series k-citelangis subalgebra of FQSym_k generated by the k-permutation $1^{\{k\}}$. However, while the fully k-rooted cuttable k-permutations still provide a combinatorial model for the basis of this messy series k-citelangis algebra similarly to Proposition 5.101, the compositions $\circ_i$ of the messy series k-citelangis operad $\mathsf{MCit}^{\ddagger}_k$ on fully k-rooted cuttable k-permutations are more intricate than Proposition 5.103.

5.2.4. *Operations of* $\mathsf{MCit}^{\ddagger}_k$ *on k-rooted k-posets.* In this section, we observe that any messy series permutation evaluation is the sum of all linear extensions of a well-chosen k-poset. This observation allows us to encode the messy series permutation evaluation by an alternative combinatorial model and to study directly on this model the action of the messy series k-citelangis operad $\mathsf{MCit}^{\ddagger}_k$. It also motivates the introduction of the series k-poset operad that will be studied in Section 5.2.5.

Operations on k-rooted k-posets and series poset evaluations. We first define some operations on the following k-posets.

Definition 5.116. *A multiposet $\le_M$ is k-rooted if it contains a chain (i.e. a totally ordered) submultiposet of size k whose elements are smaller than any other element. For $j \le k$, we denote by $\min_j(\le_M)$ the j-th element of this chain, by $\mathrm{Root}_j(\le_M)$ the chain formed by the first j elements of this chain, and by $\le_{M_{*j}}$ the submultiposet of $\le_M$ induced by $M_{*j} := M \smallsetminus \mathrm{Root}_j(\le_M)$.*

Definition 5.117. *Consider an operator $\mathfrak{b} \in \mathfrak{B}_k := \{\prec, \succ\}^k$ and two k-rooted k-posets $\le_M$ and $\le_N$ of degrees m and n respectively. For $j \le k$, denote by ℓ_j (resp. r_j) the number of $\prec$ (resp. $\succ$) signals among the first j signals of $\mathfrak{b}$. In particular, $\ell := \ell_k$ (resp. $r := r_k$) is the number of $\prec$ (resp. $\succ$) signals in $\mathfrak{b}$. We define a k-rooted k-poset $\le_P := \le_M \mathfrak{b} \le_N$ by:*

- *$P := M \sqcup N[m]$ where $N[m]$ is the k-set $(m+1)^{\{k\}} \dots (m+n)^{\{k\}}$ obtained from N by shifting each element by the degree m of M as in Definition 2.54,*
- *in $\le_P$, the elements of M (respectively of $N[m]$) are ordered by $\le_M$ (resp. by $\le_{N[m]}$), and the only other relations are such that $\mathrm{Root}_\ell(\le_M)\, \mathfrak{b}\, \mathrm{Root}_r(\le_N)$ is a k-root of $\le_P$. Formally, the comparison $x \le_P y$ holds for $x, y \in P$ if and only if one of the following statements holds:*
  - *$x \in M$, $y \in M$ and $x \le_M y$, or*
  - *$x \in N[m]$, $y \in N[m]$ and $x \le_{N[m]} y$, or*
  - *$x = \min_{\ell_j}(\le_M)$ and $y \in N[m]_{*r_j}$ for $j \le k$ such that $\mathfrak{b}_j = \prec$, or*
  - *$x \in M_{*\ell_j}$ and $y = \min_{r_j}(\le_N [m])$ for $j \le k$ such that $\mathfrak{b}_j = \succ$.*

Remark 5.118. Definition 5.117 can be conveniently rephrased in terms of ordered sums and disjoint unions of posets presented in Definition 2.58. Indeed,

$$\le_M \mathfrak{b} \le_N = \big(\mathrm{Root}_\ell(\le_M)\ \mathfrak{b}\ \mathrm{Root}_r(\le_N)\big) + \big(\le_{M_{*\ell}} \sqcup \le_{N[m]_{*r}}\big),$$

where $m = |M|$, and ℓ and r are the number of $\prec$ and $\succ$ signals in $\mathfrak{b}$.

For example, let $\le_{I_k}$ denote the only k-chain on $1^{\{k\}}$ represented in Figure 15 (left). Then for any operator $\mathfrak{b} \in \mathfrak{B}_k$ with ℓ signals $\prec$ and r signals $\succ$, the k-poset $\le_{I_k}$ forms a "Y" where the bottom branch is the chain $1^{\{\ell\}}\, \mathfrak{b}\, 2^{\{r\}}$, the left branch is the chain on $1^{\{k-\ell\}}$, and the right branch is the chain on $2^{\{k-r\}}$. This is illustrated when $k = 2$ in Figure 15. See Figure 16 for another example. The next statement clearly follows from Remark 5.118.

Lemma 5.119. *For any operator $\mathfrak{b} \in \mathfrak{B}_k$ and any two k-rooted k-posets $\le_M$ and $\le_N$ of degrees m and n respectively, $\le_M\ \mathfrak{b}\ \le_N$ is a k-rooted k-poset of degree $m + n$.*

This statement allows to define the evaluation of a syntax tree on posets. See Figure 16.

Definition 5.120. *Denote by* $\mathrm{PEval}^\ddagger(\mathfrak{t}; \le_{M_1}, \dots, \le_{M_p})$ *the evaluation of a syntax tree* $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_k)$ *of arity* p *on* p k*-rooted* k*-posets* $\le_{M_1}, \dots, \le_{M_p}$ *using the operators of Definition 5.117. The* series poset evaluation *of* $\mathfrak{t}$ *is then* $\mathrm{PEval}^\ddagger(\mathfrak{t}) := \mathrm{PEval}^\ddagger(\mathfrak{t}; \le_{I_k}, \dots, \le_{I_k})$*, where* $\le_{I_k}$ *is the* k*-chain on* $1^{\{k\}}$.

Remark 5.121. It is not difficult to check that, for any operator $\mathfrak{b} \in \mathfrak{B}_k$ and any two k-rooted k-posets $\le_M$ and $\le_N$,

- if $\le_M$ and $\le_N$ are trees, then $\le_M \mathfrak{b} \le_N$ is a tree (see Definition 2.59), and
- if $\le_M$ and $\le_N$ are interval labelled, then $\le_M \mathfrak{b} \le_N$ is interval labeled (see Definition 2.60).

Therefore, for any $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_k)$, the series poset evaluation $\mathrm{PEval}^\ddagger(\mathfrak{t})$ is a k-rooted interval labelled tree. We will characterize the series poset evaluations later in Proposition 5.129.

k-rooted cuts in k-posets. We now aim at characterizing series poset evaluations. As in Section 5.2.2, this understanding goes through k-rooted cuts in k-rooted k-posets.

Definition 5.122. *Let* $\le_M$ *be a* k*-rooted* k*-poset of degree* n *and* $\gamma \in [n-1]$*. We say that* γ *is a* k-rooted cut *of* $\le_M$ *if we can decompose* M *into* $M = \mathrm{Root}_k(\le_M) \sqcup L \sqcup R$ *such that, for all* $\ell \in L$ *and* $r \in R$*, we have* $\ell \le \gamma < r$ *and* ℓ *and* r *are incomparable for* $\le_M$*. We denote by* $\mathrm{rcuts}_k(\le_M)$ *the set of* k*-rooted cuts of* $\le_M$*. We say that the* k*-rooted* k*-poset* $\le_M$ *is* k-rooted cuttable *if it admits a* k*-rooted cut, and* k-rooted uncuttable *if it admits no* k*-rooted cut.*

The following statements are similar to Lemma 5.74 and Proposition 5.75.

Lemma 5.123. *Consider* $L \subseteq [m]$ *and* $\gamma \in [\min(L), \max(L) - 1]$*, and let* $\gamma^{|L} := |[\gamma] \cap L|$ *denote the number of elements of* L *between* 1 *and* γ*. If* γ *is a* k*-rooted cut of a* k*-rooted* k*-poset* $\le_M$ *of degree* m*, then* $\gamma^{|L}$ *is a* k*-rooted cut of its restriction* $\le_{M^{|L}}$*.*

Proposition 5.124. *The following conditions are equivalent for a* k*-rooted* k*-poset of degree* n*:*

(i) its restriction to any interval of $[n]$ *of size at least* 2 *is* k*-rooted cuttable,*
(ii) its restriction to any subset of $[n]$ *of size at least* 2 *is* k*-rooted cuttable.*

Definition 5.125. *A* k*-rooted* k*-poset is* fully k-rooted cuttable *if it satisfies the equivalent conditions of Proposition 5.124.*

The following statements are similar to Lemmas 5.91 and 5.92.

Lemma 5.126. *For any* k*-rooted* k*-posets* $\le_M$ *and* $\le_N$*, and any operator* $\mathfrak{b} \in \mathfrak{B}_k$*, the degree* m *of* $\le_M$ *is a* k*-rooted cut of* $\le_M \ \mathfrak{b} \ \le_N$*.*

Lemma 5.127. *For any* k*-rooted* k*-poset* $\le_P$ *of degree* p *and any* k*-rooted cut* $\gamma \in \mathrm{rcuts}_k(\le_P)$*, there is a unique* $\mathfrak{b} \in \mathfrak{B}_k$ *(defined by* $\mathfrak{b}_i := \prec$ *if* $\mathrm{Root}_i(\le_P) \le \gamma$ *and* $\mathfrak{b}_i := \succ$ *if* $\mathrm{Root}_i(\le_P) > \gamma$*) such that* $\le_P \, = \, \le_{P^{|[\gamma]}} \mathfrak{b} \le_{P^{|[p] \smallsetminus [\gamma]}}$*.*

Remark 5.128. Similarly to Remark 5.93, observe that Lemma 5.127 gives an inductive algorithm to compute all decompositions of a given k-rooted k-poset $\le_P$ as an evaluation of the form $\le_P = \mathrm{PEval}^\ddagger(\mathfrak{t}; \le_{M_1}, \dots, \le_{M_p})$. Namely, $\le_P$ admits

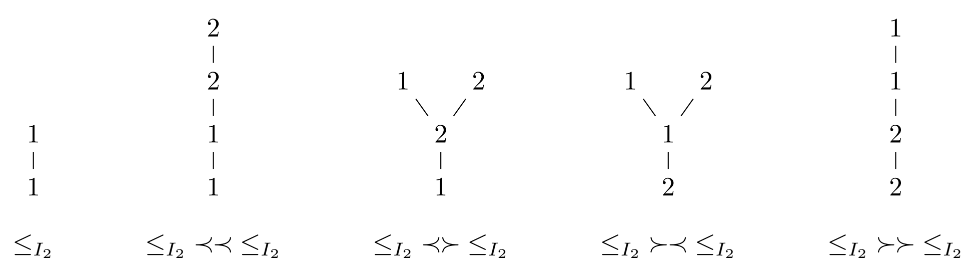


FIGURE 15. Four series operations on 2-rooted 2-posets. See Definition 5.117.

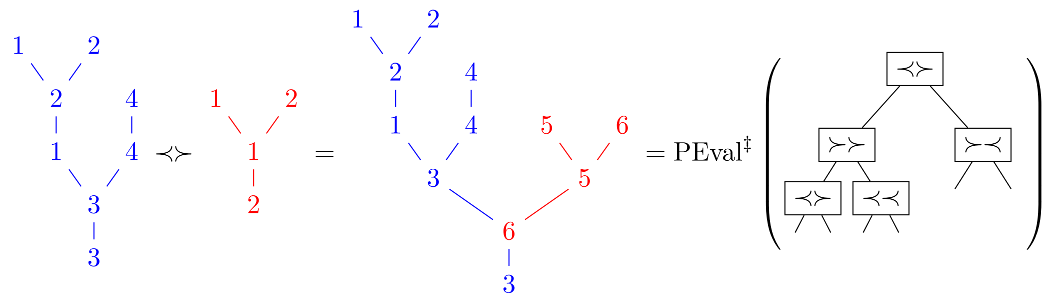


FIGURE 16. Another example of series operation on 2-rooted 2-posets. See Definition 5.117.

- the trivial evaluation $\le_0 = \mathrm{PEval}^{\ddagger}(\mathbb{1}; \le_0)$, where $\mathbb{1}$ is the unit syntax tree with no node and a single leaf, and
- the evaluation $\le_P = \mathrm{PEval}^{\ddagger}(\mathfrak{t}; \le_{M_1}, \dots, \le_{M_l}, \le_{N_1}, \dots, \le_{N_r})$, for any $\gamma \in \mathrm{rcuts}_k(\le_P)$ and any $\le_{P|[\gamma]} = \mathrm{PEval}^{\ddagger}(\mathfrak{l}; \le_{M_1}, \dots, \le_{M_l})$ and $\le_{P|[\ell]\smallsetminus[\gamma]} = \mathrm{PEval}^{\ddagger}(\mathfrak{r}; \le_{N_1}, \dots, \le_{N_r})$, where $\mathfrak{t}$ is the syntax tree with root $\mathfrak{b} \in \mathfrak{B}_k$ defined by Lemma 5.127 and with subtrees $\mathfrak{l}$ and $\mathfrak{r}$.

This algorithm implies the existence of decompositions of the form $\le_P = \mathrm{PEval}^{\ddagger}(\mathfrak{t}; \le_{M_1}, \dots, \le_{M_p})$ where $\le_{M_1}, \dots, \le_{M_p}$ are k-rooted uncuttable.

Note that not all k-rooted k-posets are obtained by evaluating syntax trees on $\mathfrak{B}_k$. For instance, the evaluation the syntax trees on $\mathfrak{B}_2$ of arity 2 only produces the four 2-rooted 2-posets of Figure 15, thus only two of the six linear 2-rooted 2-posets of degree 2. We now characterize the k-rooted k-posets which are series poset evaluations of syntax trees.

Proposition 5.129. *The series poset evaluations of the syntax trees of* $\mathbf{Trees}(\mathfrak{B}_k)$ *are precisely the fully k-rooted cuttable k-posets.*

Proof. The proof is exactly the same as Proposition 5.95. Consider first a series poset evaluation $\le_P = \mathrm{PEval}^{\ddagger}(\mathfrak{t})$ with $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_k)(\ell)$. We prove by induction on ℓ that $\le_P$ is fully k-rooted cuttable. If $\ell = 1$, there is nothing to prove. Assume that $\ell \ge 2$ and let $1 \le a < b \le \ell$. Let $\mathfrak{l}$ and $\mathfrak{r}$ denote the left and right subtrees of $\mathfrak{t}$, and let γ be the arity of $\mathfrak{l}$, so that $\le_{P|[\gamma]} = \mathrm{PEval}^{\ddagger}(\mathfrak{l})$ and $\le_{M|[\ell]\smallsetminus[\gamma]} = \mathrm{PEval}^{\ddagger}(\mathfrak{r})$. We distinguish three cases:

- Assume that $b \le \gamma$. Since $\le_{M|[\ell]} = \mathrm{PEval}^{\ddagger}(\mathfrak{l})$ is fully k-rooted cuttable by induction hypothesis and k-rooted cuts are preserved by restriction by Lemma 5.123, we obtain that $\le_{P|[a,b]} = (\le_{P|[\gamma]})^{|[a,b]}$ is k-rooted cuttable.
- Assume that $\gamma \le a$. The argument is similar since $\le_{P|[a,b]} = (\le_{P|[\ell]\smallsetminus[\gamma]})^{|[a-\gamma,b-\gamma]}$.
- Assume finally that $a < \gamma < b$. By Lemma 5.126, γ is a k-rooted cut of $\le_P$. Therefore, $\gamma - a$ is a k-rooted cut of $\le_{P|[a,b]}$ by Lemma 5.123.

Conversely, consider now a fully k-rooted cuttable k-poset $\le_P$ of degree ℓ. Similarly to Remark 5.128, we prove by induction on ℓ that $\le_P$ is the series poset evaluation of a syntax tree. If $\ell = 1$, then $\le_P = \mathrm{PEval}^{\ddagger}(\mathbb{1})$. If $\ell \ge 2$, then $\le_P$ admits at least one k-rooted cut γ by assumption. Moreover, $\le_{P|[\gamma]}$ and $\le_{P|[\ell]\smallsetminus[\gamma]}$ are both fully k-rooted cuttable by Lemma 5.123. By induction, we obtain that $\le_{P|[\gamma]} = \mathrm{PEval}^{\ddagger}(\mathfrak{l})$ and $\le_{P|[\ell]\smallsetminus[\gamma]} = \mathrm{PEval}^{\ddagger}(\mathfrak{r})$. Then $\le_P = \mathrm{PEval}^{\ddagger}(\mathfrak{t})$, where $\mathfrak{t}$ is the syntax tree with root $\mathfrak{b} \in \mathfrak{B}_k$ defined by Lemma 5.92 and with subtrees $\mathfrak{l}$ and $\mathfrak{r}$. □

Connections between $\mathrm{TEval}^{\ddagger}$, $\mathrm{MEval}^{\ddagger}$ and $\mathrm{PEval}^{\ddagger}$. We have seen in Lemma 5.112 that $\mathrm{TEval}^{\ddagger}(\mathfrak{t}) = \mathrm{LexMin}\big(\mathrm{MEval}^{\ddagger}(\mathfrak{t})\big)$ for any syntax tree $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_k)$. We now compare with the series poset evaluation $\mathrm{PEval}^{\ddagger}(\mathfrak{t})$.

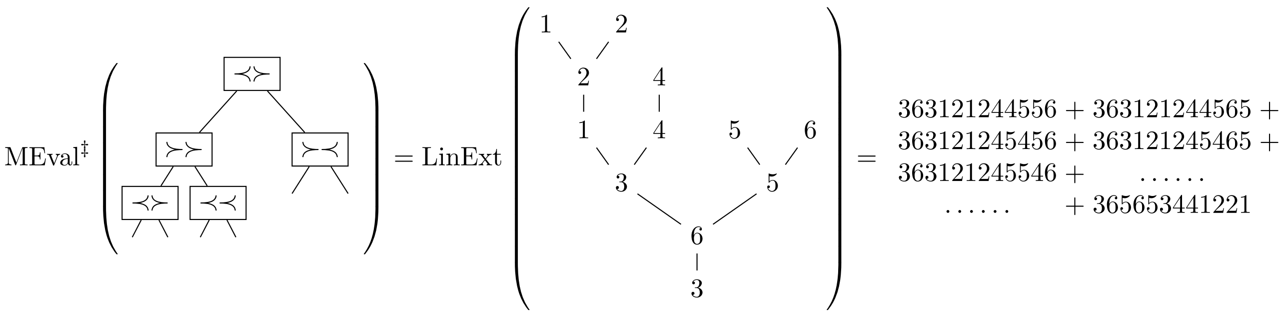


FIGURE 17. Illustration of Lemma 5.130.

First, as in Section 5.1.4, we get back the messy series permutation evaluation by projecting posets to the sum of their linear extensions. Recall that for a k-poset $\leq_M$, we let

$$\mathrm{LinExt}(\leq_M) \coloneqq \sum_{\mu \in \mathcal{L}(\leq_M)} \mu$$

be the sum of all linear extensions of $\leq_M$. We obtain the following statement, illustrated in Figure 17.

Lemma 5.130. *For any syntax tree* $\mathfrak{t} \in \mathbf{Trees}(\mathfrak{B}_k)$*, we have* $\mathrm{MEval}^{\ddagger}(\mathfrak{t}) = \mathrm{LinExt}\big(\mathrm{PEval}^{\ddagger}(\mathfrak{t})\big)$.

Proof. Remark 5.118 and Lemma 2.64 prove that

$$\mathrm{LinExt}(\leq_M)\ \mathfrak{b}\ \mathrm{LinExt}(\leq_N) = \mathrm{LinExt}(\leq_M \mathfrak{b} \leq_N)$$

for any operator $\mathfrak{b} \in \mathfrak{B}_k$ and any k-rooted k-posets $\leq_M$ and $\leq_N$. The statement then follows by induction on syntax trees. □

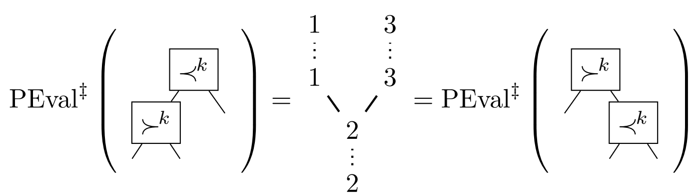
Conversely, we can obtain $\mathrm{PEval}^{\ddagger}(\mathfrak{t})$ from $\mathrm{MEval}^{\ddagger}(\mathfrak{t})$ by intersecting all linear orders corresponding to the k-permutations that appear in $\mathrm{MEval}^{\ddagger}(\mathfrak{t})$.

We now describe the connection between $\mathrm{TEval}^{\ddagger}(\mathfrak{t})$ and $\mathrm{PEval}^{\ddagger}(\mathfrak{t})$. By Lemmas 5.112 and 5.130, the tidy series permutation evaluation $\mathrm{TEval}^{\ddagger}(\mathfrak{t})$ is the minimal linear extension of $\mathrm{PEval}^{\ddagger}(\mathfrak{t})$. It can thus be obtained from $\mathrm{PEval}^{\ddagger}(\mathfrak{t})$ by repeatedly reading and deleting the minimal source.

Conversely, there are several syntax trees $\mathfrak{t}$ whose series poset evaluation $\mathrm{PEval}^{\ddagger}(\mathfrak{t})$ has the same minimal linear extension. Given a k-rooted cuttable k-permutation σ, we can find all k-posets $\mathrm{PEval}^{\ddagger}(\mathfrak{t})$ whose minimal linear extension is σ by first using Remark 5.90 to compute all possible syntax trees $\mathfrak{t}$ such that $\sigma = \mathrm{TEval}^{\ddagger}(\mathfrak{t})$, and then applying Definitions 5.117 and 5.120 to get their series poset evaluations $\mathrm{PEval}^{\ddagger}(\mathfrak{t})$.

Relations among series poset evaluations. We now study which syntax trees evaluate to the same k-rooted k-poset. Observe first the following quadratic relation:

(10)



Warning! The operators of Definition 5.117 on k-rooted k-posets do not verify the other messy series k-citelangis relations, and thus do not define a messy series k-citelangis algebra. In fact, we will now show that Equation (10) is the only relation that leads to the same series poset evaluations.

Lemma 5.131. *For any operators $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}_k$ and any k-rooted k-posets $\le_M, \le_N, \le_{M'}, \le_{N'}$, we have $\le_M \mathfrak{b} \le_N = \le_{M'} \mathfrak{b}' \le_{N'}$ if and only if*

- *either $\mathfrak{b} = \mathfrak{b}'$, $\le_M = \le_{M'}$ and $\le_N = \le_{N'}$,*
- *or $\mathfrak{b} = \prec^k$ and $\mathfrak{b}' = \succ^k$, and there exists a k-rooted k-poset $\le_O$ such that $\le_M = \le_{M'} \succ^k \le_O$ and $\le_{N'} = \le_O \prec^k \le_N$.*

Proof. The "if" direction is immediate as the second case was already observed in Equation (10).

For the "only if" direction, consider a k-rooted k-poset $\le_P := \le_M \mathfrak{b} \le_N$ obtained from an operation $\mathfrak{b} \in \mathfrak{B}_k$ and two k-rooted k-posets $\le_M$ and $\le_N$, and let p be the degree of $\le_P$.

When $\mathfrak{b} \notin \{\prec^k, \succ^k\}$, the element $\min_k(\le_P)$ is covered by precisely two elements, and all values of $\le_P$ appear outside of its k-root. Therefore, $\le_P$ has a unique k-rooted cut γ, so that $\mathfrak{b}$, $\le_M$ and $\le_N$ are all determined by Lemmas 5.126 and 5.127.

Observe now that when $\mathfrak{b} = \prec^k$, we have $\le_M = \le_{P|[p]\smallsetminus R}$ and $\le_N = \le_{P|R}$ where R be the upper order ideal of $\le_P$ generated by the maximal value covering $\min_k(\le_P)$. Similarly, when $\mathfrak{b} = \succ^k$, we have $\le_M = \le_{P|L}$ and $\le_N = \le_{P|[p]\smallsetminus L}$ where L be the upper order ideal of $\le_P$ generated by the minimal value covering $\min_k(\le_P)$. Finally, these two situations are simultaneously possible only in the case when $\le_M = \le_{M'} \succ^k \le_O$ and $\le_{N'} = \le_O \prec^k \le_N$ where $\le_{M'}$, $\le_{N'}$ and $\le_O$ are the sub-k-posets of $\le_P$ respectively induced by L, R and $P \smallsetminus (L \cup R)$. ☐

Remark 5.132. Lemma 5.131 fails for multiposets, as we used that the multiplicity of each value in a k-poset is at least the cardinality k of the root chain.

Finally, the next statement is similar to Proposition 5.98.

Proposition 5.133. *For any syntax trees $\mathfrak{t}, \mathfrak{t}' \in \mathbf{Trees}(\mathfrak{B}_k)$ of arity p and p' respectively, and any k-rooted uncuttable k-posets $\le_{M_1}, \dots, \le_{M_p}$, $\le_{M'_1}, \dots, \le_{M'_{p'}}$, if*

$$\mathrm{TEval}^\ddagger(\mathfrak{t}; \le_{M_1}, \dots, \le_{M_p}) = \mathrm{TEval}^\ddagger(\mathfrak{t}'; \le_{M'_1}, \dots, \le_{M'_{p'}}),$$

then $p = p'$, $\mathfrak{t} = \mathfrak{t}'$ modulo rewritings using Equation (10) and $\le_{M_i} = \le_{M'_i}$ for all $i \in [p]$.

Proof. Immediate by induction from Lemma 5.131. ☐

Series poset evaluations of series k-citelangis normal forms. As the operators of Definition 5.117 on k-rooted k-posets do not verify all the messy series k-citelangis relations, the series poset evaluations of all syntax trees result to two many different k-posets. To obtain an alternative combinatorial model for the messy series k-citelangis operad, we thus need to restrict our attention to series poset evaluations of normal forms of the messy series k-citelangis rewriting system described in Section 4.3.2.

Definition 5.134. *A series normal k-poset is a k-poset that can be obtained as the series poset evaluation of a normal form of the messy series k-citelangis rewriting system.*

In particular, by Remark 5.121 and Proposition 5.129, a series normal k-poset is an interval labeled and fully k-rooted cuttable tree.

Example 5.135. When $k = 1$, the series normal 1-posets are just interval labeled trees such that:

- any node has at most one smaller child, and
- if a node is larger than its parent, then it is larger than all its children.

We call them *willow posets*. These posets are illustrated in Figure 18 for $n = 4$. In each poset, we have colored the increasing edges in red and the decreasing ones in blue. The conditions defining willow posets translate to the following color rule: blue edges cannot appear in parallel (no node appears as the bottom of two blue covering relations) while red edges cannot appear in series (no node appears both at the bottom and at the top of red covering relations).

There is an obvious bijection sending a binary tree t to a willow poset $w(t)$. Given a binary tree t, we label its nodes using the infix labeling where all labels in the left (resp. right) subtree of the i-th node are smaller (resp. larger) than i. Given a node x in t, we define:

- the *right origin* of x: a node and its right child have the same right origin, and a node which is not a right child is its own right origin.
- the *right outgrowth* of x: all nodes with right origin x, except x itself.

We define the tree $w(t)$ where the children of a node are its left child in t (if any) together with its right outgrowth in t. We claim that $w(t)$ is a willow poset:

- for any node, its only smaller child in $w(t)$ is its left child in t, and
- if a node is larger than its parent, then its parent is its right origin, so that this node has no right outgrowth.

The reader can compare Figures 1 and 18 for an illustration of this bijection.

In contrast to the $k = 1$ case discussed in Example 5.135, there seems to be no simple characterization of the series normal k-posets for $k > 1$. There are still some necessary conditions on the increasing and decreasing edges, but they are more difficult to express.

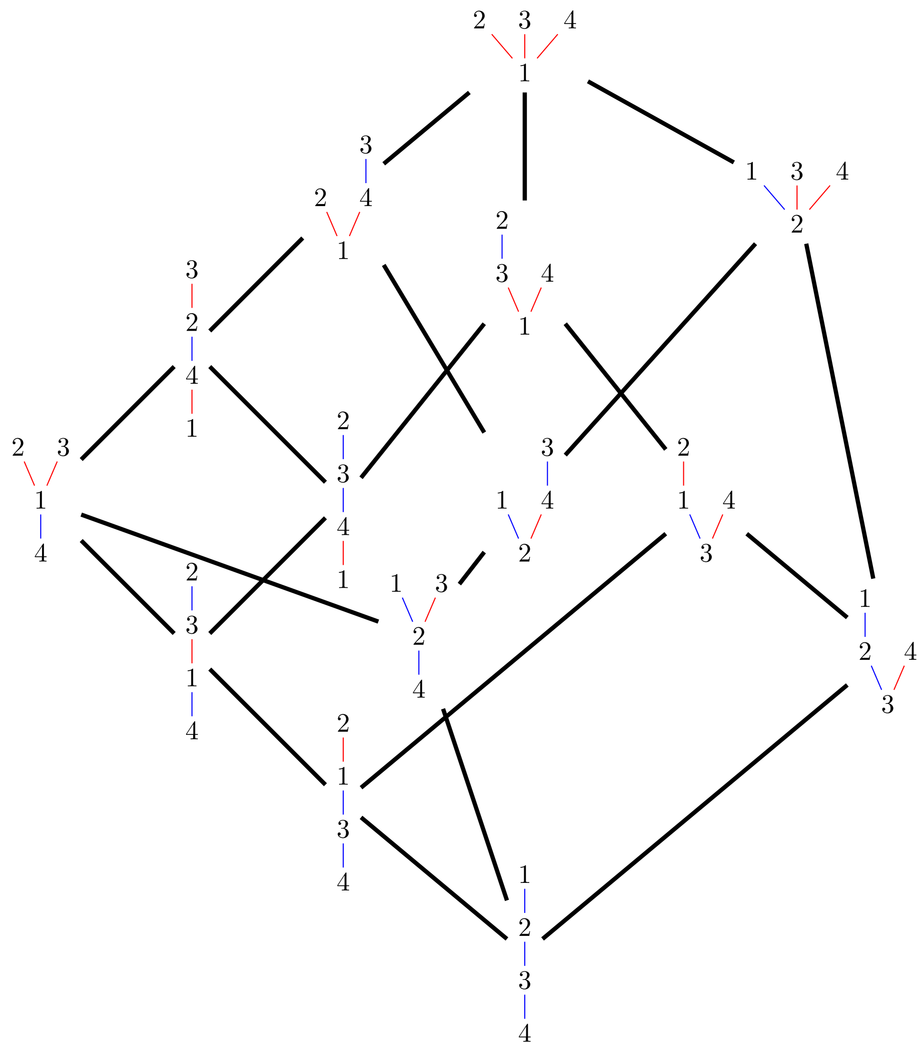
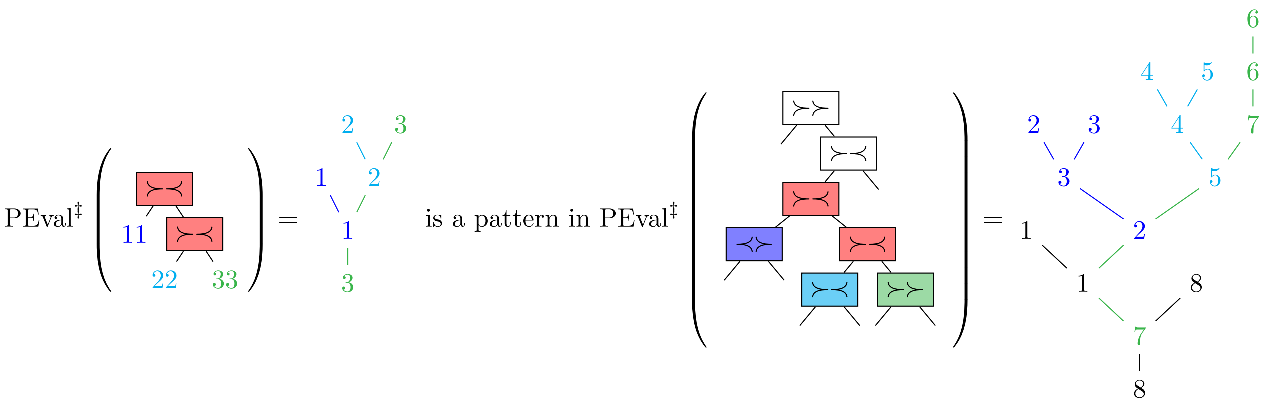


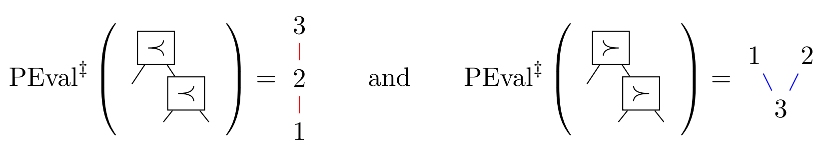
FIGURE 18. The Tamari lattice on willow posets, defined in Example 5.135. Increasing edges are colored red, while decreasing ones are colored blue, and there is no two blue edges in parallel nor two red edges in series. Compare to Figure 1 for an illustration of the bijection between binary trees and willow posets.

There is however a simple algorithm to decide whether a given fully k-rooted cuttable k-poset $\le_P$ is series normal. Namely, compute its minimal linear extension σ, then the normal form $\mathfrak{t}$ of the tidy series k-citelangis rewriting system such that $\sigma = \mathrm{TEval}^{\ddagger}(\mathfrak{t})$ by Remark 5.99, and check whether $\le_P = \mathrm{PEval}^{\ddagger}(\mathfrak{t})$.

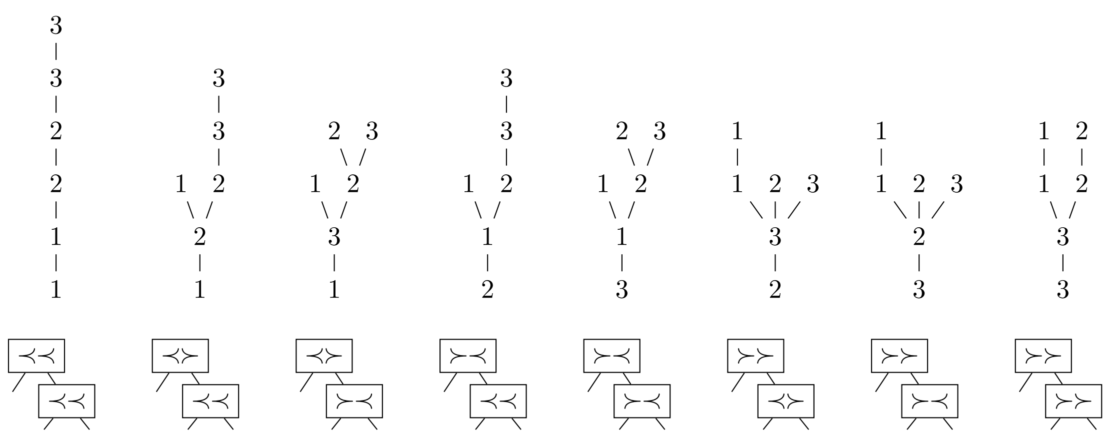
Finally, one can also consider the list of all series poset evaluations of the $3^k - 1$ quadratic series k-signaletic combs distinct from that of Equation (10). A fully k-rooted cuttable k-poset is series normal if it does not contain any of these forbidden k-posets as patterns. The meaning of patterns in k-rooted k-posets should be clear with the poset compositions defined in Definition 5.136. Here, we prefer to illustrate intuitively the notion on an example:



Note that for $k = 1$, the forbidden posets are



which leads to the description of Example 5.135. For $k = 2$, the forbidden 2-posets are



were we have represented below each 2-poset the series k-signaletic comb from which it was evaluated.

5.2.5. *Series poset operad.* As observed above, the operators $\mathfrak{B}_k$ on k-rooted k-posets defined in Definition 5.117 do not verify the messy series k-citelangis relations. However, they satisfy a subset of these relations which in turn defines an operad on posets. This operad can be seen as a suboperad of an operad on all k-rooted k-posets defined by the following composition rules, illustrated in Figure 19.

Definition 5.136. *Let $\le_M$ and $\le_N$ be two k-rooted k-posets of degrees m and n respectively, and let $i \in [m]$. Let $i_1, i_2, \dots, i_k$ denote the k copies of i in M such that $i_1 <_M i_2 <_M \cdots <_M i_k$. We define the composition $\le_P := \le_M \circ_i \le_N$ by:*

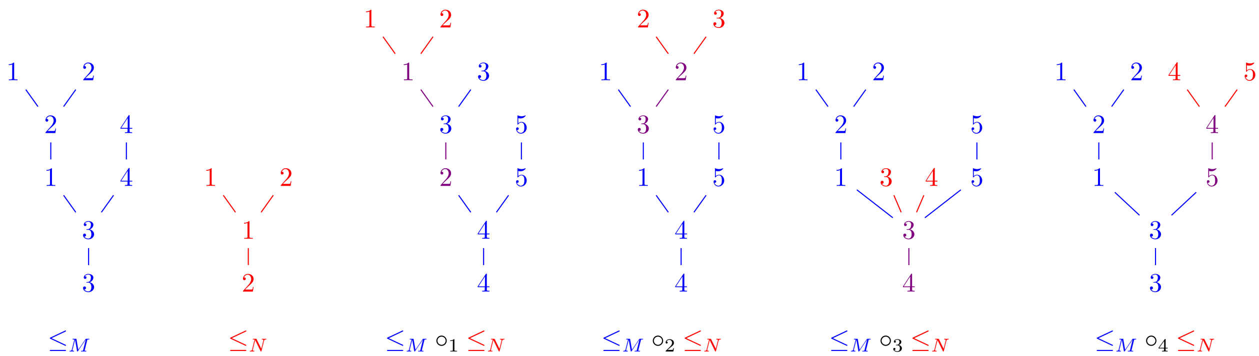


FIGURE 19. Examples of compositions in the series 2-poset operad. See Definition 5.136.

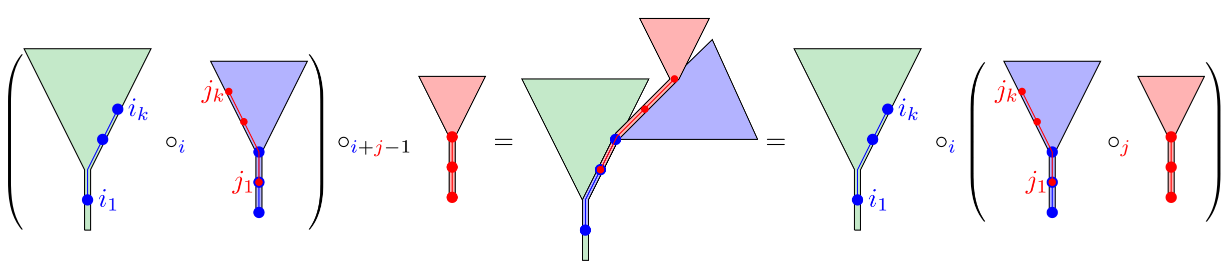
- $P := 1^{\{k\}} \dots (m+n-1)^{\{k\}}$,
- $\le_P$ *is obtained by inserting* $\le_N$ *in* $\le_M$ *by placing* $\min_j(\le_N)$ *at* i_j *for all* $j \in [k]$*, and performing the appropriate shift. More precisely, using the notations of Definitions 2.37 and 2.38, the comparison* $x \le_P y$ *holds for* $x, y \in P$ *if and only if one of the following statements holds:*
  - $x \in M[i,n]$, $y \in M[i,n]$ *and* $\overline{x} \le_M \overline{y}$*, or*
  - $x \in N[i-1]$, $y \in N[i-1]$ *and* $\overline{x} \le_N \overline{y}$*, or*
  - $x \in M[i,n]$, $y \in N[i-1]$, $\overline{x} \le_M i_j$ *and* $\min_j(\le_N) \le_N y$ *for some* $j \in [k]$ *or*
  - $x \in N[i-1]$, $y \in M[i,n]$, $\overline{x} = \min_j(\le_N)$ *and* $i_j \le_M \overline{y}$ *for some* $j \in [k]$.

The following statement is left to the reader.

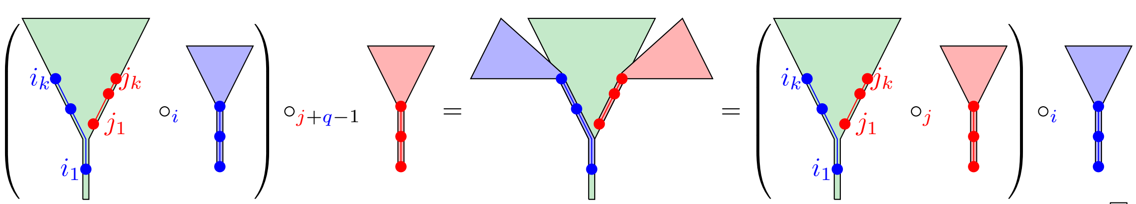
Lemma 5.137. *For any two* k*-rooted* k*-posets* $\le_M$ *and* $\le_N$ *of degrees* m *and* n *respectively, and any* $i \in [m]$*, the composition* $\le_M \circ_i \le_N$ *is a* k*-rooted* k*-poset of degree* $m+n-1$.

Proposition 5.138. *The compositions* $\circ_i$ *of Definition 5.136 define an operad structure* $\mathsf{RP}_k^{\ddagger}$ *on* k*-rooted* k*-posets.*

Proof. Rather than a formal proof, we illustrate the argument on pictures. For the the series compositions (see Section 2.1), we have:



For the the parallel compositions (see Section 2.1), we have:



□

The following statements connect this operad $\mathsf{RP}_k^{\ddagger}$ with the series poset evaluation of Definition 5.120.

Lemma 5.139. *For any operator* $\mathfrak{b} \in \mathfrak{B}_k$ *and any* k*-rooted* k*-posets* $\le_M$ *and* $\le_N$*, we have*

$$\le_M \mathfrak{b} \le_N = \mathrm{PEval}^{\ddagger}(\mathfrak{b}) \circ (\le_M, \le_N),$$

where the posets $\mathrm{PEval}^{\ddagger}(\mathfrak{b}) := \le_{I_k} \mathfrak{b} \le_{I_k}$ *are illustrated in Figure 15 for* $k = 2$.

Proof. By Definition 5.136, the composition $\mathrm{PEval}^{\ddagger}(\mathfrak{b}) \circ (\le_M, \le_N)$ places $\min_j(\le_M)$ at 1_j and $\min_j(\le_N)$ at 2_j in the k-poset $\mathrm{PEval}^{\ddagger}(\mathfrak{b})$ for all $j \in [k]$. This coincides with the result of the operation $\le_M \mathfrak{b} \le_N$ described in Definition 5.117 and Remark 5.118. □

Lemma 5.139 enables us to interpret the operations on k-rooted k-posets of the previous section as a suboperad of $\mathsf{RP}_k^{\ddagger}$. We say that two syntax trees $\mathfrak{t}, \mathfrak{t}'$ on $\mathfrak{B}_k$ with the same arity are *series poset equivalent* and we write $\mathfrak{t} \simeq^{\ddagger} \mathfrak{t}'$ if they have the same series poset evaluation.

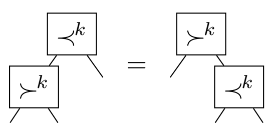
Theorem 5.140. *The series poset equivalence is compatible with the grafting of syntax trees: for any syntax trees* $\mathfrak{t} \simeq^{\ddagger} \mathfrak{t}'$ *of arity* p *and* $\mathfrak{s} \simeq^{\ddagger} \mathfrak{s}'$ *of arity* q *and* $i \in [p]$*, we have* $\mathfrak{t} \circ_i \mathfrak{s} \simeq^{\ddagger} \mathfrak{t}' \circ_i \mathfrak{s}'$*. Therefore, there exists a* series k-poset operad

$$\mathsf{Pos}_k^{\ddagger} := \mathrm{PEval}^{\ddagger}\big(\mathbf{Trees}(\mathfrak{B}_k)\big).$$

Proof. By induction on Lemma 5.139, the map $\mathrm{PEval}^{\ddagger}$ coincides with the unique operad morphism from the free operad $\mathbf{Free}(\mathfrak{B}_k)$ to $\mathsf{RP}_k^{\ddagger}$ that sends a generator $\mathfrak{b} \in \mathfrak{B}_k$ to $\mathrm{PEval}^{\ddagger}(\mathfrak{b})$. Therefore, $\mathsf{Pos}_k^{\ddagger}$ is the suboperad of $\mathsf{RP}_k^{\ddagger}$ generated by $\big\{\mathrm{PEval}^{\ddagger}(\mathfrak{b}) \mid \mathfrak{b} \in \mathfrak{B}_k\big\}$. □

By construction, the series k-poset operad $\mathsf{Pos}_k^{\ddagger}$ satisfies the relation of Equation (10). Proposition 5.133 implies that it is the only relation in $\mathsf{Pos}_k^{\ddagger}$. This generalizes the L-algebras of P. Leroux [Ler11].

Theorem 5.141. *The series* k*-poset operad* $\mathsf{Pos}_k^{\ddagger}$ *is generated by* $\mathfrak{B}_k$ *with the unique relation*



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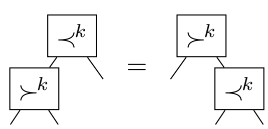
Corollary 5.142. *For any* k *the operad* $\mathsf{Pos}_k^{\ddagger}$ *is Koszul. Its Koszul dual* $(\mathsf{Pos}_k^{\ddagger})^!$ *is a finite operad generated by* $\mathfrak{B}_k$ *with a unique arity* 3 *element*



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all the other arity 3 *compositions being zero. In particular its Hilbert series is*

$$\mathcal{H}_{(\mathsf{Pos}_k^{\ddagger})^!}(t) = t + 2^k t^2 + t^3.$$

Proof. The only non trivial fact is the Koszulity of $\mathsf{Pos}_k^{\ddagger}$. We prove it on the dual $(\mathsf{Pos}_k^{\ddagger})^!$ where it is plain thanks to the Poincaré – Birkhoff – Witt basis rewriting criterion. Indeed, all trees of arity 4 rewrites ultimately to 0. □

Remark 5.143. Combinatorially, the dual series k-poset operad $(\mathsf{Pos}_k^{\ddagger})^!$ can be realized as a quotient of the messy series k-signaletic operad $\mathsf{MSig}_k^{\ddagger}$ spanned by the destination vectors of the form $(\!| 1^k |\!)_1$, $(\!| w |\!)_2$ with $w \in \{1,2\}^k$, and $(\!| 2^k |\!)_3$, which we call *L-destination vectors*. The surjective morphism $\mathsf{MSig}_k^{\ddagger} \longrightarrow (\mathsf{Pos}_k^{\ddagger})^!$ sends a destination vector to itself if it is an L-destination vector and to 0 otherwise.

Thanks to the preceding corollary, we can now identify $\mathsf{Pos}_k^{\ddagger}$ using Manin powers (see Section 2.5).

Proposition 5.144. *The operad* $\mathsf{Pos}_k^{\ddagger}$ *is isomorphic to the* k*-th black or white power of the operad* $\mathsf{Pos}_1^{\ddagger}$ *of* L*-algebras:*

$$\mathsf{Pos}_k^{\ddagger} \simeq \mathsf{Pos}_1^{\ddagger\,\square k} \simeq \mathsf{Pos}_1^{\ddagger\,\blacksquare k}.$$

$k\backslash n$	1	2	3	4	5	6	7	8
1	1	2	7	30	143	728	3876	21318
2	1	4	31	300	3251	37744	459060	5773548
3	1	8	127	2520	56003	1333472	33262836	858011352
4	1	16	511	20400	912131	43696576	2193011700	113813345712
5	1	32	2047	163680	14658563	1406534528	141388074996	14697175640928

TABLE 4. The values of $\dim\left(\mathsf{Pos}^{\ddagger}_k(n)\right)$ for $k \in [5]$ and $n \in [8]$.

Proof. We consider the map $\mathbf{Free}(\mathfrak{B}_k) \to \mathbf{Free}(\mathfrak{B})^{\otimes k}$ which sends $\mathfrak{b}$ to $\mathfrak{b}_1 \otimes \cdots \otimes \mathfrak{b}_k$. This map is compatible with the relations of $\mathsf{Pos}^{\ddagger}_k$ (Theorem 5.141) as well as the relations of $(\mathsf{Pos}^{\ddagger}_k)^!$ (Corollary 5.142) giving two maps

$$\mathsf{Pos}^{\ddagger}_k \to \mathsf{Pos}^{\ddagger \otimes k}_1 \qquad \text{and} \qquad (\mathsf{Pos}^{\ddagger}_k)^! \to (\mathsf{Pos}^{\ddagger}_1)^{! \otimes k}.$$

Thanks to Proposition 2.29, the statement amount to the fact that these two maps are injective. This is clear from the relations. □

Remark 5.145. The map $\mathsf{Pos}^{\ddagger}_k \to \mathsf{Pos}^{\ddagger \otimes k}_1$ of the preceeding proof can be seen combinatorially on k-posets. Indeed, it sends a k-poset $\leq_P$ of degree n to the tensor product $\leq^{(1)}_P \otimes \cdots \otimes \leq^{(k)}_P$ where $\leq^{(i)}_P$ is the restriction of $\leq_P$ to the set $\{1_i, \dots, n_i\}$ of the i-th copy of the integers of $[n]$. Thanks to the relation this map is injective. This is not obvious on posets.

5.2.6. *Hilbert series and numerology in the series k-poset operads.* We now explore some numerologic facts about the series k-poset operads $\mathsf{Pos}^{\ddagger}_k$ defined in Section 5.2.5. Thanks to Corollary 5.142, one can compute its Hilbert series by Koszul duality.

Corollary 5.146. *The Hilbert series $\mathcal{H}_{\mathsf{Pos}^{\ddagger}_k}(t)$ it the unique solution of the form $H = t + O(t^2)$ of the algebraic equation $H^3 - 2^k H^2 + H - t = 0$.*

It is well known that any algebraic univariate power series is holonomic (some author call them D-finite), which means that it satisfies a linear differential equation whose coefficients are in the field $\mathbb{C}(t)$ of rational functions over $\mathbb{C}$. Moreover, the differential equation can be automatically computed [FS09, Remark B.12]. In the situation of Corollary 5.146, this can actually be done in a more general setting by replacing 2^k by any number $K \geq 2$. Thus we set $H \coloneqq t + \sum_{n>1} u_n t^n$ to be the solution of the equation $H^3 - KH^2 + H - t = 0$. Then H is the solution of the differential equation

$$(27t^2 + 2Ct - D)\frac{\partial^2 H}{\partial t^2} + (27t + C)\frac{\partial H}{\partial t} - 3H + K = 0,$$

where $C = K(2K^2 - 9)$ and $D = K^2 - 4$ An extraction of coefficients shows that u_n satisfies the recurrence

$$D(n+1)(n+2)u_{n+2} = C(2n+1)(n+1)u_{n+1} + 3(3n-1)(3n+1)u_n,$$

for $n > 1$, together with the initial conditons $u_1 = 1$, $u_2 = K$. Table 4 gives the first values of $\dim\left(\mathsf{Pos}^{\ddagger}_k(n)\right)$. Note that in the special case when $K = 2$ (that is $k = 1$), we have $C = -2$ and $D = 0$, so that this recurrence degenerates into

$$2(2n+1)(n+1)u_{n+1} = 3(3n-1)(3n+1)u_n.$$

which yields the closed form

$$u_n = \frac{1}{n}\binom{3n-2}{n-1}.$$

Remark 5.147. This raise the open question whether there are interesting combinatorial operads whose Hilbert series corresponds to arbitrary integer $K \geq 2$.

Remark 5.148. The numerology presented here is also valid for the parallel 2-poset operad $\mathsf{Pos}^{\parallel}_2$ defined in Section 5.1.5 since the operads $\mathsf{Pos}^{\parallel}_2$ and $\mathsf{Pos}^{\ddagger}_2$ are isomorphic by Theorems 5.64 and 5.141.

5.2.7. *Series k-Zinbiel operads.* As in Section 5.1.6, we look for an operad $\mathsf{MZin}^{\ddagger}_k$ on k-permutations which contains the operad $\mathsf{MCit}^{\ddagger}_k$ as a suboperad generated in degree 2 and which moreover closes the following diagram of operad morphisms. We will also find a tidy version $\mathsf{TZin}^{\ddagger}_k$, closing the right square where the dashed arrows are not operad morphisms, but bijections of normal forms.

$$\begin{array}{ccccc} \mathsf{RP}^{\ddagger}_k & \xrightarrow{\text{LinExt}} & \mathsf{MZin}^{\ddagger}_k & \xleftarrow{\text{LexMin}} \dashrightarrow & \mathsf{TZin}^{\ddagger}_k \\ \uparrow & & \uparrow & & \uparrow \\ \mathsf{Pos}^{\ddagger}_k & \xrightarrow{\text{LinExt}} & \mathsf{MCit}^{\ddagger}_k & \xleftarrow{\text{LexMin}} \dashrightarrow & \mathsf{TCit}^{\ddagger}_k \end{array}$$

Messy series k-Zinbiel operad. We start with the messy version.

Definition 5.149. *Let $\sigma \in \mathsf{Perm}_k(m)$ and $\tau \in \mathsf{Perm}_k(n)$ be two k-permutations and $i \in [m]$. Write $\sigma = \omega_0\, i\, \omega_1\, i\, \omega_2 \cdots \omega_{k-1}\, i\, \omega_k$ and $\tau = \tau_1 \cdots \tau_k\, \theta$. Then the i-th composition of σ and μ is defined by*

$$\sigma \circ_i \tau = \omega_0[i,n]\, \tau_1[i-1]\, \omega_1[i,n]\, \tau_2[i-1]\, \omega_2[i,n] \cdots \omega_{k-1}[i,n]\, \tau_k[i-1]\, (\omega_k[i,n] \sqcup \theta[i-1]).$$

We extends this definition by linearity.

Here are some examples of compositions on k-permutations:

$$\begin{aligned} 31232144 \circ_1 313122 &= 5345413122 66 + 5345413126 26 + 534541312662 + \cdots + 534541361262 \\ &\quad + \cdots (15 \text{ terms}) \cdots + 534541636122 + 534541663122, \\ 31232144 \circ_2 313122 &= 514524233166 + 514524231366 + 514524231636 + \cdots + 514524123636 \\ &\quad + \cdots (35 \text{ terms}) \cdots + 534541636122 + 534541663122, \\ 31232144 \circ_3 313122 &= 512353442166 + 512353424166 + 512353421466 + \cdots + 512352341646 + \\ &\quad + \cdots (70 \text{ terms}) \cdots + 512321656344 + 512321665344, \\ 31232144 \circ_4 313122 &= 312321646455. \end{aligned}$$

The following two lemmas relate the compositions of k-permutations and the compositions of k-rooted k-posets. They play the same role for the composition as Lemma 5.130 played for the operations.

Lemma 5.150. *For any two k-permutations $\sigma \in \mathsf{Perm}_k(m)$ and $\tau \in \mathsf{Perm}_k(n)$ and $i \in [m]$,*

$$\sigma \circ_i \tau = \text{LinExt}\left(\le_\sigma \circ_i \le_\tau\right),$$

where $\le_\mu$ denotes the total k-rooted k-poset associated with the k-permutation μ, and the composition $\circ_i$ on the left is the composition on k-permutations of Definition 5.149, while the composition $\circ_i$ on the right is the composition on k-rooted k-posets of Definition 5.136.

Proof. This follows from Lemma 2.64 and the decomposition of the poset $\le_\sigma \circ_i \le_\tau$ as basic manipulations on posets. Details are left to the reader. □

Lemma 5.151. *For any two k-rooted k-posets $\le_M \in \mathsf{RP}^{\ddagger}_k(m)$ and $\le_N \in \mathsf{RP}^{\ddagger}_k(n)$ and $i \in [m]$,*

$$\text{LinExt}(\le_M) \circ_i \text{LinExt}(\le_N) = \text{LinExt}(\le_M \circ_i \le_N),$$

where the composition $\circ_i$ on the left is the composition on k-permutations of Definition 5.149, while the composition $\circ_i$ on the right is the composition on k-posets of Definition 5.136.

Proof. Fix three integers m, n, and $i \in [m]$. Remark that if ν is a permutation appearing in the composition $\sigma \circ_i \tau$ for some $\sigma \in \mathsf{Perm}_k(m)$ and $\tau \in \mathsf{Perm}_k(n)$, one can recover uniquely these two permutations from ν. As a consequence, $\text{LinExt}(\le_M) \circ_i \text{LinExt}(\le_N)$ cannot have multiplicities, so that we can argue by double inclusion. We conclude by Lemma 2.64. □

As a consequence, Definition 5.149 actually defines an operad composition.

Proposition 5.152. *The family $\left(\mathbb{K}\mathsf{Perm}_k(n)\right)_{n>0}$ endowed with the composition rules of Definition 5.149 defines an operad $\mathsf{MZin}^{\ddagger}_k$, called the messy series k-Zinbiel operad. Moreover,* LinExt *is a surjective operad morphism from $\mathsf{RP}^{\ddagger}_k$ to $\mathsf{MZin}^{\ddagger}_k$.*

Proof. Thanks to Lemmas 5.150 and 5.151, we have

$$\sigma \circ_i (\tau \circ_j \mu) = \mathrm{LinExt}\left(\leq_\sigma \circ_i (\leq_\tau \circ_j \leq_\mu)\right).$$

Using similar equalities for the other compositions, we prove the operad axioms. The morphism property is just Lemma 5.150. The surjectivity follows from $\mathrm{LinExt}(\leq_\sigma) = \sigma$. □

Proposition 5.153. *The operad* $\mathsf{MCit}_k^{\ddagger}$ *is the suboperad of* $\mathsf{MZin}_k^{\ddagger}$ *generated by the four elements* 1122*,* $1212 + 1221$*,* $2112 + 2121$ *and* 2211*.*

Proof. The generators are given by the linear extensions of the k-rooted k-posets of Figure 15:

$$\begin{array}{lcl} \mathrm{LinExt}(\leq_I \prec\prec \leq_I) = 1122, & & \mathrm{LinExt}(\leq_I \prec\succ \leq_I) = 1212 + 1221, \\ \mathrm{LinExt}(\leq_I \succ\prec \leq_I) = 2112 + 2121 & \text{and} & \mathrm{LinExt}(\leq_I \succ\succ \leq_I) = 2211. \end{array}$$ □

Tidy series k-Zinbiel operad. The goal of this section is to generalize to any k-permutations the composition formula of Proposition 5.103 for fully k-rooted cuttable k-permutations, as suggested by Remark 5.104.

Definition 5.154. *Let* $\sigma \in \mathsf{Perm}_k(m)$ *and* $\tau \in \mathsf{Perm}_k(n)$ *be two* k*-permutations and* $i \in [m]$*. Write* $\sigma = \omega_0\, i\, \omega_1\, i\, \omega_2 \cdots \omega_{k-1}\, i\, \mu\, \nu$ *where* μ *is the maximal factor such that* $\mu_p < i$ *for all* $p \in [|\mu|]$*. Therefore either* ν *is empty or* $i < \nu_1$*. Write moreover* $\tau = \tau_1 \cdots \tau_k\, \theta$*. Then the* i*-th composition of* σ *and* μ *is defined by*

$$\begin{aligned} \sigma \circ_i \tau &:= \omega_0[i,n]\, \tau_1[i-1]\, \omega_1[i,n]\, \tau_2[i-1]\, \omega_2[i,n] \cdots \omega_{k-1}[i,n]\, \tau_k[i-1]\, \mu[i,n]\, \theta[i-1]\, \nu[i,n] \\ &= \omega_0[i,n]\, \tau_1[i-1]\, \omega_1[i,n]\, \tau_2[i-1]\, \omega_2[i,n] \cdots \omega_{k-1}[i,n]\, \tau_k[i-1]\, \mu \qquad \theta[i-1]\, \nu[n-1]. \end{aligned}$$

Note that this is the same composition formula as in Proposition 5.103. Here are some examples of compositions on k-permutations:

$$312321|44 \circ_1 313122 = 534541312266,$$
$$312321|44 \circ_2 313122 = 514521423366,$$
$$312321|44 \circ_3 313122 = 512321534466,$$
$$31232144| \circ_4 313122 = 312321646455,$$

where we have materialized by a vertical bar the separation $\mu|\nu$.

Similar to Lemma 5.112, there is a relation between messy and tidy compositions.

Lemma 5.155. *For any two* k*-permutations* $\sigma \in \mathsf{Perm}_k(m)$ *and* $\tau \in \mathsf{Perm}_k(n)$ *and* $i \in [m]$*,*

$$\sigma \circ_i \tau = \mathrm{LexMin}(\sigma \circ_i \tau),$$

where the composition $\circ_i$ *on the left is the tidy composition on* k*-permutations of Definition 5.154, while the composition* $\circ_i$ *on the right is the messy composition on* k*-permutations of Definition 5.149.*

As a consequence, the composition rules of Definition 5.154 define an operad.

Proposition 5.156. *The family* $\left(\mathsf{Perm}_k(n)\right)_{n>0}$ *endowed with the composition rules of Definition 5.154 defines a non-symmetric set operad* $\mathsf{TZin}_k^{\ddagger}$*, called the* tidy series k-Zinbiel operad*. Moreover* $\mathsf{TCit}_k^{\ddagger}$ *is the suboperad of* $\mathsf{TZin}_k^{\ddagger}$ *given by fully* k*-rooted cuttable* k*-permutations.*

6. Series versus parallel

In this section, we show that the series and parallel circulation rules strongly disagree but are reconcilable.

6.1. Series and parallel are not isomorphic.

In this section, we study witnesses for the non-isomorphism of the four signaletic operads $\mathsf{MSig}^{\|}_k$, $\mathsf{TSig}^{\|}_k$, $\mathsf{MSig}^{\ddagger}_k$ and $\mathsf{TSig}^{\ddagger}_k$ for $k \geq 2$. These witnesses are isomorphism invariants that turn out to differ in these four operads. We have seen in Corollary 3.30 that these four operads have the same Hilbert series, so we need finer invariants. For this, we study two families of binary operations in these operads:

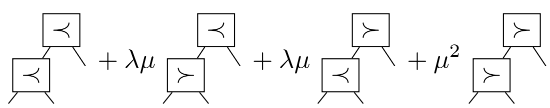
- *associative* operations such that $\mathfrak{a} \circ_1 \mathfrak{a} = \mathfrak{a} \circ_2 \mathfrak{a}$,
- *left-bipotent* (resp. *right-bipotent*) operations such that $\mathfrak{a} \circ_1 \mathfrak{a} = \mathfrak{o}_3$ (resp. $\mathfrak{a} \circ_2 \mathfrak{a} = \mathfrak{o}_3$), where $\mathfrak{o}_n$ is the zero operation of arity n.

6.1.1. *The $k = 1$ case.* As a warm-up, we start with the $k = 1$ case. In this case, there is no distinction between the parallel and series situations, and we therefore abbreviate $\mathsf{MSig}_1 = \mathsf{MSig}^{\|}_1 = \mathsf{MSig}^{\ddagger}_1 = \mathsf{Diass}$ and $\mathsf{TSig}_c = \mathsf{TSig}^{\|}_c = \mathsf{TSig}^{\ddagger}_c = \mathsf{Dup}^{!}_c$ for $c \in \{\prec, \odot \succ\}$. We first express explicitly the two self-compositions of a generic binary operation in these four operads.

Lemma 6.1. *The self-compositions of a generic operation $\mathfrak{a} := \lambda \prec + \mu \succ$ (with $\lambda, \mu \in \mathbb{K}$) in the 1-signaletic operads MSig_1, $\mathsf{TSig}_\prec$, $\mathsf{TSig}_\odot$ and $\mathsf{TSig}_\succ$ extends in terms of destination vectors as*

	$\mathfrak{a} \circ_1 \mathfrak{a}$	$\mathfrak{a} \circ_2 \mathfrak{a}$
in MSig_1	$\lambda^2 \llparenthesis 1 \rrparenthesis_3 + \lambda\mu \llparenthesis 2 \rrparenthesis_3 + (\lambda\mu + \mu^2) \llparenthesis 3 \rrparenthesis_3$	$(\lambda^2 + \lambda\mu) \llparenthesis 1 \rrparenthesis_3 + \lambda\mu \llparenthesis 2 \rrparenthesis_3 + \mu^2 \llparenthesis 3 \rrparenthesis_3$
in $\mathsf{TSig}_\prec$	$\lambda^2 \llparenthesis 1 \rrparenthesis_3 + \lambda\mu \llparenthesis 2 \rrparenthesis_3 + \lambda\mu \llparenthesis 3 \rrparenthesis_3$	$\lambda^2 \llparenthesis 1 \rrparenthesis_3 + \lambda\mu \llparenthesis 2 \rrparenthesis_3 + \mu^2 \llparenthesis 3 \rrparenthesis_3$
in $\mathsf{TSig}_\odot$	$\lambda^2 \llparenthesis 1 \rrparenthesis_3 + \lambda\mu \llparenthesis 2 \rrparenthesis_3 + \mu^2 \llparenthesis 3 \rrparenthesis_3$	$\lambda^2 \llparenthesis 1 \rrparenthesis_3 + \lambda\mu \llparenthesis 2 \rrparenthesis_3 + \mu^2 \llparenthesis 3 \rrparenthesis_3$
in $\mathsf{TSig}_\succ$	$\lambda^2 \llparenthesis 1 \rrparenthesis_3 + \lambda\mu \llparenthesis 2 \rrparenthesis_3 + \mu^2 \llparenthesis 3 \rrparenthesis_3$	$\lambda\mu \llparenthesis 1 \rrparenthesis_3 + \lambda\mu \llparenthesis 2 \rrparenthesis_3 + \mu^2 \llparenthesis 3 \rrparenthesis_3$

Proof. We just need to expand the composition and compute the corresponding destination vectors. For instance, in MSig_1 we have



$$\mathfrak{a} \circ_1 \mathfrak{a} = \lambda^2 [\cdot] + \lambda\mu [\cdot] + \lambda\mu [\cdot] + \mu^2 [\cdot] = \lambda^2 \llparenthesis 1 \rrparenthesis_3 + \lambda\mu \llparenthesis 2 \rrparenthesis_3 + (\lambda\mu + \mu^2) \llparenthesis 3 \rrparenthesis_3.$$

The proof is identical for the seven other compositions. □

By coefficients comparisons, we immediately derive from Lemma 6.1 the following statements.

Proposition 6.2. *Up to scalar multiples, the only associative operations are precisely*

- *the operations $\prec$ and $\succ$ in the messy 1-signaletic operad MSig_1,*
- *the operations $\prec$ and $\prec + \succ$ in the $\prec$-tidy signaletic operad $\mathsf{TSig}_\prec$,*
- *any operation in the $\odot$-tidy signaletic operad $\mathsf{TSig}_\odot$ (i.e. dual duplicial operad $\mathsf{Dup}^{!}$),*
- *the operations $\succ$ and $\prec + \succ$ in the $\succ$-tidy signaletic operad $\mathsf{TSig}_\succ$.*

Proposition 6.3. *Up to scalar multiples, the only non-trivial left-bipotent or right-bipotent operations in the 1-signaletic operads MSig_1, $\mathsf{TSig}_\prec$, $\mathsf{TSig}_\odot$ and $\mathsf{TSig}_\succ$ are the following:*

- *the operation $\succ$ is left-bipotent in the $\prec$-tidy signaletic operad $\mathsf{TSig}_\prec$,*
- *the operation $\prec$ is right-bipotent in the $\succ$-tidy signaletic operad $\mathsf{TSig}_\succ$.*

As the dimensions of the spaces of associative, left-bipotent and right-bipotent operations are operad isomorphism invariants, Propositions 6.2 and 6.3 enable to distinguish between the messy and tidy 1-signaletic operads.

Corollary 6.4. *The messy and tidy 1-signaletic operads MSig_1, $\mathsf{TSig}_\prec$, $\mathsf{TSig}_\odot$ and $\mathsf{TSig}_\succ$ are not isomorphic.*

Remark 6.5. By Koszul duality, Corollary 6.4 implies a similar statement on the 1-citelangis operads. In contrast to our analysis on 1-signaletic operads, we note that the spaces of associative, left-bipotent and right-bipotent would not suffice to distinguish the 1-citelangis operads.

6.1.2. *The $k = 2$ case.* To prepare the general case, we now consider the $k = 2$ situation. Note that since the Manin product is commutative up to operad isomorphism, the tidy parallel 2-signaletic operads $\mathsf{TSig}^{\parallel}_{cd} = \mathsf{Dup}^{!}_{c} \,\square\, \mathsf{Dup}^{!}_{d}$ and $\mathsf{TSig}^{\parallel}_{dc} = \mathsf{Dup}^{!}_{d} \,\square\, \mathsf{Dup}^{!}_{c}$ are isomorphic for any $c, d \in \{\prec, \odot, \succ\}$. We show below that these are the only isomorphisms between the 2-signaletic operads. Our argument is again based on the associative, left-bipotent and right-bipotent binary operations in these operads.

Proposition 6.6. *All associative operations in the 2-signaletic operads are given in Figure 20.*

Proof. Consider a generic associative operation $\mathfrak{a} := \lambda \prec\prec + \mu \prec\succ + \nu \succ\prec + \omega \succ\succ$ with $\lambda, \mu, \nu, \omega \in \mathbb{K}$ in the messy parallel 2-signaletic operad $\mathsf{MSig}^{\parallel}_{2}$. Comparing the coefficients of the destination vectors that appear in the expansions of $\mathfrak{a} \circ_1 \mathfrak{a}$ and $\mathfrak{a} \circ_2 \mathfrak{a}$, we obtain the following 9 equations:

$$\begin{aligned} \lambda(\lambda+\mu+\nu+\omega) &= \lambda\lambda, & \mu(\lambda+\nu) &= \mu\lambda, & \mu(\mu+\omega) &= (\lambda+\mu)\mu, \\ \nu(\lambda+\mu) &= \nu\lambda, & \omega\lambda &= \omega\lambda, & \omega\mu &= (\nu+\omega)\mu, \\ \nu(\nu+\omega) &= (\lambda+\nu)\nu, & \omega\nu &= (\mu+\omega)\nu, & \omega\omega &= (\lambda+\mu+\nu+\omega)\omega. \end{aligned}$$

Up to scaling, this leads to 6 solutions for $(\lambda, \mu, \nu, \omega)$:

$$(1,0,0,0), \quad (0,1,0,0), \quad (0,0,1,0), \quad (0,0,0,1), \quad (1,-1,0,1), \quad \text{and} \quad (1,0,-1,1).$$

The proof is identical for the other cases. □

Proposition 6.7. *All left-bipotent and right-bipotent operations in the 2-signaletic operads are given in Figure 20.*

Proof. They are again obtained by comparisons of the coefficients of the destination vectors in the expansions of $\mathfrak{a} \circ_1 \mathfrak{a}$ and $\mathfrak{a} \circ_2 \mathfrak{a}$. □

As the dimensions of the spaces of associative, left-bipotent, and right-bipotent operations are operad isomorphism invariants, Propositions 6.6 and 6.7 enable to distinguish between the parallel and series and the messy and tidy 2-signaletic operads.

Corollary 6.8. *Among all 2-signaletic operads of Figure 20, the only isomorphisms are*

$$\mathsf{TSig}^{\parallel}_{\prec\odot} \simeq \mathsf{TSig}^{\parallel}_{\odot\prec}, \qquad \mathsf{TSig}^{\parallel}_{\prec\succ} \simeq \mathsf{TSig}^{\parallel}_{\succ\prec} \qquad \textit{and} \qquad \mathsf{TSig}^{\parallel}_{\odot\succ} \simeq \mathsf{TSig}^{\parallel}_{\succ\odot}.$$

Remark 6.9. By Koszul duality, Corollary 6.8 implies a similar statement on the 2-citelangis operads. In contrast to our analysis on 2-signaletic operads, we note that the spaces of associative, left-bipotent and right-bipotent would not suffice to distinguish the 2-citelangis operads.

6.1.3. *The $k \geq 3$ case.* As k grows, the description of the associative, left-bipotent and right-bipotent operations become more and more complicated. We now collect some relevant observations about these operations.

Associative operations in parallel. For the parallel situation, we can make use of the insertion operators $\mathsf{Ins}^{\mathfrak{a}}_{i}$ of Remarks 3.9 and 3.16 to create associative and bipotent operators in arity k from those in arity ℓ for $\ell < k$. The following two statements are immediate from the fact that these insertion operators are operad morphisms.

Proposition 6.10. *For any $k \geq 1$ and $i \in [k+1]$, if $\mathfrak{a}$ is associative in $\mathsf{MSig}^{\parallel}_{k}$, then $\mathsf{Ins}^{\prec}_{i}(\mathfrak{a})$ and $\mathsf{Ins}^{\succ}_{i}(\mathfrak{a})$ are associative in $\mathsf{MSig}^{\parallel}_{k+1}$. Therefore, any operation of $\{\prec, \succ\}^{k}$ is associative in $\mathsf{MSig}^{\parallel}_{k}$.*

Proposition 6.11. *For any $\mathbf{c} \in \{\prec, \odot, \succ\}^{k}, \mathbf{d} \in \{\prec, \odot, \succ\}^{\ell}$ and $\mathfrak{a} \in \{\prec, \succ\}^{k}, \mathfrak{b} \in \{\prec, \succ\}^{\ell}$, if the operation $\mathfrak{a}\mathfrak{b}$ is associative in $\mathsf{TSig}^{\parallel}_{\mathbf{cd}}$, then $\mathfrak{a}{\prec}\mathfrak{b}$ and $\mathfrak{a}{\prec}\mathfrak{b} + \mathfrak{a}{\succ}\mathfrak{b}$ (resp. $\mathfrak{a}{\prec}\mathfrak{b}$ and $\mathfrak{a}{\succ}\mathfrak{b}$, resp. $\mathfrak{a}{\succ}\mathfrak{b}$ and $\mathfrak{a}{\prec}\mathfrak{b} + \mathfrak{a}{\succ}\mathfrak{b}$) are associative in $\mathsf{TSig}^{\parallel}_{\mathbf{c}\prec\mathbf{d}}$ (resp. in $\mathsf{TSig}^{\parallel}_{\mathbf{c}\odot\mathbf{d}}$, resp. in $\mathsf{TSig}^{\parallel}_{\mathbf{c}\succ\mathbf{d}}$).*

It would be interesting to characterize which operations in $\mathsf{MSig}^{\parallel}_{k}$ and $\mathsf{TSig}^{\parallel}_{\mathbf{c}}$ are non-trivially associative, in the sense that they do not arise as consequences of Propositions 6.10 and 6.11. For instance, when $k = 2$, the only non-trivially associative operations in parallel 2-signaletic operads are up to symmetries:

- the operations $\prec\prec - \prec\succ + \succ\succ$ and $\prec\prec - \succ\prec + \succ\succ$ in $\mathsf{MSig}^{\parallel}_{2}$,

Figure 20. The spaces of associative, left-bipotent and right-bipotent operations in all 2-signaletic operads.

	associative operations	left-bipotent operations	right-bipotent operations
in $\mathsf{MSig}_2^{\parallel}$	$\mathbb{K}\prec\prec$, $\mathbb{K}\prec\succ$, $\mathbb{K}\succ\prec$, $\mathbb{K}\succ\succ$, $\mathbb{K}(\prec\prec - \prec\succ + \succ\succ)$ and $\mathbb{K}(\prec\prec - \succ\prec + \succ\succ)$	$\{\mathfrak{o}_2\}$	$\{\mathfrak{o}_2\}$
in $\mathsf{TSig}^{\parallel}_{\prec\prec}$	$\mathbb{K}\prec\prec$, $\mathbb{K}(\prec\prec + \prec\succ)$, $\mathbb{K}(\prec\prec + \succ\prec)$, $\mathbb{K}(\prec\prec + \succ\succ)$, and $\mathbb{K}(\prec\prec + \prec\succ + \succ\prec + \succ\succ)$	$\mathbb{K}\prec\succ + \mathbb{K}\succ\succ$ and $\mathbb{K}\succ\prec + \mathbb{K}\succ\succ$	$\{\mathfrak{o}_2\}$
in $\mathsf{TSig}^{\parallel}_{\prec\odot}$	$\mathbb{K}\prec\prec + \mathbb{K}\prec\succ$, $\mathbb{K}\prec\prec + \mathbb{K}(\prec\succ + \succ\succ)$, and $\mathbb{K}(\prec\prec + \succ\prec) + \mathbb{K}(\prec\succ + \succ\succ)$	$\mathbb{K}\succ\prec + \mathbb{K}\succ\succ$	$\{\mathfrak{o}_2\}$
in $\mathsf{TSig}^{\parallel}_{\prec\succ}$	$\mathbb{K}\prec\succ$, $\mathbb{K}(\prec\succ + \succ\succ)$, $\mathbb{K}(\prec\prec + \prec\succ)$, and $\mathbb{K}\succ\prec + \mathbb{K}(\prec\prec + \prec\succ + \succ\succ)$	$\mathbb{K}\succ\prec + \mathbb{K}\succ\succ$	$\mathbb{K}\prec\prec + \mathbb{K}\succ\prec$
in $\mathsf{TSig}^{\parallel}_{\odot\prec}$	$\mathbb{K}\prec\prec + \mathbb{K}\succ\prec$, $\mathbb{K}\prec\prec + \mathbb{K}(\succ\prec + \succ\succ)$, and $\mathbb{K}(\prec\prec + \prec\succ) + \mathbb{K}(\succ\prec + \succ\succ)$	$\mathbb{K}\prec\succ + \mathbb{K}\succ\succ$	$\{\mathfrak{o}_2\}$
in $\mathsf{TSig}^{\parallel}_{\odot\odot}$	$\mathbb{K}\prec\prec + \mathbb{K}\prec\succ + \mathbb{K}\succ\prec + \mathbb{K}\succ\succ$	$\{\mathfrak{o}_2\}$	$\{\mathfrak{o}_2\}$
in $\mathsf{TSig}^{\parallel}_{\odot\succ}$	$\mathbb{K}\prec\succ + \mathbb{K}\succ\succ$, $\mathbb{K}(\prec\prec + \prec\succ) + \mathbb{K}\succ\succ$, and $\mathbb{K}(\prec\prec + \prec\succ) + \mathbb{K}(\succ\prec + \succ\succ)$	$\{\mathfrak{o}_2\}$	$\mathbb{K}\prec\prec + \mathbb{K}\succ\prec$
in $\mathsf{TSig}^{\parallel}_{\succ\prec}$	$\mathbb{K}\succ\prec$, $\mathbb{K}(\prec\prec + \succ\prec)$, $\mathbb{K}(\succ\prec + \succ\succ)$, and $\mathbb{K}(\prec\prec + \succ\prec + \succ\succ) + \mathbb{K}\prec\succ$	$\mathbb{K}\prec\succ + \mathbb{K}\succ\succ$	$\mathbb{K}\prec\prec + \mathbb{K}\prec\succ$
in $\mathsf{TSig}^{\parallel}_{\succ\odot}$	$\mathbb{K}\succ\prec + \mathbb{K}\succ\succ$, $\mathbb{K}(\prec\prec + \succ\prec) + \mathbb{K}\succ\succ$, and $\mathbb{K}(\prec\prec + \succ\prec) + \mathbb{K}(\prec\succ + \succ\succ)$	$\{\mathfrak{o}_2\}$	$\mathbb{K}\prec\prec + \mathbb{K}\prec\succ$
in $\mathsf{TSig}^{\parallel}_{\succ\succ}$	$\mathbb{K}\succ\succ$, $\mathbb{K}(\prec\prec + \succ\succ)$, $\mathbb{K}(\prec\succ + \succ\succ)$, $\mathbb{K}(\succ\prec + \succ\succ)$ and $\mathbb{K}(\prec\prec + \prec\succ + \succ\prec + \succ\succ)$	$\{\mathfrak{o}_2\}$	$\mathbb{K}\prec\prec + \mathbb{K}\prec\succ$ and $\mathbb{K}\prec\prec + \mathbb{K}\succ\prec$
in $\mathsf{MSig}_2^{\ddagger}$	$\mathbb{K}\prec\prec$ and $\mathbb{K}\succ\succ$	$\mathbb{K}(\succ\prec - \succ\succ)$	$\mathbb{K}(\prec\prec - \prec\succ)$
in $\mathsf{TSig}_2^{\ddagger}$	$\mathbb{K}\prec\prec$, $\mathbb{K}(\prec\prec + \succ\succ)$, and $\mathbb{K}(\prec\prec + \succ\prec + \succ\succ) + \mathbb{K}\prec\succ$	$\mathbb{K}\prec\succ + \mathbb{K}\succ\succ$	$\mathbb{K}\prec\succ$

- the operation $\prec\prec + \succ\succ$ in $\mathsf{TSig}^{\parallel}_{\prec\prec}$,
- all operations of $\mathbb{K}\succ\prec + \mathbb{K}(\prec\prec + \prec\succ + \succ\succ)$ in $\mathsf{TSig}^{\parallel}_{\prec\succ}$.

Associative operations in series. For the series situation, we have no simple way to construct associative operations. We can however make the following observations, whose proofs are immediate on destination vectors.

Proposition 6.12. *For any $k \ge 1$, the operations $\prec^k$ and $\succ^k$ are associative in $\mathsf{MSig}^{\ddagger}_k$. All other operations of $\{\prec,\succ\}^k$ are not associative.*

Proposition 6.13. *For any $k \ge 1$, the operations $\prec^k$ and $\prec^k + \succ^k$ are associative in $\mathsf{TSig}^{\ddagger}_k$.*

Note that there are other associative operations in the series k-signaletic operads. For instance:

- the associative operations in the tidy series 3-signaletic operad $\mathsf{TSig}^{\ddagger}_3$ are given by:
$$\mathbb{K}\prec\prec\prec, \qquad \mathbb{K}\succ\succ\succ, \qquad \text{and} \qquad \mathbb{K}(\succ\prec\prec - \succ\prec\succ) + \mathbb{K}(\prec\succ\prec - \prec\succ\succ).$$
In fact, the compositions of any two operations in $\mathbb{K}(\succ\prec\prec - \succ\prec\succ) + \mathbb{K}(\prec\succ\prec - \prec\succ\succ)$ vanish.
- the associative operations in the tidy series 3-signaletic operad $\mathsf{TSig}^{\ddagger}_3$ are given by:
$$\mathbb{K}\prec\prec\prec, \qquad \mathbb{K}(\prec\prec\prec + \succ\succ\succ), \qquad \mathbb{K}(\prec\prec\prec + \succ\prec\prec + \succ\succ\prec + \succ\succ\succ),$$
$$\mathbb{K}\prec\prec\succ + \mathbb{K}\prec\succ\succ + \mathbb{K}\succ\prec\succ, \qquad \mathbb{K}(\prec\prec\prec + \succ\succ\prec) + \mathbb{K}\prec\succ\succ + \mathbb{K}\succ\prec\succ, \qquad \text{and}$$
$$\big\{\lambda(\prec\prec\prec + \succ\prec\prec + \succ\succ\prec + \succ\succ\succ) + \mu(\prec\prec\succ + \prec\succ\succ) \pm \sqrt{\lambda\mu}(\prec\succ\prec + \succ\prec\succ) \mid \lambda, \mu \in \mathbb{K}\big\}.$$
Note that these are not anymore linear spaces.

Messy parallel and messy series are not isomorphic. We finally prove that the messy parallel and series k-signaletic operads are not isomorphic, studying their left-bipotent and right-bipotent operations. On the one hand, we observe that the messy parallel k-signaletic operad have no such operations.

Proposition 6.14. *The messy parallel k-signaletic operad $\mathsf{MSig}^{\parallel}_k$ has no non-trivial left-bipotent or right-bipotent operation.*

Proof. The proof works by induction on k and heavily relies on Remark 3.22. The result was already stated in Proposition 6.3 when $k = 1$ and in Proposition 6.7 and Figure 20 when $k = 2$. Assume now that we have proved the statement for the parallel $(k-1)$-signaletic operad. Consider a generic left-bipotent binary operation $\mathfrak{a} := \sum_{\mathfrak{b}\in\{\prec,\succ\}^k} \lambda_{\mathfrak{b}}\, \mathfrak{b}$ of the messy parallel k-signaletic operad. We observe that

- $\lambda_{\prec^k} = 0$ since the coefficient of $(\!| 1^{\{k\}} |\!)_3$ in $\mathfrak{a} \circ_1 \mathfrak{a}$ is $(\lambda_{\prec^k})^2$.
- for any $i \in [k]$, applying the restriction morphism to the positions $[k] \smallsetminus \{i\}$ as defined in Remark 3.22, we obtain an operator
$$\mathfrak{a}' := \mathsf{Res}_{[k]\smallsetminus\{i\}}(\mathfrak{a}) = \sum_{\substack{\mathfrak{c}\in\{\prec,\succ\}^{i-1} \\ \mathfrak{d}\in\{\prec,\succ\}^{k-i}}} \big(\lambda_{\mathfrak{c}\prec\mathfrak{d}} + \lambda_{\mathfrak{c}\succ\mathfrak{d}}\big)\mathfrak{c}\mathfrak{d}$$
of the parallel $(k-1)$-signaletic operad satisfying $\mathfrak{a}' \circ_1 \mathfrak{a}' = \mathfrak{o}_3$. This shows that any two operators of $\{\prec,\succ\}^k$ which only differ at position $i \in [k]$ have the opposite coefficients in $\mathfrak{a}$.

We conclude that all coefficients of $\mathfrak{a}$ vanish, so that the only left-bipotent operation in $\mathsf{MSig}^{\parallel}_k$ is the zero operation $\mathfrak{o}_2$. The proof is symmetric for right-bipotent operations. □

In contrast, we observe that the messy parallel k-signaletic operad has (many) non-trivial left-bipotent and right-bipotent operations. This requires the following observation, whose immediate proof is left to the reader.

Lemma 6.15. *For any two binary operations $\mathfrak{a}$ and $\mathfrak{b}$ of the k-signaletic operad and any $\ell, r \ge 0$,*

- *the destination vector of* $\left(\prec^{\ell}\mathfrak{a}\succ^{r}\right)\circ_1\left(\prec^{\ell}\mathfrak{b}\succ^{r}\right)$ *is obtained from the destination vector of* $\mathfrak{a}\circ_1\mathfrak{b}$ *by adding the prefix* $1^{\{\ell\}}$ *and the suffix* $3^{\{r\}}$*,*
- *the destination vector of* $\left(\succ^{\ell}\mathfrak{a}\prec^{r}\right)\circ_2\left(\succ^{\ell}\mathfrak{b}\prec^{r}\right)$ *is obtained from the destination vector of* $\mathfrak{a}\circ_2\mathfrak{b}$ *by adding the prefix* $3^{\{\ell\}}$ *and the suffix* $1^{\{r\}}$*.*

Proposition 6.16. *For any* $\ell, r\in\mathbb{N}$*, the binary operation* $\prec^{\ell}\succ\prec\succ^{r} - \prec^{\ell}\succ\succ\succ^{r}$ *is left-bipotent and the operation* $\succ^{\ell}\prec\succ\prec^{r} - \succ^{\ell}\prec\prec\prec^{r}$ *is right-bpotent in the messy series* k*-signaletic operad* $\mathsf{MSig}^{\ddagger}_{k}$*.*

Proof. This was already observed in Proposition 6.7 and Figure 20 when $p = q = 0$. The general case follows by Lemma 6.15. □

From Propositions 6.14 and 6.16, we managed to distinguished the series and parallel messy signaletic operads.

Corollary 6.17. *For any* $k\geq 1$*, the messy parallel and series* k*-signaletic operads* $\mathsf{MSig}^{\parallel}_{k}$ *and* $\mathsf{MSig}^{\ddagger}_{k}$ *are not isomorphic.*

Based on computer experiments, we conjecture that there are in fact no non-trivial isomorphisms between all our k-signaletic operads.

Conjecture 6.18. *For any* $k\geq 1$*, the only isomorphisms among all* k*-signaletic operads are of the form*

$$\mathsf{TSig}^{\parallel}_{\mathbf{c}}\simeq\mathsf{TSig}^{\parallel}_{\sigma(\mathbf{c})}$$

for some $\mathbf{c}\in\{\prec,\odot,\succ\}^{k}$ *and* $\sigma\in\mathfrak{S}_k$*.*

Remark 6.19. By Koszul duality, we have shown that the messy parallel and series k-citelangis operads $\mathsf{MCit}^{\parallel}_{k}$ and $\mathsf{MCit}^{\ddagger}_{k}$ are not isomorphic.

6.2. **Parallel-series and series-parallel operads.** In this section, we briefly indicate how to reconcile the series and parallel circulation rules.

6.2.1. *Parallel-series operads.* First of all, it is not difficult to define parallel-series operads. The input is a sequence of integers $\mathbf{k} = (k_1,\dots,k_\ell)$ whose sum is denoted by $|\mathbf{k}|$. We then consider syntax trees on $\mathfrak{B}_{|\mathbf{k}|}\simeq\mathfrak{B}_{k_1}\otimes\cdots\otimes\mathfrak{B}_{k_\ell}$, where each node is a traffic signal decomposed into ℓ blocks of sizes $k_1,\dots,k_\ell$. As usual, $|\mathbf{k}|$ cars drive through the syntax tree, starting at its root, and following the traffic signals at each node. But the cars are colored with ℓ colors: there are k_i cars colored i for all $i\in[\ell]$. Cars of the same color follow the series traffic rule on their block: the k_i cars of color i start one after the others and follow and erase the first remaining traffic signal of the i-th block in each node, never crossing barriers between blocks. This defines two ***k**-signaletic parallel-series operads* $\mathsf{MSig}^{\parallel\ddagger}_{\mathbf{k}}$ and $\mathsf{TSig}^{\parallel\ddagger}_{\mathbf{k}}$ according to the ordering rules of the traffic signals not traversed by a car. The complete Section 3.4 on Hilbert series and Kozulity applies with the corresponding definition for right **k**-signaletic combs.

Theorem 6.20. *The* $\mathbf{k}$*-signaletic parallel-series operads* $\mathsf{MSig}^{\parallel\ddagger}_{\mathbf{k}}$ *and* $\mathsf{TSig}^{\parallel\ddagger}_{\mathbf{k}}$ *are quadratic and Koszul with Hilbert series* $\sum_{p\geq 1} p^{|\mathbf{k}|}t^p$*.*

As a consequence, we can consider their dual ***k**-citelangis parallel-series operads* $\mathsf{MCit}^{\parallel\ddagger}_{\mathbf{k}}$ and $\mathsf{TCit}^{\parallel\ddagger}_{\mathbf{k}}$ which share the same Hilbert series with $\mathsf{Cit}_{|\mathbf{k}|}$. The definition clearly translates into the following description of these operads in terms of Manin products.

Proposition 6.21. *The* $\mathbf{k}$*-signaletic and* $\mathbf{k}$*-citelangis parallel-series operads are Manin products of series signaletic and citelangis operads:*

$$\mathsf{MSig}^{\parallel\ddagger}_{\mathbf{k}} = \mathop{\square}_{i\in[\ell]}\mathsf{MSig}^{\ddagger}_{k_i},\quad \mathsf{TSig}^{\parallel\ddagger}_{\mathbf{k}} = \mathop{\square}_{i\in[\ell]}\mathsf{TSig}^{\ddagger}_{k_i},\quad \mathsf{MCit}^{\parallel\ddagger}_{\mathbf{k}} = \mathop{\blacksquare}_{i\in[\ell]}\mathsf{MCit}^{\ddagger}_{k_i}\quad\text{and}\quad \mathsf{TCit}^{\parallel\ddagger}_{\mathbf{k}} = \mathop{\blacksquare}_{i\in[\ell]}\mathsf{TCit}^{\ddagger}_{k_i}.$$

The particular case where $\mathbf{k} = (k,\ell)$ is of length 2 has interesting combinatorial properties. It acts on $k+\ell$ multipermutations where the first k signals chose the first k letters and the last ℓ signals chose the last ℓ letters of the shuffles. Most of all the results of Section 5 can be adapted in this setting.

Theorem 6.22. *The algebra* $\mathsf{FQSym}_{k+\ell}$ *can be endowed with both messy and tidy parallel-series* (k,ℓ)*-citelangis algebra structures which are both free.*

There is also a notion of (k,ℓ)-rooted bounded posets and the messy and tidy (k,ℓ)-Zinbiel operads, getting the following diagram:

$$\begin{array}{ccccc} \mathsf{BP}^{\|}_{k,\ell} & \xrightarrow{\text{LinExt}} & \mathsf{MZin}^{\|}_{k,\ell} & \overset{\text{LexMin}}{\hookleftarrow\dashrightarrow} & \mathsf{TZin}^{\|}_{k,\ell} \\ \uparrow & & \uparrow & & \uparrow \\ \mathsf{Pos}^{\|}_{k,\ell} & \xrightarrow{\text{LinExt}} & \mathsf{MCit}^{\|}_{k,\ell} & \overset{\text{LexMin}}{\hookleftarrow\dashrightarrow} & \mathsf{TCit}^{\|}_{k,\ell} \end{array}$$

Details are left to the reader.

6.2.2. *Series-parallel operads.* Under appropriate conditions, we can also adapt the series setting to work with blocks of parallel operads. In a first version, the input is a decreasing sequence of integers $\mathbf{k} = (k_1 \geq \cdots \geq k_\ell)$ whose sum is denoted by $|\mathbf{k}|$. We consider again syntax trees on $\mathfrak{B}_{|\mathbf{k}|} \simeq \mathfrak{B}_{k_1} \otimes \cdots \otimes \mathfrak{B}_{k_\ell}$, where each node is a traffic signal decomposed into ℓ blocks of sizes $k_1, \ldots, k_\ell$. As before, $|\mathbf{k}|$ cars drive through the syntax tree, starting at its root, and following the traffic signals at each node. But the traffic rule is now as follows. The first k_1 cars start in parallel at the root and follow the signals in parallel in the first block. After this first round, we erase the entire block of k_1 letters at each node that has been traversed by at least one car. The next k_2 cars start again in parallel from the root and follow the signals in parallel in the first remaining block. Note that in the nodes not traversed in the first round, the cars of the second round are reading traffic signal from the first block. This causes no problem since the size of the first block is at least k_2. Again, we erase the entire block (now of size k_1 or k_2) at each node that has been traversed by at least one car. We repeat until the ℓ-th round. This defines two $\mathbf{k}$-*signaletic series-parallel operads* $\mathsf{MSig}^{\ddagger\|}_{\mathbf{k}}$ and $\mathsf{TSig}^{\ddagger\|}_{\mathbf{k}}$. Again, the complete Section 3.4 on Hilbert series and Koszulity applies with the corresponding definition for right $\mathbf{k}$-signaletic combs.

Theorem 6.23. *The* $\mathbf{k}$*-signaletic series-parallel operads* $\mathsf{MSig}^{\ddagger}_{\mathbf{k}}$ *and* $\mathsf{TSig}^{\ddagger}_{\mathbf{k}}$ *are quadratic and Koszul with hilbert series* $\sum_{p\geq 1} p^{|\mathbf{k}|} t^p$*.*

As a consequence, we can consider their dual $\mathbf{k}$-*citelangis series-parallel operads* $\mathsf{MCit}^{\ddagger\|}_{\mathbf{k}}$ and $\mathsf{TCit}^{\ddagger\|}_{\mathbf{k}}$ which share the same Hilbert series with $\mathsf{Cit}_{|\mathbf{k}|}$. However it is not clear how these $\mathbf{k}$-citelangis series-parallel operads can act on multipermutations. Finally, thanks to Remark 3.9, we can release the decreasing condition on $\mathbf{k}$, by adding some $\prec$ or $\succ$ (not changing the operad) to comply to the decreasing assumption.

7. Open questions

This works raises a lot of open questions. We present some of them below.

7.1. **A general notion of series Manin products?** All the parallel constructions presented here are k-th powers of certain operads for Manin products. The similarities between the parallel and series situations suggest that the series constructions could be k-th powers of the same operads for a general twisted Manin product defined for any non-symmetric operads or only in some specific cases. It would be interesting to have a general construction or at least some more examples with a similar behavior.

7.2. **Symmetric k-Zinbiel operads?** In this paper, we did not provide a thorough analysis of the k-Zinbiel operads and in particular the messy ones. The main reason is that we think that they should be investigated as symmetric operads. Indeed, in the $k = 1$ case, the non-symmetric suboperad generated in degree 2 within the Zinbiel operad is the dendriform operad. One has to use the symmetric groups action to generate all permutations. For larger values of k, the series k-citelangis operad is generated by a strict subspace of degree 2. For example for $k = 2$, there are four generators but six 2-permutations. So it may be interesting to investigate the suboperad generated by the full component of degree 2. However, we think that an in depth study of the series k-Zinbiel operad as a symmetric operad might be much more interesting. Is it Koszul? What is its dual (some kind of k-Leibniz operad)?

7.3. **k-tridendriform operads?** A straightforward extension of the signaletic interpretation of the diassociative operad is obtained by adding a splitting signal $\perp$. When a car reads a signal $\perp$, it splits into two cars, one following the left branch and the other one following the right branch. In this setting, for a tree of arity n, each destination is replaced by a non-empty subset of $[n]$ (because at least one car arrives at the leaves of the tree). This operad was already considered by J.-L. Loday and M. Ronco under the name of triassociative operad [LR04]. It is quadratic, Koszul and its dual is knows as the tridendriform operad. Typical tridendriform algebras include quasi-shuffle algebras and in particular the algebra of quasi-symmetric functions. Its usual Manin square was considered by P. Leroux under the name of ennea operad [Ler04].

It is very likely that both parallel and series stories generalize in this setting. Moreover, J.-L. Loday and M. Ronco considered the triduplicial operad which after some adaptation should provide the base case for the tidy setting. With respect to action and combinatorial realization, the analogue of FQSym and the Zinbiel operad should be provided by a suitable k-version of the algebra of word quasi-symmetric function WQSym [DHT02, MNT13] where permutations are replaced by packed words or equivalently ordered set partitions.

Finally, a similar extension would be to add a stoping signal $\ominus$. A car that reads a signal $\ominus$ stops, so that the destination set of each car can now be empty. It would be worth to study the parallel and series extensions of this rule as well.

7.4. **k-duplicial operads?** As discussed in Sections 3.2.2 and 4.1.2, there are natural ways to extend any of the three duplicial operads Dup, $\mathsf{Dup}_{\prec}$ and $\mathsf{Dup}_{\succ}$ and their duals in the parallel situation. In contrast, in Sections 3.3.2 and 4.2.2, we could only extend the twisted duplicial operad $\mathsf{Dup}_{\prec}$ and its dual in the series situation.

A natural question is thus to look for a generalization of the classical duplicial operad Dup and its dual in series. The idea would be to force all signals not traversed by the series route to point towards these series route. The difficulty is that given a signal not traversed by the series routes, there is no natural way to decide towards which series route it should point. Natural choices include the following:

- in each node, the i-th letter points towards the i-th route,
- in each node v, the i-th free letter points towards the i-th route not passing through v,
- in each node, each letter goes up in the tree until it reaches a route, and then points towards this route,
- all signals that are on the left (resp. right) of all routes to point to the right (resp. left).

However, none of these rules lead to an operad. An appropriate rule still remains to be found.

7.5. **Interpolating with q-analogues between messy and tidy?** Recall that there exists a q-deformation of the shuffle product defined inductively by

$$X \sqcup\!\sqcup_q \varepsilon = X, \qquad \varepsilon \sqcup\!\sqcup_q Y = Y, \qquad \text{and} \qquad xX \sqcup\!\sqcup_q yY := x(X \sqcup\!\sqcup_q yY) + q^{|xX|}\, y(xX \sqcup\!\sqcup Y),$$

for any two words X, Y and any two letters x, y. This quantum shuffle, which is the simplest case of M. Rosso's construction [Ros98], is associative and interpolates between the concatenation when $q = 0$ and the shuffle when $q = 1$. When q is not a root of the unity, the shuffle algebra is isomorphic to the concatenation algebra. One can similarly define a shifted q-shuffle of two permutations. This gives a quantum deformation FQSym_q of the algebra of free quasi-symmetric functions [TU96, HNT07].

The tidy rules return 0 or a single permutation defined using a concatenation, whereas the messy rules involve a shuffle. The comparison with the q-shuffle suggests that it may be possible to interpolate between the tidy and the messy rules for the three families of operads: signaletic, citelangis and Zinbiel.

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