

THE WEAK ORDER ON INTEGER POSETS

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ABSTRACT. We explore lattice structures on integer binary relations (*i.e.* binary relations on the set $\{1, 2, \dots, n\}$ for a fixed integer n) and on integer posets (*i.e.* partial orders on the set $\{1, 2, \dots, n\}$ for a fixed integer n). We first observe that the weak order on the symmetric group naturally extends to a lattice structure on all integer binary relations. We then show that the subposet of this weak order induced by integer posets defines as well a lattice. We finally study the subposets of this weak order induced by specific families of integer posets corresponding to the elements, the intervals, and the faces of the permutahedron, the associahedron, and some recent generalizations of those.

The *weak order* is the lattice on the symmetric group $\mathfrak{S}(n)$ defined as the inclusion order of inversions, where an *inversion* of $\sigma \in \mathfrak{S}(n)$ is a pair of values $a < b$ such that $\sigma^{-1}(a) > \sigma^{-1}(b)$. It is a fundamental tool for the study of the symmetric group, in connection to reduced expressions of permutations as products of simple transpositions. Its Hasse diagram can also be seen as a certain acyclic orientation of the skeleton of the permutahedron (the convex hull of all permutations of $\mathfrak{S}(n)$ seen as vectors in \mathbb{R}^n).

This paper extends the weak order to all integer binary relations, *i.e.* binary relations on the set $[n] := \{1, 2, \dots, n\}$ for a fixed integer n . A permutation $\sigma \in \mathfrak{S}(n)$ is seen as an binary relation \triangleleft on $[n]$ where $u \triangleleft v$ when u appears before v in σ . Inversions of σ then translates to decreasing relations of \triangleleft , *i.e.* elements $a < b$ such that $b \triangleleft a$. This interpretation enables to naturally extend the weak order to all binary relations on $[n]$ as follows. For any two binary relations R, S on $[n]$, we define

$$R \preceq S \iff R^{\text{Inc}} \supseteq S^{\text{Inc}} \text{ and } R^{\text{Dec}} \subseteq S^{\text{Dec}},$$

where $R^{\text{Inc}} := \{(a, b) \in R \mid a \leq b\}$ and $R^{\text{Dec}} := \{(b, a) \in R \mid a \leq b\}$ respectively denote the increasing and decreasing subrelations of R . We call this order the *weak order* on integer binary relations, see Figure 1. The central result of this paper is the following statement, see Figure 5.

Theorem 1. *For any $n \in \mathbb{N}$, the weak order restricted to the set of all posets on $[n]$ is a lattice.*

Our motivation for this result is that many relevant combinatorial objects can be interpreted by specific integer posets, and the subposets of the weak order induced by these specific integer posets often correspond to classical lattice structures on these combinatorial objects. To illustrate this, we study specific integer posets corresponding to the elements, to the intervals, and to the faces in the classical weak order, the Tamari and Cambrian lattices [MHPS12, Rea06], the boolean lattice, and other related lattices defined in [PP18]. By this systematic approach, we rediscover and shed light on lattice structures studied by G. Chatel and V. Pons on Tamari interval posets [CP15], by G. Chatel and V. Pilaud on Cambrian and Schröder-Cambrian trees [CP17], by D. Krob, M. Latapy, J.-C. Novelli, H.-D. Phan and S. Schwer on pseudo-permutations [KLN+01], and by P. Palacios and M. Ronco [PR06] and J.-C. Novelli and J.-Y. Thibon [NT06] on plane trees.

Part 1. The weak order on integer posets

1.1. THE WEAK ORDER ON INTEGER BINARY RELATIONS

1.1.1. Integer binary relations. Our main object of focus are binary relations on integers. An *integer (binary) relation* of size n is a binary relation on $[n] := \{1, \dots, n\}$, that is, a subset R of $[n]^2$. As usual, we write equivalently $(u, v) \in R$ or $u R v$, and similarly, we write equivalently $(u, v) \notin R$ or $u \not R v$. Recall that a relation $R \in [n]^2$ is called:

- *reflexive* if $u R u$ for all $u \in [n]$,
- *transitive* if $u R v$ and $v R w$ implies $u R w$ for all $u, v, w \in [n]$,
- *symmetric* if $u R v$ implies $v R u$ for all $u, v \in [n]$,
- *antisymmetric* if $u R v$ and $v R u$ implies $u = v$ for all $u, v \in [n]$.

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From now on, we only consider reflexive relations. We denote by $\mathcal{R}(n)$ (resp. $\mathcal{T}(n)$, resp. $\mathcal{S}(n)$, resp. $\mathcal{A}(n)$) the collection of all reflexive (resp. reflexive and transitive, resp. reflexive and symmetric, resp. reflexive and antisymmetric) integer relations of size n . We denote by $\mathcal{E}(n)$ the set of *integer equivalences* of size n , that is, reflexive transitive symmetric integer relations, and by $\mathcal{P}(n)$ the collection of *integer posets* of size n , that is, reflexive transitive antisymmetric integer relations. In all these notations, we forget the n when we consider a relation without restriction on its size.

A *subrelation* of $R \in \mathcal{R}(n)$ is a relation $S \in \mathcal{R}(n)$ such that $S \subseteq R$ as subsets of $[n]^2$. We say that S *coarsens* R and R *extends* S . The extension order defines a graded lattice structure on $\mathcal{R}(n)$ whose meet and join are respectively given by intersection and union. The complementation $R \mapsto \{(u, v) \mid u = v \text{ or } u \not R v\}$ is an antiautomorphism of $(\mathcal{R}(n), \subseteq, \cap, \cup)$ and makes it an ortho-complemented lattice.

Note that $\mathcal{T}(n)$, $\mathcal{S}(n)$ and $\mathcal{A}(n)$ are all stable by intersection, while only $\mathcal{S}(n)$ is stable by union. In other words, $(\mathcal{S}(n), \subseteq, \cap, \cup)$ is a sublattice of $(\mathcal{R}(n), \subseteq, \cap, \cup)$, while $(\mathcal{T}(n), \subseteq)$ and $(\mathcal{A}(n), \subseteq)$ are meet-semisublattices of $(\mathcal{R}(n), \subseteq, \cap)$ but not sublattices of $(\mathcal{R}(n), \subseteq, \cap, \cup)$. However, $(\mathcal{T}(n), \subseteq)$ is a lattice. To see it, consider the *transitive closure* of a relation $R \in \mathcal{R}(n)$ defined by

$$R^{\text{tc}} := \{(u, w) \in [n]^2 \mid \exists v_1, \dots, v_p \in [n] \text{ such that } u = v_1 R v_2 R \dots R v_{p-1} R v_p = w\}.$$

The transitive closure R^{tc} is the coarsest transitive relation containing R . It follows that $(\mathcal{T}(n), \subseteq)$ is a lattice where the meet of $R, S \in \mathcal{R}(n)$ is given by $R \cap S$ and the join of $R, S \in \mathcal{R}(n)$ is given by $(R \cup S)^{\text{tc}}$. Since the transitive closure preserves symmetry, the subposet $(\mathcal{E}(n), \subseteq)$ of integer equivalences is a sublattice of $(\mathcal{T}(n), \subseteq)$.

1.1.2. Weak order. From now on, we consider both a relation R and the natural order $<$ on $[n]$ simultaneously. To limit confusions, we try to stick to the following convention throughout the paper. We denote integers by letters a, b, c when we know that $a < b < c$ in the natural order. In contrast, we prefer to denote integers by letters u, v, w when we do not know their relative order. This only helps avoid confusions and is always specified.

Let $I_n := \{(a, b) \in [n]^2 \mid a \leq b\}$ and $D_n := \{(b, a) \in [n]^2 \mid a \leq b\}$. Observe that $I_n \cup D_n = [n]^2$ while $I_n \cap D_n = \{(a, a) \mid a \in [n]\}$. We say that the relation $R \in \mathcal{R}(n)$ is *increasing* (resp. *decreasing*) when $R \subseteq I_n$ (resp. $R \subseteq D_n$). We denote by $\mathcal{I}(n)$ (resp. $\mathcal{D}(n)$) the collection of all increasing (resp. decreasing) relations on $[n]$. The *increasing* and *decreasing subrelations* of an integer relation $R \in \mathcal{R}(n)$ are the relations defined by:

$$R^{\text{Inc}} := R \cap I_n = \{(a, b) \in R \mid a \leq b\} \in \mathcal{I}(n) \quad \text{and} \quad R^{\text{Dec}} := R \cap D_n = \{(b, a) \in R \mid a \leq b\} \in \mathcal{D}(n).$$

In our pictures, we always represent an integer relation $R \in \mathcal{R}(n)$ as follows: we write the numbers $1, \dots, n$ from left to right and we draw the increasing relations of R above in blue and the decreasing relations of R below in red. Although we only consider reflexive relations, we always omit the relations (i, i) in the pictures (as well as in our explicit examples). See *e.g.* Figure 1.

Besides the extension lattice mentioned above in Section 1.1.1, there is another natural poset structure on $\mathcal{R}(n)$, whose name will be justified in Section 2.1.

Definition 2. The *weak order* on $\mathcal{R}(n)$ is the order defined by $R \preceq S$ if $R^{\text{Inc}} \supseteq S^{\text{Inc}}$ and $R^{\text{Dec}} \subseteq S^{\text{Dec}}$.

The weak order on $\mathcal{R}(3)$ is illustrated in Figure 1. Observe that the weak order is obtained by combining the extension lattice on increasing subrelations with the coarsening lattice on decreasing subrelations. In other words, $\mathcal{R}(n)$ is the square of an $\binom{n}{2}$ -dimensional boolean lattice. It explains the following statement.

Proposition 3. The weak order $(\mathcal{R}(n), \preceq)$ is a graded lattice whose meet and join are given by

$$R \wedge_{\mathcal{R}} S = (R^{\text{Inc}} \cup S^{\text{Inc}}) \cup (R^{\text{Dec}} \cap S^{\text{Dec}}) \quad \text{and} \quad R \vee_{\mathcal{R}} S = (R^{\text{Inc}} \cap S^{\text{Inc}}) \cup (R^{\text{Dec}} \cup S^{\text{Dec}}).$$

Proof. The weak order is clearly a poset (antisymmetry comes from the fact that $R = R^{\text{Inc}} \cup R^{\text{Dec}}$). Its cover relations are all of the form $R \preceq R \setminus \{(a, b)\}$ for $a R^{\text{Inc}} b$ or $R \setminus \{(b, a)\} \preceq R$ with $b R^{\text{Dec}} a$. Therefore, the weak order is graded by $R \mapsto |R^{\text{Dec}}| - |R^{\text{Inc}}|$. To check that it is a lattice, consider $R, S \in \mathcal{R}(n)$. Observe first that $R \wedge_{\mathcal{R}} S$ is indeed below both R and S in weak order. Moreover, if $T \preceq R$ and $T \preceq S$, then $T^{\text{Inc}} \supseteq R^{\text{Inc}} \cup S^{\text{Inc}}$ and $T^{\text{Dec}} \subseteq R^{\text{Dec}} \cap S^{\text{Dec}}$, so that $T \preceq R \wedge_{\mathcal{R}} S$. This proves that $R \wedge_{\mathcal{R}} S$ is indeed the meet of R and S . The proof is similar for the join. \square

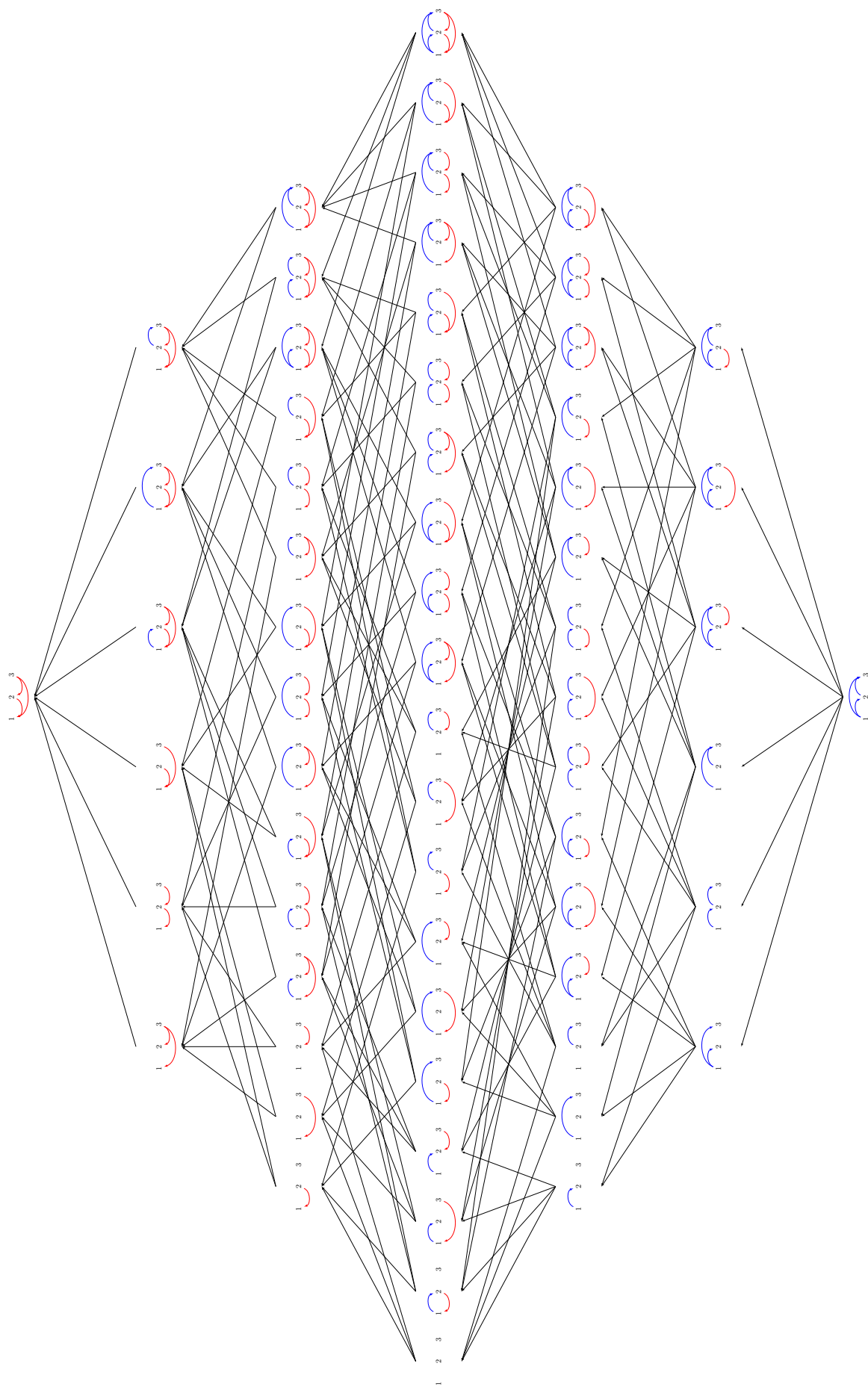


FIGURE 1. The weak order on (reflexive) integer binary relations of size 3. All reflexive relations (i, i) for $i \in [n]$ are omitted.

Remark 4. Define the *reverse* of a relation $R \in \mathcal{R}$ as $R^{\text{rev}} := \{(u, v) \in [n]^2 \mid (v, u) \in R\}$. Observe that $(R^{\text{rev}})^{\text{Inc}} = (R^{\text{Dec}})^{\text{rev}}$ and $(R^{\text{rev}})^{\text{Dec}} = (R^{\text{Inc}})^{\text{rev}}$. Therefore, the reverse map $R \mapsto R^{\text{rev}}$ defines an antiautomorphism of the weak order $(\mathcal{R}(n), \preceq, \wedge_{\mathcal{R}}, \vee_{\mathcal{R}})$. Note that it preserves symmetry, antisymmetry and transitivity.

1.2. THE WEAK ORDER ON INTEGER POSETS

In this section, we show that the three subposets of the weak order $(\mathcal{R}(n), \preceq)$ induced by antisymmetric relations, by transitive relations, and by posets are all lattices (although the last two are not sublattices of $(\mathcal{R}(n), \preceq, \wedge_{\mathcal{R}}, \vee_{\mathcal{R}})$).

1.2.1. Antisymmetric relations. We first treat the case of antisymmetric relations. Figure 2 shows the meet and join of two antisymmetric relations, and illustrates the following statement.

Proposition 5. *The meet $\wedge_{\mathcal{R}}$ and the join $\vee_{\mathcal{R}}$ both preserve antisymmetry. Thus, the antisymmetric relations induce a sublattice $(\mathcal{A}(n), \preceq, \wedge_{\mathcal{R}}, \vee_{\mathcal{R}})$ of the weak order $(\mathcal{R}(n), \preceq, \wedge_{\mathcal{R}}, \vee_{\mathcal{R}})$.*

Proof. Let $R, S \in \mathcal{A}(n)$. Let $a < b \in [n]$ be such that $(b, a) \in R \wedge_{\mathcal{R}} S$. Since (b, a) is decreasing and $(R \wedge_{\mathcal{R}} S)^{\text{Dec}} = R^{\text{Dec}} \cap S^{\text{Dec}}$, we have $b R^{\text{Dec}} a$ and $b S^{\text{Dec}} a$. By antisymmetry of R and S , we obtain that $a R^{\text{Inc}} b$ and $a S^{\text{Inc}} b$. Therefore, $(a, b) \notin R^{\text{Inc}} \cup S^{\text{Inc}} = (R \wedge_{\mathcal{R}} S)^{\text{Inc}}$. We conclude that $(b, a) \in R \wedge_{\mathcal{R}} S$ implies $(a, b) \notin R \wedge_{\mathcal{R}} S$ and thus that $R \wedge_{\mathcal{R}} S$ is antisymmetric. The proof is identical for $\vee_{\mathcal{R}}$. \square

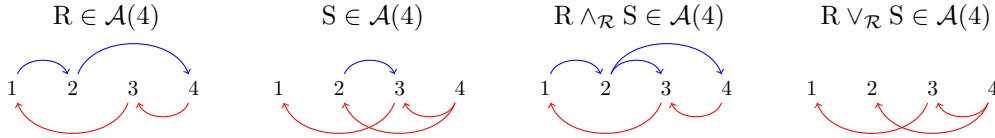


FIGURE 2. The meet $R \wedge_{\mathcal{R}} S$ and join $R \vee_{\mathcal{R}} S$ of two antisymmetric relations R, S .

Our next two statements describe all cover relations in $(\mathcal{A}(n), \preceq)$.

Proposition 6. *All cover relations in $(\mathcal{A}(n), \preceq)$ are cover relations in $(\mathcal{R}(n), \preceq)$. In particular, $(\mathcal{A}(n), \preceq)$ is still graded by $R \mapsto |R^{\text{Dec}}| - |R^{\text{Inc}}|$.*

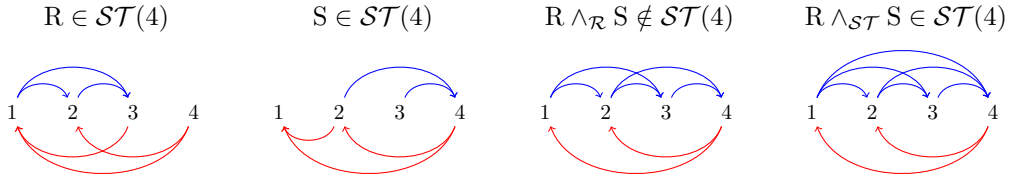
Proof. Consider a cover relation $R \preceq S$ in $(\mathcal{A}(n), \preceq)$. We have $R^{\text{Inc}} \supseteq S^{\text{Inc}}$ and $R^{\text{Dec}} \subseteq S^{\text{Dec}}$ where at least one of the inclusions is strict. Suppose first that $R^{\text{Inc}} \neq S^{\text{Inc}}$. Let $(a, b) \in R^{\text{Inc}} \setminus S^{\text{Inc}}$ and $T := R \setminus \{(a, b)\}$. Note that T is still antisymmetric as it is obtained by removing an arc from an antisymmetric relation. Moreover, we have $R \neq T$ and $R \preceq T \preceq S$. Since S covers R , this implies that $S = T = R \setminus \{(a, b)\}$. We prove similarly that if $R^{\text{Dec}} \neq S^{\text{Dec}}$, there exists $a < b$ such that $S = R \cup \{(b, a)\}$. In both cases, $R \preceq S$ is a cover relation in $(\mathcal{R}(n), \preceq)$. \square

Corollary 7. *In the weak order $(\mathcal{A}(n), \preceq)$, the antisymmetric relations that cover a given antisymmetric relation $R \in \mathcal{A}(n)$ are precisely the relations*

- $R \setminus \{(a, b)\}$ for $a < b$ such that $a R b$,
- $R \cup \{(b, a)\}$ for $a < b$ such that $a \not R b$ and $b \not R a$.

1.2.2. Transitive relations. We now consider transitive relations. Observe first that the subposet $(\mathcal{T}(n), \preceq)$ of $(\mathcal{R}(n), \preceq)$ is not a sublattice since $\wedge_{\mathcal{R}}$ and $\vee_{\mathcal{R}}$ do not preserve transitivity (see e.g. Figure 4). When R and S are transitive, we need to transform $R \wedge_{\mathcal{R}} S$ to make it a transitive relation $R \wedge_{\mathcal{T}} S$. We proceed in two steps described below.

SEMITRANSITIVE RELATIONS Before dealing with transitive relations, we introduce the intermediate notion of semitransitivity. We say that a relation $R \in \mathcal{R}$ is *semitransitive* when both R^{Inc} and R^{Dec} are transitive. We denote by $\mathcal{ST}(n)$ the collection of all semitransitive relations of size n . Figure 3 illustrates the following statement.

FIGURE 3. Two semi-transitive relations R, S and their meets $R \wedge_{\mathcal{R}} S$ and $R \wedge_{\mathcal{ST}} S$.

Proposition 8. *The weak order $(\mathcal{ST}(n), \preceq)$ is a lattice whose meet and join are given by*

$$R \wedge_{\mathcal{ST}} S = (R^{\text{Inc}} \cup S^{\text{Inc}})^{\text{tc}} \cup (R^{\text{Dec}} \cap S^{\text{Dec}}) \quad \text{and} \quad R \vee_{\mathcal{ST}} S = (R^{\text{Inc}} \cap S^{\text{Inc}}) \cup (R^{\text{Dec}} \cup S^{\text{Dec}})^{\text{tc}}.$$

Proof. Let $R, S \in \mathcal{ST}(n)$. Observe first that $R \wedge_{\mathcal{ST}} S$ is indeed semitransitive and below both R and S . Moreover, if a semitransitive relation T is such that $T \preceq R$ and $T \preceq S$, then $T^{\text{Inc}} \supseteq R^{\text{Inc}} \cup S^{\text{Inc}}$ and $T^{\text{Dec}} \subseteq R^{\text{Dec}} \cap S^{\text{Dec}}$. By semitransitivity of T , we get $T^{\text{Inc}} \supseteq (R^{\text{Inc}} \cup S^{\text{Inc}})^{\text{tc}}$, so that $T \preceq R \wedge_{\mathcal{ST}} S$. This proves that $R \wedge_{\mathcal{ST}} S$ is indeed the meet of R and S . The proof is similar for the join. \square

As in the previous section, we describe all cover relations in $(\mathcal{ST}(n), \preceq)$.

Proposition 9. *All cover relations in $(\mathcal{ST}(n), \preceq)$ are cover relations in $(\mathcal{R}(n), \preceq)$. In particular, $(\mathcal{ST}(n), \preceq)$ is still graded by $R \mapsto |R^{\text{Dec}}| - |R^{\text{Inc}}|$.*

Proof. Consider a cover relation $R \preceq S$ in $(\mathcal{ST}(n), \preceq)$. We have $R^{\text{Inc}} \supseteq S^{\text{Inc}}$ and $R^{\text{Dec}} \subseteq S^{\text{Dec}}$ where at least one of the inclusions is strict. Suppose first that $R^{\text{Inc}} \neq S^{\text{Inc}}$. Let $(a, b) \in R^{\text{Inc}} \setminus S^{\text{Inc}}$ be such that $b - a$ is minimal, and let $T := R \setminus \{(a, b)\}$. Observe that there is no $a < i < b$ such that $a R i R b$. Otherwise, by minimality of $b - a$, we would have $a S i$ and $i S b$ while $a \not S b$, contradicting the transitivity of S^{Inc} . It follows that T^{Inc} is still transitive. Since $T^{\text{Dec}} = R^{\text{Dec}}$ is also transitive, we obtain that T is semitransitive. Moreover, we have $R \neq T$ and $R \preceq T \preceq S$. Since S covers R , this implies that $S = T \cup \{(a, b)\}$. We prove similarly that if $R^{\text{Dec}} \neq S^{\text{Dec}}$, there exists (b, a) such that $S = R \cup \{(b, a)\}$: in this case, one needs to pick $(b, a) \in S^{\text{Dec}} \setminus R^{\text{Dec}}$ with $b - a$ maximal. In both cases, $R \preceq S$ is a cover relation in $(\mathcal{R}(n), \preceq)$. \square

Corollary 10. *In the weak order $(\mathcal{ST}(n), \preceq)$, the semitransitive relations that cover a given semitransitive relation $R \in \mathcal{ST}(n)$ are precisely the relations*

- $R \setminus \{(a, b)\}$ for $a < b$ such that $a R b$ and there is no $a < i < b$ with $a R i R b$,
- $R \cup \{(b, a)\}$ for $a < b$ such that $b \not R a$ and there is no $i < a$ with $a R i$ but $b \not R i$ and similarly no $b < j$ with $j R b$ but $j \not R a$.

TRANSITIVE RELATIONS Now we consider transitive relations.

Note that $\mathcal{T}(n) \subseteq \mathcal{ST}(n)$ but $\mathcal{ST}(n) \not\subseteq \mathcal{T}(n)$. In particular, $R \wedge_{\mathcal{ST}} S$ and $R \vee_{\mathcal{ST}} S$ may not be transitive even if R and S are (see Figure 4). To see that the subposet of the weak order induced by transitive relations is indeed a lattice, we therefore need operations which ensure transitivity and are compatible with the weak order. For $R \in \mathcal{R}$, define the *transitive decreasing deletion* of R as

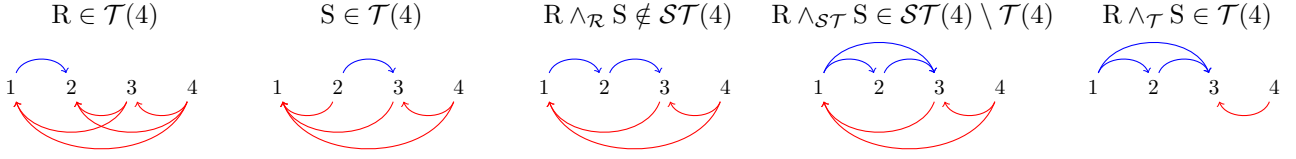
$$R^{\text{tdd}} := R \setminus \{(b, a) \in R^{\text{Dec}} \mid \exists i \leq b \text{ and } j \geq a \text{ such that } i R b R a R j \text{ while } i \not R j\},$$

and the *transitive increasing deletion* of R as

$$R^{\text{tid}} := R \setminus \{(a, b) \in R^{\text{Inc}} \mid \exists i \geq a \text{ and } j \leq b \text{ such that } i R a R b R j \text{ while } i \not R j\}.$$

Note that in these definitions, i and j may coincide with a and b (since we assumed that all our relations are reflexive). Figure 4 illustrates the transitive decreasing deletion: the rightmost relation $R \wedge_{\mathcal{T}} S$ is indeed obtained as $(R \wedge_{\mathcal{ST}} S)^{\text{tdd}}$. Observe that two decreasing relations have been deleted: $(3, 1)$ (take $i = 2$ and $j = 1$, or $i = 3$ and $j = 2$) and $(4, 1)$ (take $i = 4$ and $j = 2$).

Remark 11. The idea of the transitive decreasing deletion is to delete all decreasing relations which prevent the binary relation to be transitive. It may thus seem more natural to assume in the definition of R^{tdd} that either $i = b$ or $j = a$. However, this would not suffice to rule out all non-transitive relations, consider for example the relation $[4]^2 \setminus \{(2, 3), (3, 2)\}$. We would therefore

FIGURE 4. Two transitive relations R, S and their meets $R \wedge_{\mathcal{R}} S$, $R \wedge_{\mathcal{ST}} S$ and $R \wedge_{\mathcal{T}} S$.

need to iterate the deletion process, which would require to prove a converging property. Our definition of R^{tdd} simplifies the presentation as it requires only one deletion step.

Lemma 12. *For any relation $R \in \mathcal{R}$, we have $R^{\text{tdd}} \preceq R \preceq R^{\text{tid}}$.*

Proof. R^{tdd} is obtained from R by deleting decreasing relations. Therefore $(R^{\text{tdd}})^{\text{inc}} = R^{\text{inc}}$ and $(R^{\text{tdd}})^{\text{dec}} \subseteq R^{\text{dec}}$ and thus $R^{\text{tdd}} \preceq R$ by definition of the weak order. The argument is similar for R^{tid} . \square

Lemma 13. *If $R \in \mathcal{R}$ is semitransitive, then R^{tdd} and R^{tid} are transitive.*

Proof. We prove the result for R^{tdd} , the proof being symmetric for R^{tid} . Set

$$U := \{(b, a) \in R^{\text{dec}} \mid \exists i \leq b \text{ and } j \geq a \text{ such that } i R b R a R j \text{ while } i \not R j\},$$

so that $R^{\text{tdd}} = R \setminus U$ with $(R^{\text{tdd}})^{\text{inc}} = R^{\text{inc}}$ and $(R^{\text{tdd}})^{\text{dec}} = R^{\text{dec}} \setminus U$. Let $u, v, w \in [n]$ be such that $u R^{\text{tdd}} v$ and $v R^{\text{tdd}} w$. We want to prove that $u R^{\text{tdd}} w$. We distinguish six cases according to the relative order of u, v, w :

- (i) If $u < v < w$, then $u R^{\text{inc}} v$ and $v R^{\text{inc}} w$. Thus $u R^{\text{inc}} w$ by transitivity of R^{inc} , and thus $u R^{\text{tdd}} w$.
- (ii) If $u < w < v$, then $u R^{\text{inc}} v$ and $v R^{\text{dec}} w$. Since $v \not U w$, we have $u R^{\text{inc}} w$ and thus $u R^{\text{tdd}} w$.
- (iii) If $v < u < w$, then $u R^{\text{dec}} v$ and $v R^{\text{inc}} w$. Since $u \not U v$, we have $u R^{\text{inc}} w$ and thus $u R^{\text{tdd}} w$.
- (iv) If $v < w < u$, then $u R^{\text{dec}} v$ and $v R^{\text{inc}} w$. Since $u \not U v$, we have $u R^{\text{dec}} w$. Assume by contradiction that $u U w$. Hence there is $i \leq u$ and $j \geq w$ such that $i R u R w R j$ but $i \not R j$. Since $v R^{\text{inc}} w$ and $w R^{\text{inc}} j$, the transitivity of R^{inc} ensures that $v R j$. We obtain that $u U v$, a contradiction. Therefore, $u \not U w$ and $u R^{\text{tdd}} w$.
- (v) If $w < u < v$, then $u R^{\text{inc}} v$ and $v R^{\text{dec}} w$. Since $v \not U w$, we have $u R^{\text{dec}} w$. Assume by contradiction that $u U w$. Hence there is $i \leq u$ and $j \geq w$ such that $i R u R w R j$ but $i \not R j$. Since $i R^{\text{inc}} u$ and $u R^{\text{inc}} v$, the transitivity of R^{inc} ensures that $i R v$. We obtain that $v U w$, a contradiction. Therefore, $u \not U w$ and $u R^{\text{tdd}} w$.
- (vi) If $w < v < u$, then $u R^{\text{dec}} v$ and $v R^{\text{dec}} w$, so that $u R^{\text{dec}} w$ by transitivity of R^{dec} . Assume by contradiction that $u U w$. Hence there is $i \leq u$ and $j \geq w$ such that $i R u R w R j$ but $i \not R j$. Since $u \not U v$ and $v \not U w$, we obtain that $i R v$ and $v R j$. If $i \leq v$, then we have $i \leq v$ and $j \geq w$ with $i R v R w R j$ and $i \not R j$ contradicting the fact that $v \not U w$. Similarly, if $j \geq v$, we have $i \leq u$ and $j \geq v$ with $i R u R v R j$ and $i \not R j$ contradicting the fact that $u \not U v$. Finally, if $j < v < i$, we have $i R^{\text{dec}} v R^{\text{dec}} j$ and $i \not R^{\text{dec}} j$ contradicting the transitivity of R^{dec} . \square

Remark 14. We observed earlier that the transitive closure R^{tc} is the coarsest transitive relation containing R . For $R \in \mathcal{ST}$, Lemmas 12 and 13 show that R^{tdd} is a transitive relation below R in weak order. However, there might be other transitive relations S with $S \preceq R$ and which are not comparable to R^{tdd} in weak order. For example, consider $R := \{(1, 3), (3, 2)\}$ and $S := \{(1, 2), (1, 3), (3, 2)\}$. Then S is transitive and $S \preceq R$ while S is incomparable to $R^{\text{tdd}} = \{(1, 3)\}$ in weak order.

We use the maps $R \mapsto R^{\text{tdd}}$ and $R \mapsto R^{\text{tid}}$ to obtain the main result of this section. Figure 4 illustrates all steps of a meet computation in $\mathcal{T}(4)$.

Proposition 15. *The weak order $(\mathcal{T}(n), \preceq)$ is a lattice whose meet and join are given by*

$$R \wedge_{\mathcal{T}} S = ((R^{\text{inc}} \cup S^{\text{inc}})^{\text{tc}} \cup (R^{\text{dec}} \cap S^{\text{dec}}))^{\text{tdd}} \quad \text{and} \quad R \vee_{\mathcal{T}} S = ((R^{\text{inc}} \cap S^{\text{inc}}) \cup (R^{\text{dec}} \cup S^{\text{dec}})^{\text{tc}})^{\text{tid}}.$$

Before proving the proposition, we state the following technical Lemma which we will use repeatedly in our proofs.

Lemma 16. *Let R and S be two transitive relations, let $M = R \wedge_{\mathcal{T}} S$, and let $1 \leq a < b \leq n$ such that $b M a$ and $b \not M^{\text{tdd}} a$. By definition of M^{tdd} , there exist $i \leq b$ and $j \geq a$ such that $i M b M a M j$ while $i \not M j$. Then we have*

- either $i \neq b$ or $j \neq a$,
- if $i \neq b$, there is $a < k < b$ such that $i M k M b$ with $(k, b) \in R \cup S$ and $k \not M^{\text{tdd}} a$,
- if $j \neq a$, there is $a < k < b$ such that $a M k M j$ with $(a, k) \in R \cup S$ and $b \not M^{\text{tdd}} k$,

Besides if R and S are also antisymmetric, then in both cases, $b \not M^{\text{tdd}} k \not M^{\text{tdd}} a$.

Proof. Since $b M a$ and $i \not M j$, we cannot have both $i = b$ and $j = a$. By symmetry, we can assume that $i \neq b$. Since $(i, b) \in M^{\text{inc}} = (R^{\text{inc}} \cup S^{\text{inc}})^{\text{tc}}$, there exists $i \leq k < b$ such that $(i, k) \in M^{\text{inc}}$ and $(k, b) \in R^{\text{inc}} \cup S^{\text{inc}}$. Assume without loss of generality that $k R^{\text{inc}} b$. We obtain that $k R b R a$ and thus that $k R a$ by transitivity of R . We want to prove that $k > a$.

Assume that $k \leq a$. We then have $(k, a) \in R^{\text{inc}} \subseteq M^{\text{inc}}$ and thus that $i M^{\text{inc}} k M^{\text{inc}} a M^{\text{inc}} j$ while $i \not M^{\text{inc}} j$ contradicting the transitivity of M^{inc} . We then have $a < k < b$. There is left to prove that $k \not M^{\text{tdd}} a$. Suppose that we have $k M^{\text{tdd}} a$, then we have $i M k M a M j$ which implies $i M j$ because (k, a) is not deleted by the transitive decreasing deletion. This contradicts our initial statement $i \not M j$.

Besides, if R is antisymmetric, then $k R b$ implies $b \not R k$ which in turns gives $b \not M^{\text{tdd}} k$. \square

Proof of Proposition 15. The weak order $(\mathcal{T}(n), \preceq)$ is a subposet of $(\mathcal{R}(n), \preceq)$. It is also clearly bounded: the weak order minimal transitive relation is $I_n = \{(a, b) \in [n]^2 \mid a \leq b\}$ while the weak order maximal transitive relation is $D_n = \{(b, a) \in [n]^2 \mid a \leq b\}$. Therefore, we only have to show that any two transitive relations admit a meet and a join. We prove the result for the meet, the proof for the join being symmetric.

Let $R, S \in \mathcal{T}(n)$ and $M := R \wedge_{\mathcal{T}} S = (R^{\text{inc}} \cup S^{\text{inc}})^{\text{tc}} \cup (R^{\text{Dec}} \cap S^{\text{Dec}})$, so that $R \wedge_{\mathcal{T}} S = M^{\text{tdd}}$. First we have $M \preceq R$ so that $R \wedge_{\mathcal{T}} S = M^{\text{tdd}} \preceq M \preceq R$ by Lemma 12. Similarly, $R \wedge_{\mathcal{T}} S \preceq S$. Moreover, $R \wedge_{\mathcal{T}} S$ is transitive by Lemma 13. It thus remains to show that $R \wedge_{\mathcal{T}} S$ is larger than any other transitive relation smaller than both R and S .

Consider thus another transitive relation $T \in \mathcal{T}(n)$ such that $T \preceq R$ and $T \preceq S$. We need to show that $T \preceq R \wedge_{\mathcal{T}} S = M^{\text{tdd}}$. Observe that $T \preceq M$ since T is semitransitive and $M = R \wedge_{\mathcal{T}} S$ is larger than any semitransitive relation smaller than both R and S . It implies in particular that $T^{\text{inc}} \supseteq M^{\text{inc}} = (M^{\text{tdd}})^{\text{inc}}$ and that $T^{\text{Dec}} \subseteq M^{\text{Dec}}$.

Assume by contradiction that $T \not\preceq M^{\text{tdd}}$. Since $T^{\text{inc}} \supseteq (M^{\text{tdd}})^{\text{inc}}$, this means that there exist $(b, a) \in T^{\text{Dec}} \setminus M^{\text{tdd}}$. We choose $(b, a) \in T^{\text{Dec}} \setminus M^{\text{tdd}}$ such that $b - a$ is minimal. Since $T^{\text{Dec}} \subseteq M^{\text{Dec}}$, we have $(b, a) \in M^{\text{Dec}} \setminus M^{\text{tdd}}$. By definition of M^{tdd} , there exists $i \leq b$ and $j \geq a$ such that $i M b M a M j$ while $i \not M j$. We use Lemma 16 and assume without loss of generality that there is $a < k < b$ with $(k, b) \in R \cup S$ and $k \not M^{\text{tdd}} a$. Since $(k, b) \in (R^{\text{inc}} \cup S^{\text{inc}}) \subseteq T^{\text{inc}}$ and $b T a$ we have $k T a$ by transitivity of T . Since $k > a$, we get $(k, a) \in T^{\text{Dec}} \subseteq M^{\text{Dec}}$. But by Lemma 16, $(k, a) \notin M^{\text{tdd}}$: it has been deleted by the transitive decreasing deletion and thus contradicts the minimality of $b - a$. \square

Remark 17. In contrast to Propositions 6 and 9 and Corollaries 7 and 10, the cover relations in $(\mathcal{T}(n), \preceq)$ are more complicated to describe. In fact, the lattice $(\mathcal{T}(n), \preceq)$ is not graded as soon as $n \geq 3$. Indeed, consider the maximal chains from I_3 to D_3 in $(\mathcal{T}(3), \preceq)$. Those chains passing through the trivial reflexive relation $\{(i, i) \mid i \in [n]\}$ have all length 6, while those passing through the full relation $[3]^2$ all have length 4.

1.2.3. Integer posets. We finally arrive to the subposet of the weak order induced by integer posets. The weak order on $\mathcal{P}(3)$ is illustrated in Figure 5. We now have all tools to show Theorem 1 announced in the introduction.

Proposition 18. *The transitive meet $\wedge_{\mathcal{T}}$ and the transitive join $\vee_{\mathcal{T}}$ both preserve antisymmetry. In other words, $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$ is a sublattice of $(\mathcal{T}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$.*

Proof. Let $R, S \in \mathcal{P}(n)$. Let $M := R \wedge_{\mathcal{T}} S = (R^{\text{inc}} \cup S^{\text{inc}})^{\text{tc}} \cup (R^{\text{Dec}} \cap S^{\text{Dec}})$, so that $R \wedge_{\mathcal{T}} S = M^{\text{tdd}}$. Assume that M^{tdd} is not antisymmetric. Let $a < c \in [n]$ be such that $\{(a, c), (c, a)\} \subseteq M^{\text{tdd}}$ with

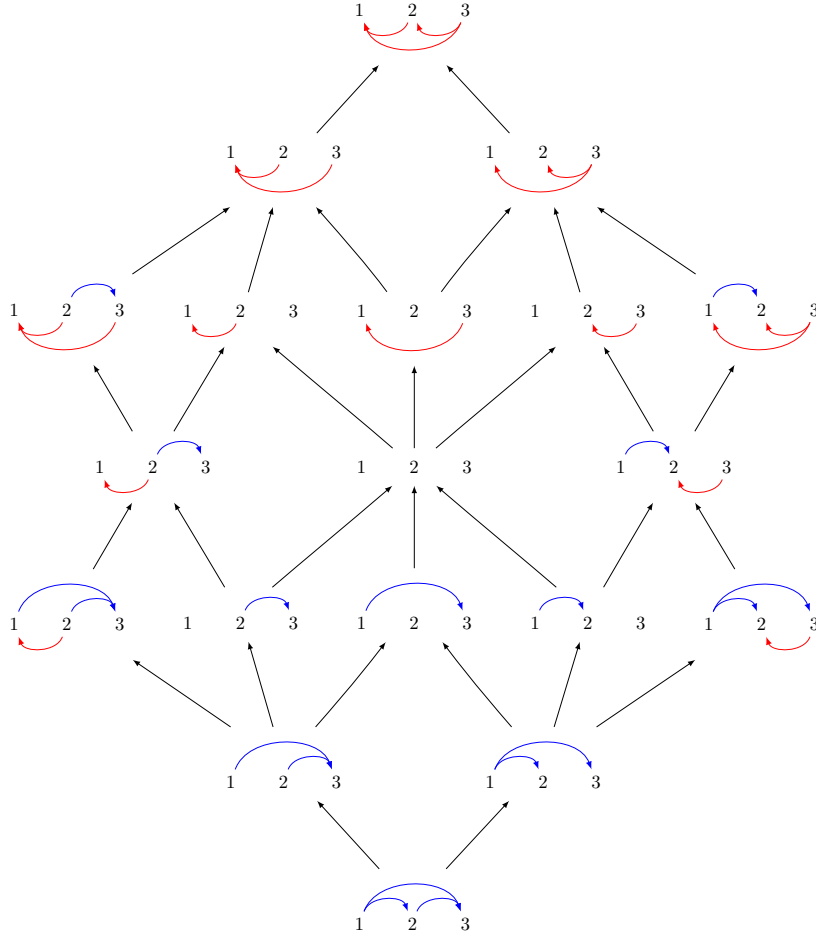


FIGURE 5. The weak order on integer posets of size 3.

$c - a$ minimal. Since $(c, a) \in (M^{\text{tdd}})^{\text{Dec}} \subseteq M^{\text{Dec}} = R^{\text{Dec}} \cap S^{\text{Dec}}$, we have $(a, c) \notin R^{\text{Inc}} \cup S^{\text{Inc}}$ by antisymmetry of R and S . Since $(a, c) \in (R^{\text{Inc}} \cup S^{\text{Inc}})^{\text{tc}} \setminus (R^{\text{Inc}} \cup S^{\text{Inc}})$, there exists $a < b < c$ such that $\{(a, b), (b, c)\} \subseteq (R^{\text{Inc}} \cup S^{\text{Inc}})^{\text{tc}}$. Since $c M^{\text{tdd}} a M^{\text{tdd}} b$, we obtain by transitivity of M^{tdd} that $\{(b, c), (c, b)\} \subseteq M^{\text{tdd}}$, contradicting the minimality of $c - a$. \square

Remark 19. In contrast, there is no guarantee that the semitransitive meet of two transitive antisymmetric relations is antisymmetric. For example in Figure 4, R and S are antisymmetric but $M = R \wedge_{\mathcal{ST}} S$ is not as it contains both $(1, 3)$ and $(3, 1)$. However, the relation $(3, 1)$ is removed by the transitive decreasing delation and the result $M^{\text{tdd}} = R \wedge_{\mathcal{T}} S$ is antisymmetric.

As in Propositions 6 and 9 and Corollaries 7 and 10, the next two statements describe all cover relations in $(\mathcal{P}(n), \preceq)$.

Proposition 20. *All cover relations in $(\mathcal{P}(n), \preceq)$ are cover relations in $(\mathcal{R}(n), \preceq)$. In particular, $(\mathcal{P}(n), \preceq)$ is still graded by $R \mapsto |R^{\text{Dec}}| - |R^{\text{Inc}}|$.*

Proof. Consider a cover relation $R \preceq S$ in $(\mathcal{P}(n), \preceq)$. We have $R^{\text{Inc}} \supseteq S^{\text{Inc}}$ and $R^{\text{Dec}} \subseteq S^{\text{Dec}}$ where at least one of the inclusions is strict. Suppose first that $R^{\text{Inc}} \neq S^{\text{Inc}}$. Consider the set $X := \{(a, b) \in R^{\text{Inc}} \setminus S^{\text{Inc}} \mid \nexists a < i < b \text{ with } a R i R b\}$. This set X is nonempty as it contains any (a, b) in $R^{\text{Inc}} \setminus S^{\text{Inc}}$ with $b - a$ minimal. Consider now $(a, b) \in X$ with $b - a$ maximal and let $T := R \setminus \{(a, b)\}$. We claim that T is still a poset. It is clearly still reflexive and antisymmetric. For transitivity, assume by means of contradiction that there is $j \in [n] \setminus \{a, b\}$ such that $a R j R b$. Since $(a, b) \in X$, we know that $j < a$ or $b < j$. As these two options are symmetric, assume for

instance that $j < a$ and choose j so that $a - j$ is minimal. We claim that there is no $j < i < b$ such that $j R i R b$. Otherwise, since $a R j R i$ and R is transitive, we have $a R i R b$. Now, if $i < a$, we have $a R i R b$ and $j < i < a$ contradicting the minimality of $a - j$ in our choice of j . If $i > a$, we have $a R i R b$ and $a < i < b$ contradicting the fact that $(a, b) \in X$. Finally, if $i = a$, we have $a R j R a$ contradicting the antisymmetry of R . This proves that there is no $j < i < b$ such that $j R i R b$. By maximality of $b - a$ in our choice of (a, b) this implies that $j S b$. Since $(a, j) \in R^{\text{Dec}} \subseteq S^{\text{Dec}}$, we therefore obtain that $a S j S b$ while $a \not S b$, contradicting the transitivity of S . This proves that T is transitive and it is thus a poset. Moreover, we have $R \neq T$ and $R \preceq T \preceq S$. Since S covers R , this implies that $S = T = R \setminus \{(a, b)\}$. We prove similarly that if $R^{\text{Dec}} \neq S^{\text{Dec}}$, there exists (b, a) such that $S = R \cup \{(b, a)\}$. In both cases, $R \preceq S$ is a cover relation in $(\mathcal{R}(n), \preceq)$. \square

Corollary 21. *In the weak order $(\mathcal{P}(n), \preceq)$, the posets that cover a given integer poset $R \in \mathcal{P}(n)$ are precisely the posets*

- the relations $R \setminus \{(a, b)\}$ for $a < b$ such that $a R b$ and there is no $i \in [n]$ with $a R i R b$,
- the relations $R \cup \{(b, a)\}$ for $a < b$ such that $a \not R b$ and $b \not R a$ and there is no $i \neq a$ with $a R i$ but $b \not R i$ and similarly no $j \neq b$ with $j R b$ but $j \not R a$.

Part 2. Weak order induced by some relevant families of posets

In the rest of the paper, we present our motivation to study Theorem 1. We observe that many relevant combinatorial objects (for example permutations, binary trees, binary sequences, ...) can be interpreted by specific integer posets¹. Moreover, the subposets of the weak order induced by these specific integer posets often correspond to classical lattice structures on these combinatorial objects (for example the classical weak order, the Tamari lattice, the boolean lattice, ...). Table 1 summarizes the different combinatorial objects involved and a roadmap to their properties.

Rather than our previous notations R, S, M used for integer binary relations, we will denote integer posets by $\triangleleft, \blacktriangleleft, \dashv$ so that $a \triangleleft b$ (resp. $a \blacktriangleleft b$ and $a \dashv v$) means that a is in relation with b for the \triangleleft relation. These notations emphasize the notion of *order* and allow us to write $a \triangleright b$ for $b \triangleleft a$, in particular when $a < b$. To make our presentation easier to read, we have decomposed some of our proofs into technical but straightforward claims that are proved separately in Appendix A.

2.1. FROM THE PERMUTAHEDRON

We start with relevant families of posets corresponding to the elements, the intervals, and the faces of the permutahedron. Further similar families of posets will appear in Sections 2.2 and 2.3.

Let $\mathfrak{S}(n)$ denote the symmetric group on $[n]$. For $\sigma \in \mathfrak{S}(n)$, we denote by

$$\begin{aligned} \text{ver}(\sigma) &:= \{(a, b) \in [n]^2 \mid a \leq b \text{ and } \sigma^{-1}(a) \leq \sigma^{-1}(b)\} \\ \text{and } \text{inv}(\sigma) &:= \{(b, a) \in [n]^2 \mid a \leq b \text{ and } \sigma^{-1}(a) \geq \sigma^{-1}(b)\} \end{aligned}$$

the set of *versions* and *inversions* of σ respectively². Inversions are classical (although we order their entries in a strange way), while versions are borrowed from [KLR03]. Clearly, the versions of σ determine the inversions of σ and *vice versa*. The *weak order* on $\mathfrak{S}(n)$ is defined as the inclusion order of inversions, or as the clusion (reverse inclusion) order of the versions:

$$\sigma \preceq \tau \iff \text{inv}(\sigma) \subseteq \text{inv}(\tau) \iff \text{ver}(\sigma) \supseteq \text{ver}(\tau).$$

It is known that the weak order $(\mathfrak{S}(n), \preceq)$ is a lattice. We denote by $\wedge_{\mathfrak{S}}$ and $\vee_{\mathfrak{S}}$ its meet and join, and by $e := [1, 2, \dots, n]$ and $w_o := [n, \dots, 2, 1]$ the weak order minimal and maximal permutations.

¹A comment on the notations used along this section. We use different notations for the set of permutations $\mathfrak{S}(n)$ and the set of corresponding posets WOEP. Although it might look like a complicated notation for a well-known object, we want our notation to clearly distinguish between the combinatorial objects and their corresponding posets.

²Throughout the paper, we only work with versions and inversions of values (sometimes called left inversions, or coinversions). The cover relations of the weak order are thus given by transpositions of consecutive positions (sometimes called right weak order). As there is no ambiguity in the paper, we never specify this convention.

	weak order	Tamari lattice	Cambrian lattices [Rea06]	boolean lattice	Permutree lattices
Elements	combinatorial objects	permutations 2751346 	binary trees 	binary sequences -+-+--+ 	permutrees [PP18]
	notation	WOEP(n)	TOEP(n)	BOEP(n)	PEP(\mathbb{O})
	characterization	Prop. 22	Prop. 40	Prop. 63	Prop. 63
	lattice properties	Prop. 24 & Coro. 95	Prop. 42 & Coro. 91	Coro. 91	Thms. 90 & 93
enumeration	1, 2, 6, 24, 120, ... [OEI10, A000142]	1, 2, 5, 14, 42, ... [OEI10, A000108]	1, 2, 5, 14, 42, ... [OEI10, A000108]	1, 2, 4, 8, 16, ... [OEI10, A000079]	depends on the orientation \mathbb{O}
Intervals	combinatorial objects	weak order intervals [213, 321] 	Tamari lattice intervals 	Boolean lattice intervals 	Permutree lattice intervals
	notation	WOIP(n)	TOIP(n)	BOIP(n)	PIP(\mathbb{O})
	characterization	Prop. 26	Coro. 43	Coro. 58	Coro. 58
	lattice properties	Coro. 28	Coro. 45 & Prop. 47	Coro. 2.4.3 & 85	Coro. 2.4.3
enumeration	1, 3, 17, 151, 1899, ... [OEI10, A007767]	1, 3, 13, 68, 399, ... [OEI10, A000260]	depends on the signature ε	1, 3, 9, 27, 81, ... [OEI10, A000244]	depends on the orientation \mathbb{O}
Faces	combinatorial objects	ordered partitions 125 37 46 	Schröder trees 	ternary sequences 0+-+--+ 	Schröder permutrees [PP18]
	notation	WOFP(n)	TOFP(n)	BOFP(n)	PPF(\mathbb{O})
	characterization	Prop. 31	Prop. 48	Prop. 66	Prop. 66
	lattice properties	Rem. 32	Rem. 49	Rem. 69	Rem. 69
enumeration	1, 3, 13, 75, 541, ... [OEI10, A000670]	1, 3, 11, 45, 197, ... [OEI10, A001003]	1, 3, 11, 45, 197, ... [OEI10, A001003]	1, 3, 9, 27, 81, ... [OEI10, A000244]	depends on the orientation \mathbb{O}

TABLE 1. A roadmap through the combinatorial objects considered in Part 2.

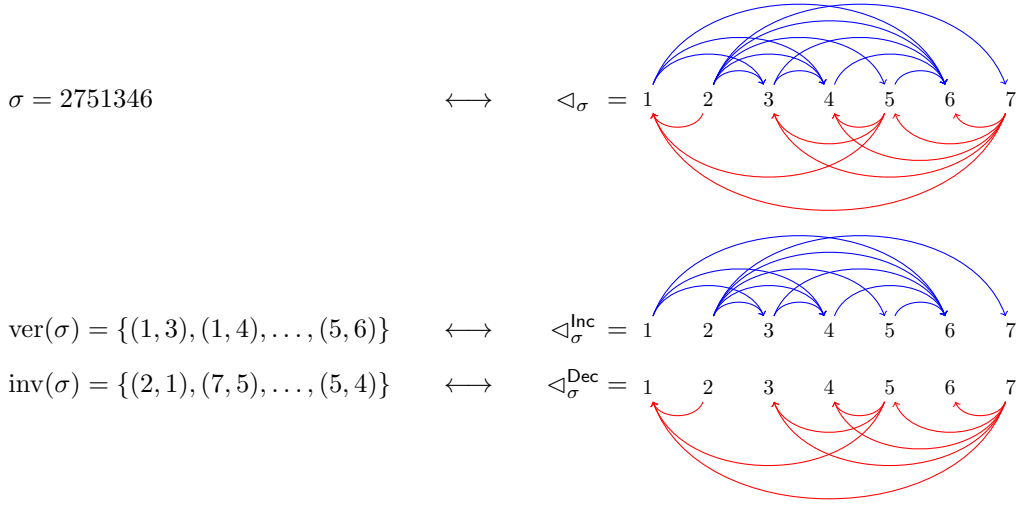


FIGURE 6. A Weak Order Element Poset (WOEP).

2.1.1. Weak Order Element Posets. We see a permutation $\sigma \in \mathfrak{S}(n)$ as a total order \triangleleft_{σ} on $[n]$ defined by $u \triangleleft_{\sigma} v$ if $\sigma^{-1}(u) \leq \sigma^{-1}(v)$ (i.e. u is before v in σ). In other words, \triangleleft_{σ} is the chain $\sigma(1) \triangleleft_{\sigma} \dots \triangleleft_{\sigma} \sigma(n)$ as illustrated in Figure 6.

We say that \triangleleft_{σ} is a *weak order element poset*, and we denote by

$$\text{WOEP}(n) := \{ \triangleleft_{\sigma} \mid \sigma \in \mathfrak{S}(n) \}$$

the set of all total orders on $[n]$. The following characterization of these elements is immediate.

Proposition 22. *A poset $\triangleleft \in \mathcal{P}(n)$ is in $\text{WOEP}(n)$ if and only if $\forall u, v \in [n]$, either $u \triangleleft v$ or $u \triangleright v$.*

In other words, the WOEP are the maximal posets, with $\binom{n}{2}$ relations (this should help spotting them on Figure 5). The following proposition connects the weak order on $\mathfrak{S}(n)$ to that on $\mathcal{P}(n)$. It justifies the term “weak order” used in Definition 2.

Proposition 23. *For $\sigma \in \mathfrak{S}(n)$, the increasing (resp. decreasing) relations of \triangleleft_{σ} are the versions (resp. inversions) of σ : $\triangleleft_{\sigma}^{\text{Inc}} = \text{ver}(\sigma)$ and $\triangleleft_{\sigma}^{\text{Dec}} = \text{inv}(\sigma)$. Therefore, for any permutations $\sigma, \sigma' \in \mathfrak{S}(n)$, we have $\sigma \preceq \sigma'$ if and only if $\triangleleft_{\sigma} \preceq \triangleleft_{\sigma'}$.*

Proof. $\triangleleft_{\sigma}^{\text{Inc}} = \{(a, b) \mid a < b \text{ and } a \triangleleft_{\sigma} b\} = \{(a, b) \mid a < b \text{ and } \sigma^{-1}(a) < \sigma^{-1}(b)\} = \text{ver}(\sigma)$. \square

We thus obtain that the subposet of the weak order $(\mathcal{P}(n), \preceq)$ induced by the set $\text{WOEP}(n)$ is isomorphic to the weak order on $\mathfrak{S}(n)$, and thus is a lattice. To conclude on $\text{WOEP}(n)$, we mention the following stronger statement which will be derived in Corollary 95.

Proposition 24. *The set $\text{WOEP}(n)$ induces a sublattice of the weak order $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$.*

2.1.2. Weak Order Interval Posets. For two permutations $\sigma, \sigma' \in \mathfrak{S}(n)$ with $\sigma \preceq \sigma'$, we denote by $[\sigma, \sigma'] := \{\tau \in \mathfrak{S}(n) \mid \sigma \preceq \tau \preceq \sigma'\}$ the weak order interval between σ and σ' . As illustrated in Figure 7, we can see such an interval as the set of linear extensions of a poset.

Proposition 25. *The permutations of $[\sigma, \sigma']$ are precisely the linear extensions of the poset*

$$\triangleleft_{[\sigma, \sigma']} := \bigcap_{\sigma \preceq \tau \preceq \sigma'} \triangleleft_{\tau} = \triangleleft_{\sigma} \cap \triangleleft_{\sigma'} = \triangleleft_{\sigma'}^{\text{Inc}} \cup \triangleleft_{\sigma}^{\text{Dec}}.$$

Proof. We first prove that the three expressions for $\triangleleft_{[\sigma, \sigma']}$ coincide. Indeed we have

$$\bigcap_{\sigma \preceq \tau \preceq \sigma'} \triangleleft_{\tau} = \left(\bigcap_{\sigma \preceq \tau \preceq \sigma'} \triangleleft_{\tau}^{\text{Inc}} \right) \cup \left(\bigcap_{\sigma \preceq \tau \preceq \sigma'} \triangleleft_{\tau}^{\text{Dec}} \right) = \triangleleft_{\sigma'}^{\text{Inc}} \cup \triangleleft_{\sigma}^{\text{Dec}} = \triangleleft_{\sigma} \cap \triangleleft_{\sigma'},$$

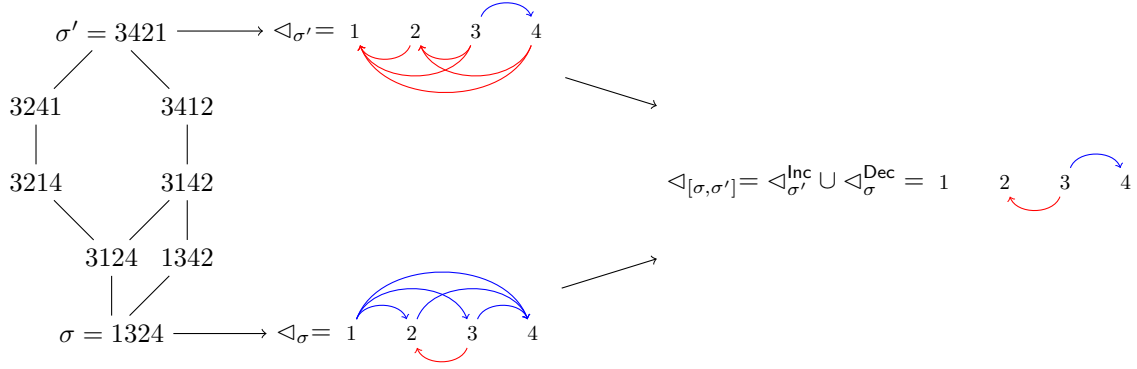


FIGURE 7. A Weak Order Interval Poset (WOIP).

where the first equality is obtained by restriction to the increasing and decreasing relations, the second equality holds since $\sigma \preceq \tau \preceq \sigma' \iff \triangleleft_{\tau}^{\text{Inc}} \supseteq \triangleleft_{\sigma'}^{\text{Inc}}$ and $\triangleleft_{\sigma}^{\text{Dec}} \subseteq \triangleleft_{\tau}^{\text{Dec}}$ by Proposition 23, and the last one follows from $\triangleleft_{\sigma}^{\text{Inc}} \supseteq \triangleleft_{\sigma'}^{\text{Inc}}$ and $\triangleleft_{\sigma}^{\text{Dec}} \subseteq \triangleleft_{\sigma'}^{\text{Dec}}$.

Consider now a permutation τ . By definition, \triangleleft_{τ} extends $\triangleleft_{\sigma'}^{\text{Inc}} \cup \triangleleft_{\sigma}^{\text{Dec}}$ if and only if $\triangleleft_{\tau}^{\text{Inc}} \supseteq \triangleleft_{\sigma'}^{\text{Inc}}$ and $\triangleleft_{\tau}^{\text{Dec}} \subseteq \triangleleft_{\sigma}^{\text{Dec}}$, which in turns is equivalent to $\sigma \preceq \tau \preceq \sigma'$ by Proposition 23. \square

We say that $\triangleleft_{[\sigma, \sigma']}$ is a *weak order interval poset*, and we denote by

$$\text{WOIP}(n) := \{ \triangleleft_{[\sigma, \sigma']} \mid \sigma, \sigma' \in \mathfrak{S}(n), \sigma \preceq \sigma' \}$$

the set of all weak order interval posets on $[n]$. The following characterization of these posets already appeared in [BW91, Thm. 6.8] and will be discussed in Section 2.1.4.

Proposition 26 ([BW91, Thm. 6.8]). *A poset $\triangleleft \in \mathcal{P}(n)$ is in $\text{WOIP}(n)$ if and only if $\forall a < b < c$,*

$$a \triangleleft c \implies a \triangleleft b \text{ or } b \triangleleft c \quad \text{and} \quad a \triangleright c \implies a \triangleright b \text{ or } b \triangleright c.$$

We now describe the weak order on $\text{WOIP}(n)$.

Proposition 27. *For any $\sigma \preceq \sigma'$ and $\tau \preceq \tau'$, we have $\triangleleft_{[\sigma, \sigma']} \preceq \triangleleft_{[\tau, \tau']} \iff \sigma \preceq \tau$ and $\sigma' \preceq \tau'$.*

Proof. From the formula of Proposition 25, we have

$$\begin{aligned} \triangleleft_{[\sigma, \sigma']} \preceq \triangleleft_{[\tau, \tau']} &\iff \triangleleft_{[\sigma, \sigma']}^{\text{Inc}} \supseteq \triangleleft_{[\tau, \tau']}^{\text{Inc}} \text{ and } \triangleleft_{[\sigma, \sigma']}^{\text{Dec}} \subseteq \triangleleft_{[\tau, \tau']}^{\text{Dec}} \\ &\iff \triangleleft_{\sigma'}^{\text{Inc}} \supseteq \triangleleft_{\tau'}^{\text{Inc}} \text{ and } \triangleleft_{\sigma}^{\text{Dec}} \subseteq \triangleleft_{\tau}^{\text{Dec}} \\ &\iff \sigma' \preceq \tau' \text{ and } \sigma \preceq \tau. \end{aligned} \quad \square$$

It follows that $(\text{WOIP}(n), \preceq)$ gets the lattice structure of a product, described in the next statement. See also Corollary 39 for an alternative description of the meet and join in this lattice.

Corollary 28. *The weak order $(\text{WOIP}(n), \preceq)$ is a lattice whose meet and join are given by*

$$\triangleleft_{[\sigma, \sigma']} \wedge_{\text{WOIP}} \triangleleft_{[\tau, \tau']} = \triangleleft_{[\sigma \wedge_{\mathfrak{S} \tau}, \sigma' \wedge_{\mathfrak{S} \tau'}]} \quad \text{and} \quad \triangleleft_{[\sigma, \sigma']} \vee_{\text{WOIP}} \triangleleft_{[\tau, \tau']} = \triangleleft_{[\sigma \vee_{\mathfrak{S} \tau}, \sigma' \vee_{\mathfrak{S} \tau'}]}.$$

Corollary 29. *The set $\text{WOEP}(n)$ induces a sublattice of the weak order $(\text{WOIP}, \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$.*

Remark 30. $(\text{WOIP}(n), \preceq, \wedge_{\text{WOIP}}, \vee_{\text{WOIP}})$ is not a sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$. For example,

$$\begin{aligned} \triangleleft_{[231, 321]} \wedge_{\mathcal{T}} \triangleleft_{[312, 321]} &= \triangleleft_{[231, 321]} \wedge_{\mathcal{T}} \triangleleft_{[312, 321]} = \triangleleft_{[123, 321]} = \emptyset \text{ (trivial poset on } [3]). \\ \triangleleft_{[231, 321]} \wedge_{\text{WOIP}} \triangleleft_{[312, 321]} &= \triangleleft_{[\sigma \wedge_{\mathfrak{S} \tau}, \sigma' \wedge_{\mathfrak{S} \tau'}]} = \triangleleft_{[123, 321]} = \emptyset \end{aligned}$$

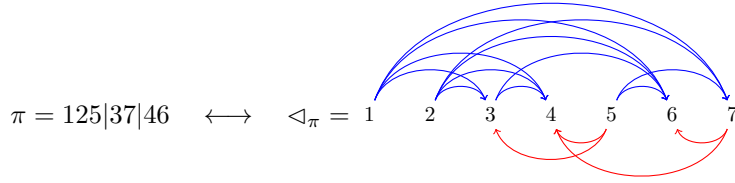


FIGURE 8. A Weak Order Face Poset (WOFP).

2.1.3. Weak Order Face Posets. The permutations of $\mathfrak{S}(n)$ correspond to the vertices of the permutahedron $\text{Perm}(n) := \text{conv} \{(\sigma(1), \dots, \sigma(n)) \mid \sigma \in \mathfrak{S}(n)\}$. We now consider all the faces of the permutahedron. The codimension k faces of $\text{Perm}(n)$ correspond to *ordered partitions* of $[n]$ into k parts, or equivalently to *surjections* from $[n]$ to $[k]$. We see an ordered partition π as a poset \triangleleft_π on $[n]$ defined by $u \triangleleft_\pi v$ if and only if $u = v$ or $\pi^{-1}(u) < \pi^{-1}(v)$, that is, the part of π containing u appears strictly before the part of π containing v . See Figure 8. Note that a permutation σ belongs to the face of the permutahedron $\text{Perm}(n)$ corresponding to an ordered partition π if and only if \triangleleft_σ is a linear extension of \triangleleft_π .

We say that \triangleleft_π is a *weak order face poset*, and we denote by

$$\text{WOFP}(n) := \{\triangleleft_\pi \mid \pi \text{ ordered partition of } [n]\}$$

the set of all weak order face posets on $[n]$. We first characterize these posets.

Proposition 31. *The following conditions are equivalent for a poset $\triangleleft \in \mathcal{P}(n)$:*

- (i) $\triangleleft \in \text{WOFP}(n)$,
- (ii) $\forall u, v, w \in [n], u \triangleleft w \implies u \triangleleft v \text{ or } v \triangleleft w$,
- (iii) $\triangleleft \in \text{WOIP}(n)$ and $\forall a < b < c$ with a, c incomparable, $a \triangleleft b \iff b \triangleright c$ and $a \triangleright b \iff b \triangleleft c$.

Proof. Assume that $\triangleleft = \triangleleft_\pi \in \text{WOFP}(n)$ for an ordered partition π of $[n]$, and let $u, v, w \in [n]$ such that $u \triangleleft w$. By definition, we have $\pi^{-1}(u) < \pi^{-1}(w)$. Therefore, we certainly have $\pi^{-1}(u) < \pi^{-1}(v)$ or $\pi^{-1}(v) < \pi^{-1}(w)$, and thus $u \triangleleft v$ or $v \triangleleft w$. This proves that (i) \implies (ii).

Assume now that \triangleleft satisfies (ii). It immediately implies that $\triangleleft \in \text{WOIP}(n)$ by the characterization of Proposition 26. Consider now $a < b < c$ such that a and c are incomparable in \triangleleft . If $a \triangleleft b$, then (ii) implies that either $a \triangleleft c$ or $c \triangleleft b$. Since we assumed that a and c are incomparable, we obtain that $b \triangleright c$. We obtain similarly that $b \triangleright c \implies a \triangleleft b$, that $a \triangleright b \implies b \triangleleft c$ and that $b \triangleleft c \implies a \triangleright b$. This shows that (ii) \implies (iii).

Finally, assume that \triangleleft satisfies (iii). Consider the \triangleleft incomparability relation \equiv defined by $u \equiv v$ when u and v are incomparable in \triangleleft . Condition (iii) ensures that \equiv is an equivalence relation. Moreover, the equivalence classes of \equiv are totally ordered. This shows that \triangleleft defines an ordered partition of $[n]$ and thus that (iii) \implies (i). \square

We now consider the weak order on $\text{WOFP}(n)$. Since $\text{WOFP}(n) \subseteq \text{WOIP}(n)$, Proposition 27 shows that we have $\triangleleft \preceq \blacktriangleleft \iff \triangleleft^{\text{minle}} \preceq \blacktriangleleft^{\text{minle}}$ and $\triangleleft^{\text{maxle}} \preceq \blacktriangleleft^{\text{maxle}}$. This order is precisely the *facial weak order* on the permutahedron $\text{Perm}(n)$ studied by A. Dermenjian, C. Hohlweg and V. Pilaud in [DHP18]. They prove in particular that this order coincides with the *pseudo-permutahedron* originally defined by D. Krob, M. Latapy, J.-C. Novelli, H.-D. Phan and S. Schwer [KLN⁺01] on ordered partitions as the transitive closure of the relations

$$\lambda_1 | \cdots | \lambda_i | \lambda_{i+1} | \cdots | \lambda_k \prec \lambda_1 | \cdots | \lambda_i \lambda_{i+1} | \cdots | \lambda_k \prec \lambda_1 | \cdots | \lambda_{i+1} | \lambda_i | \cdots | \lambda_k,$$

if $\max(\lambda_i) < \min(\lambda_{i+1})$. This order is known to be a lattice [KLN⁺01, DHP18]. We will discuss an alternative description of the meet and join in this lattice in Section 2.4.4.

Remark 32. $(\text{WOFP}(n), \preceq, \wedge_{\text{WOFP}}, \vee_{\text{WOFP}})$ is not a sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$, nor a sublattice of $(\text{WOIP}(n), \preceq, \wedge_{\text{WOIP}}, \vee_{\text{WOIP}})$. For example,

$$\triangleleft_{2|13} \wedge_{\mathcal{T}} \triangleleft_{123} = \triangleleft_{2|13} \wedge_{\text{WOIP}} \triangleleft_{123} = \{(2, 3)\} \quad \text{while} \quad \triangleleft_{2|13} \wedge_{\text{WOFP}} \triangleleft_{123} = \triangleleft_{12|3} = \{(1, 3), (2, 3)\}.$$

2.1.4. IWOIP(n) and DWOIP(n) and the WOIP deletion. We conclude our section on the permutahedron by introducing some variations on WOIP(n) which are needed later and provide a proof of the characterization of WOIP(n) given in Proposition 26.

Since the set of linear extensions of a poset is order-convex, a poset is in WOIP(n) if and only if it admits weak order minimal and maximal linear extensions. This motivates to consider separately two bigger families of posets. Denote by IWOIP(n) (resp. by DWOIP(n)) the set of posets of $\mathcal{P}(n)$ which admit a weak order maximal (resp. minimal) linear extension. Proposition 26 follows from the characterization of these posets, illustrated in Figure 9.

Proposition 33. For a poset $\triangleleft \in \mathcal{P}(n)$,

$$\begin{aligned} \triangleleft \in \text{IWOIP}(n) &\iff \forall a < b < c, a \triangleleft c \implies a \triangleleft b \text{ or } b \triangleleft c, \\ \triangleleft \in \text{DWOIP}(n) &\iff \forall a < b < c, a \triangleright c \implies a \triangleright b \text{ or } b \triangleright c. \end{aligned}$$

Proof. By symmetry, we only prove the characterization of IWOIP(n). Assume first that $\triangleleft \in \mathcal{P}(n)$ is such that $a \triangleleft c \implies a \triangleleft b$ or $b \triangleleft c$ for all $a < b < c$. Let

$$\triangleleft^{\text{maxle}} := \triangleleft \cup \{(b, a) \mid a < b \text{ incomparable in } \triangleleft\}$$

denote the binary relation obtained from \triangleleft by adding a decreasing relation between any two incomparable elements in \triangleleft (see Figure 9). The following claim is proved in Appendix A.1.

Claim A. $\triangleleft^{\text{maxle}}$ is a poset.

Moreover $\triangleleft^{\text{maxle}}$ is a total order (since any two elements are comparable in $\triangleleft^{\text{maxle}}$ by definition) which is a linear extension of \triangleleft (since $\triangleleft \subseteq \triangleleft^{\text{maxle}}$ by definition). Finally, any other linear extension of \triangleleft is smaller than $\triangleleft^{\text{maxle}}$ in weak order (since a linear extension of \triangleleft contains \triangleleft and $\triangleleft^{\text{maxle}} \setminus \triangleleft \subseteq D_n$). We conclude that $\triangleleft^{\text{maxle}}$ is the maximal linear extension of \triangleleft in weak order.

Reciprocally, assume now that there exists $a < b < c$ such that $a \triangleleft c$ while $a \not\triangleleft b$ and $b \not\triangleleft c$. The transitivity of \triangleleft implies that $b \not\triangleleft a$ and $c \not\triangleleft b$. Let $\sim := \triangleleft \cup \{(a, b), (c, b)\}$ and $\smile := \triangleleft \cup \{(b, a), (b, c)\}$. Note that \sim and \smile are still acyclic (but not necessary transitive). Indeed any cycle for example in \sim would involve either (a, b) or (c, b) , but not both. If \sim has a cycle involving for example (a, b) , then $b \triangleleft a$ by transitivity of \triangleleft , which gives a contradiction. Thus they admit linear extensions, and we consider minimal linear extensions ρ of \sim and σ of \smile . We conclude that ρ and σ are minimal linear extensions of \triangleleft incomparable in the weak order as illustrated on Figure 9. \square

Remark 34. Note that it is enough to check the conditions of Proposition 33 only for all cover relations $a \triangleleft c$ and $a \triangleright c$ of \triangleleft . Indeed, consider $a < b < c$ where $a \triangleleft c$ is not a cover relation, so that there exists $u \in [n]$ such that $a \triangleleft u \triangleleft c$. Assume for example that $b < u$, the case $u < b$ being symmetric. Hence $a < b < u$ and $a \triangleleft u$ implies that either $a \triangleleft b$ or $b \triangleleft u$ (by induction on the length of the minimal chain between a and c). If $b \triangleleft u$, we obtain that $b \triangleleft u \triangleleft c$ so that $b \triangleleft c$.

We have seen in Corollary 28 that the weak order (WOIP(n), \preceq) on interval posets forms a lattice. Using the characterization of Proposition 33, we now show that the subsets (IWOIP(n), \preceq)

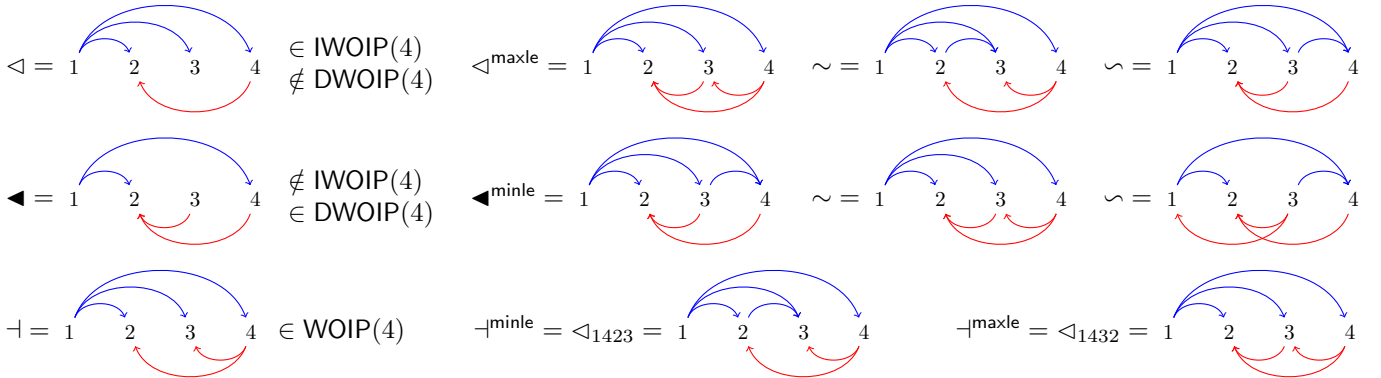


FIGURE 9. Examples and counterexamples of elements in IWOIP(4) and DWOIP(4).

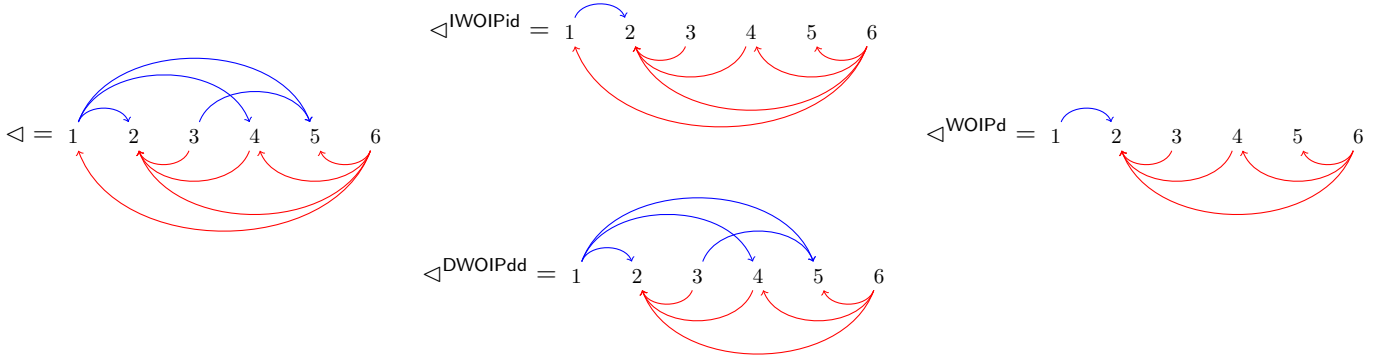


FIGURE 10. The IWOIP increasing deletion, the DWOIP decreasing deletion, and the WOIP deletion.

and $(\text{DWOIP}(n), \preceq)$ of the weak order $(\mathcal{P}(n), \preceq)$ form lattices — although there are not sublattices of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$. We define the **IWOIP increasing deletion**, the **DWOIP decreasing deletion**, and the **WOIP deletion** by

$$\begin{aligned} \triangleleft^{\text{IWOIPid}} &:= \triangleleft \setminus (\text{I}_n \setminus \triangleleft^{\text{Inc}})^{\text{tc}} = \triangleleft \setminus \{(a, c) \mid \exists a < b_1 < \dots < b_k < c, a \not\triangleleft b_1 \not\triangleleft \dots \not\triangleleft b_k \not\triangleleft c\}, \\ \triangleleft^{\text{DWOIPdd}} &:= \triangleleft \setminus (\text{D}_n \setminus \triangleleft^{\text{Dec}})^{\text{tc}} = \triangleleft \setminus \{(c, a) \mid \exists a < b_1 < \dots < b_k < c, a \not\triangleright b_1 \not\triangleright \dots \not\triangleright b_k \not\triangleright c\}, \\ \triangleleft^{\text{WOIPd}} &:= (\triangleleft^{\text{DWOIPdd}})^{\text{IWOIPid}} = (\triangleleft^{\text{IWOIPid}})^{\text{DWOIPdd}}. \end{aligned}$$

These operations are illustrated on Figure 10.

Remark 35. Similar to Remark 11, the IWOIP increasing deletion (resp. DWOIP decreasing deletion) deletes at once all increasing relations which prevent the poset to be in $\text{IWOIP}(n)$ (resp. in $\text{DWOIP}(n)$). Deleting only the relations (a, c) (resp. (c, a)) for which there exists $a < b < c$ such that $a \not\triangleleft b \not\triangleleft c$ (resp. $a \not\triangleright b \not\triangleright c$) would require several iterations. For example, we would need n iterations to obtain $\{(i, j) \mid i, j \in [n], i + 1 < j\}^{\text{IWOIPid}} = \emptyset$.

These functions satisfy the following properties.

Lemma 36. For any poset $\triangleleft \in \mathcal{P}(n)$, we have $\triangleleft^{\text{IWOIPid}} \in \text{IWOIP}(n)$ and $\triangleleft^{\text{DWOIPdd}} \in \text{DWOIP}(n)$. Moreover, $\triangleleft \in \text{DWOIP}(n) \implies \triangleleft^{\text{IWOIPid}} \in \text{WOIP}(n)$ and $\triangleleft \in \text{IWOIP}(n) \implies \triangleleft^{\text{DWOIPdd}} \in \text{WOIP}(n)$.

Proof. We prove the result for $\triangleleft^{\text{IWOIPid}}$, the proof for $\triangleleft^{\text{DWOIPdd}}$ being symmetric. The details of the following claim are given in Appendix A.1.

Claim B. $\triangleleft^{\text{IWOIPid}}$ is a poset.

Thus the characterization of Proposition 33 implies that $\triangleleft^{\text{IWOIPid}}$ is always in $\text{IWOIP}(n)$, and even in $\text{WOIP}(n)$ when $\triangleleft \in \text{DWOIP}(n)$. \square

Lemma 37. For any poset $\triangleleft \in \mathcal{P}(n)$, the poset $\triangleleft^{\text{IWOIPid}}$ (resp. $\triangleleft^{\text{DWOIPdd}}$) is the weak order minimal (resp. maximal) poset in $\text{IWOIP}(n)$ bigger than \triangleleft (resp. in $\text{DWOIP}(n)$ smaller than \triangleleft).

Proof. We prove the result for $\triangleleft^{\text{IWOIPid}}$, the proof for $\triangleleft^{\text{DWOIPdd}}$ being symmetric. Observe first that $\triangleleft \preceq \triangleleft^{\text{IWOIPid}}$ since $\triangleleft^{\text{IWOIPid}}$ is obtained from \triangleleft by deleting increasing relations. Consider now $\blacktriangleleft \in \text{IWOIP}(n)$ such that $\triangleleft \preceq \blacktriangleleft$. By definition, we have $\triangleleft^{\text{Inc}} \supseteq \blacktriangleleft^{\text{Inc}}$ and $\triangleleft^{\text{Dec}} \subseteq \blacktriangleleft^{\text{Dec}}$. Since $(\triangleleft^{\text{IWOIPid}})^{\text{Dec}} = \triangleleft^{\text{Dec}}$, it just remains to show that for any $(a, c) \in \blacktriangleleft^{\text{Inc}}$, there exist no $a < b_1 < \dots < b_k < c$ with $a \not\triangleleft b_1 \not\triangleleft b_2 \not\triangleleft \dots \not\triangleleft b_k \not\triangleleft c$. Assume otherwise and choose such a pair (a, c) with $c - a$ minimal. Since $\blacktriangleleft \in \text{IWOIP}(n)$ and $a < b_1 < c$ are such that $a \blacktriangleleft c$ while $a \not\triangleleft b_1$ (because $a \not\triangleleft^{\text{Inc}} b_1$ and $\blacktriangleleft^{\text{Inc}} \subset \triangleleft^{\text{Inc}}$), we have $b_1 \blacktriangleleft c$. But this assertion contradicts the minimality of $c - a$. \square

Proposition 38. *The subposets of the weak order $(\mathcal{P}(n), \preceq)$ induced by $\text{IWOIP}(n)$ and $\text{DWOIP}(n)$ are lattices whose meets and joins are given by*

$$\begin{aligned} \triangleleft \wedge_{\text{IWOIP}} \blacktriangleleft &= \triangleleft \wedge_{\mathcal{T}} \blacktriangleleft & \triangleleft \vee_{\text{IWOIP}} \blacktriangleleft &= (\triangleleft \vee_{\mathcal{T}} \blacktriangleleft)^{\text{IWOIPid}} \\ \triangleleft \wedge_{\text{DWOIP}} \blacktriangleleft &= (\triangleleft \wedge_{\mathcal{T}} \blacktriangleleft)^{\text{DWOIPdd}} & \triangleleft \vee_{\text{DWOIP}} \blacktriangleleft &= \triangleleft \vee_{\mathcal{T}} \blacktriangleleft. \end{aligned}$$

Proof. We prove the result for $\text{IWOIP}(n)$, the proof for $\text{DWOIP}(n)$ being symmetric. Consider $\triangleleft, \blacktriangleleft \in \text{IWOIP}(n)$. We first prove that $\dashv := \triangleleft \wedge_{\mathcal{T}} \blacktriangleleft = ((\triangleleft^{\text{Inc}} \cup \blacktriangleleft^{\text{Inc}})^{\text{tc}} \cup (\triangleleft^{\text{Dec}} \cap \blacktriangleleft^{\text{Dec}}))^{\text{td}}$ is also in $\text{IWOIP}(n)$ (see also Proposition 80 and Example 81 for a more systematic approach). For any cover relation $a \dashv c$ and $a < b < c$, we have $a \dashv^{\text{Inc}} c$ so that $a \triangleleft^{\text{Inc}} c$ or $a \blacktriangleleft^{\text{Inc}} c$ (since we have a cover relation). Since $\triangleleft, \blacktriangleleft \in \text{IWOIP}(n)$, we obtain that $a \triangleleft^{\text{Inc}} b$, or $b \triangleleft^{\text{Inc}} c$, or $a \blacktriangleleft^{\text{Inc}} b$, or $b \blacktriangleleft^{\text{Inc}} c$. Thus, $a \dashv b$ or $b \dashv c$ for any cover relation $a \dashv c$ and any $a < b < c$. Using Remark 34, we conclude that $\dashv \in \text{IWOIP}(n)$.

On the other hand, Lemma 37 asserts that $(\triangleleft \vee_{\mathcal{T}} \blacktriangleleft)^{\text{IWOIPid}}$ is the weak order minimal poset in $\text{IWOIP}(n)$ bigger than $\triangleleft \vee_{\mathcal{T}} \blacktriangleleft$. Any poset in $\text{IWOIP}(n)$ bigger than \triangleleft and \blacktriangleleft is also bigger than $\triangleleft \vee_{\mathcal{T}} \blacktriangleleft$, and thus bigger than $(\triangleleft \vee_{\mathcal{T}} \blacktriangleleft)^{\text{IWOIPid}}$. We conclude that $(\triangleleft \vee_{\mathcal{T}} \blacktriangleleft)^{\text{IWOIPid}}$ is indeed the join of \triangleleft and \blacktriangleleft . \square

We finally deduce from Proposition 38 and Lemma 36 an alternative formula for the meet and join in the weak order $(\text{WOIP}(n), \preceq)$. See also Corollary 28.

Corollary 39. *The meet and join in the weak order on $\text{WOIP}(n)$ are given by*

$$\triangleleft \wedge_{\text{WOIP}} \blacktriangleleft = (\triangleleft \wedge_{\mathcal{T}} \blacktriangleleft)^{\text{DWOIPdd}} \quad \text{and} \quad \triangleleft \vee_{\text{WOIP}} \blacktriangleleft = (\triangleleft \vee_{\mathcal{T}} \blacktriangleleft)^{\text{IWOIPid}}.$$

2.2. FROM THE ASSOCIAHEDRON

Similarly to the previous section, we now briefly discuss some relevant families of posets corresponding to the elements, the intervals, and the faces of the associahedron. Further similar families of posets arising from permutreehedra [PP18] will be discussed in Section 2.3. This section should just be considered as a simplified prototype to the next section. We therefore omit the proofs which will appear in a more general context in Sections 2.3 and 2.4.

We denote by $\mathfrak{B}(n)$ the set of *planar rooted binary trees* with n nodes, that we simply call binary trees here for short. We label the vertices of a binary tree $T \in \mathfrak{B}(n)$ through an *inorder traversal*, i.e. such that all vertices in the left (resp. right) child of a vertex v of T receive a label smaller (resp. larger) than the label of v . From now on, we identify a vertex and its label.

There is a fundamental surjection from permutations to binary trees. Namely, a permutation $\sigma := \sigma_1 \dots \sigma_n \in \mathfrak{S}(n)$ is mapped to the binary tree $\text{bt}(\sigma) \in \mathfrak{B}(n)$ obtained by successive insertions of $\sigma_n, \dots, \sigma_1$ in a binary (search) tree. The fiber of a tree T is precisely the set of linear extensions of T . It is an interval of the weak order whose minimal and maximal elements respectively avoid the patterns 312 and 132. Moreover, the fibers of bt define a lattice congruence of the weak order. Thus, the set $\mathfrak{B}(n)$ of binary trees is endowed with a lattice structure \preceq defined by

$$T \preceq T' \iff \exists \sigma, \sigma' \in \mathfrak{S}(n) \text{ such that } \text{bt}(\sigma) = T, \text{bt}(\sigma') = T' \text{ and } \sigma \preceq \sigma'$$

whose meet $\wedge_{\mathfrak{B}}$ and join $\vee_{\mathfrak{B}}$ are given by

$$T \wedge_{\mathfrak{B}} T' = \text{bt}(\sigma \wedge_{\mathfrak{S}} \sigma') \quad \text{and} \quad T \vee_{\mathfrak{B}} T' = \text{bt}(\sigma \vee_{\mathfrak{S}} \sigma')$$

for any representatives $\sigma, \sigma' \in \mathfrak{S}(n)$ such that $\text{bt}(\sigma) = T$ and $\text{bt}(\sigma') = T'$. Note that in particular, $T \preceq T'$ if and only if $\sigma \preceq \sigma'$ where σ and σ' denote the minimal (resp. maximal) linear extensions of T and T' respectively. For example, the minimal (resp. maximal) tree is the left (resp. right) comb whose unique linear extension is $e := [1, 2, \dots, n]$ (resp. $w_{\circ} := [n, \dots, 2, 1]$). This lattice structure is the *Tamari lattice* whose cover relations are given by *right rotations* on binary trees. It was introduced by D. Tamari [MHPS12] on Dyck paths, our presentation is a more modern perspective [BW91, Rea06].

2.2.1. Tamari Order Element Posets. We consider the tree T as a poset \triangleleft_T , defined by $i \triangleleft_T j$ when i is a descendant of j in T . In other words, the Hasse diagram of \triangleleft_T is the tree T oriented towards its root. An illustration is provided in Figure 11. Note that the increasing (resp. decreasing) subposet of \triangleleft_T is given by $i \triangleleft_T^{\text{Inc}} j$ (resp. $i \triangleleft_T^{\text{Dec}} j$) if and only if i belongs to the left (resp. right) subtree of j in T .

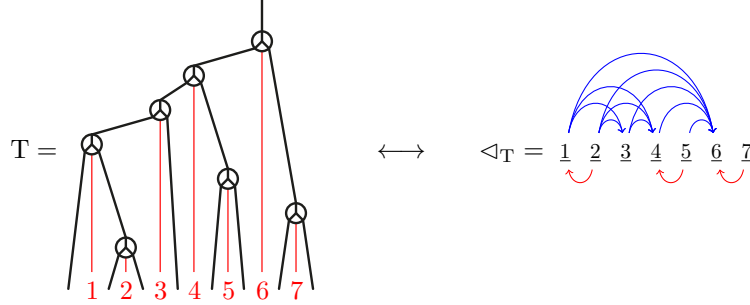


FIGURE 11. A Tamari Order Element Poset (TOEP).

We say that \triangleleft_T is a *Tamari order element poset*, and we denote by

$$\text{TOEP}(n) := \{ \triangleleft_T \mid T \in \mathfrak{B}(n) \}$$

the set of all Tamari order element posets on $[n]$. We first characterize them (see Proposition 63).

Proposition 40. *A poset $\triangleleft \in \mathcal{P}(n)$ is in $\text{TOEP}(n)$ if and only if*

- $\forall a < b < c, a \triangleleft c \implies b \triangleleft c$ and $a \triangleright c \implies a \triangleright b$,
- for all $a < c$ incomparable in \triangleleft , there exists $a < b < c$ such that $a \triangleleft b \triangleright c$.

Now we establish the relationship between the Tamari lattice on $\mathfrak{B}(n)$ and the weak order on $\text{TOEP}(n)$ (see Proposition 54).

Proposition 41. *For any binary trees $T, T' \in \mathfrak{B}(n)$, we have $T \preceq T'$ in the Tamari lattice if and only if $\triangleleft_T \preceq \triangleleft_{T'}$ in the weak order on posets.*

It follows that the subposet of the weak order (\mathcal{P}, \preceq) induced by the set $\text{TOEP}(n)$ is isomorphic to the Tamari lattice on $\mathfrak{B}(n)$, and is thus a lattice. We conclude on $\text{TOEP}(n)$ with the following stronger statement (see Theorem 90).

Proposition 42. *The set $\text{TOEP}(n)$ induces a sublattice of the weak order $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$.*

2.2.2. Tamari Order Interval Posets. For two binary trees $T, T' \in \mathfrak{B}(n)$ with $T \preceq T'$, we denote by $[T, T'] := \{ S \in \mathfrak{B}(n) \mid T \preceq S \preceq T' \}$ the Tamari order interval between T and T' . We can see this interval as the poset

$$\triangleleft_{[T, T']} := \bigcap_{T \preceq S \preceq T'} \triangleleft_S = \triangleleft_T \cap \triangleleft_{T'} = \triangleleft_T^{\text{Inc}} \cap \triangleleft_{T'}^{\text{Dec}}.$$

See Figure 12 for an example.

This poset $\triangleleft_{[T, T']}$ was introduced in [CP15] with the motivation that its linear extensions are precisely the linear extensions of all binary trees in the interval $[T, T']$. We say that $\triangleleft_{[T, T']}$ is a *Tamari order interval poset*, and we denote by

$$\text{TOIP}(n) := \{ \triangleleft_{[T, T']} \mid T, T' \in \mathfrak{B}(n), T \preceq T' \}$$

the set of all Tamari order interval posets on $[n]$. The following characterization of these posets (see Proposition 58) already appeared in [CP15, Thm. 2.8].

Corollary 43 ([CP15, Thm. 2.8]). *A poset $\triangleleft \in \mathcal{P}(n)$ is in $\text{TOIP}(n)$ if and only if $\forall a < b < c$,*

$$a \triangleleft c \implies b \triangleleft c \quad \text{and} \quad a \triangleright c \implies a \triangleright b.$$

Now we describe the weak order on $\text{TOIP}(n)$ (see Proposition 60, Corollary 2.4.3).

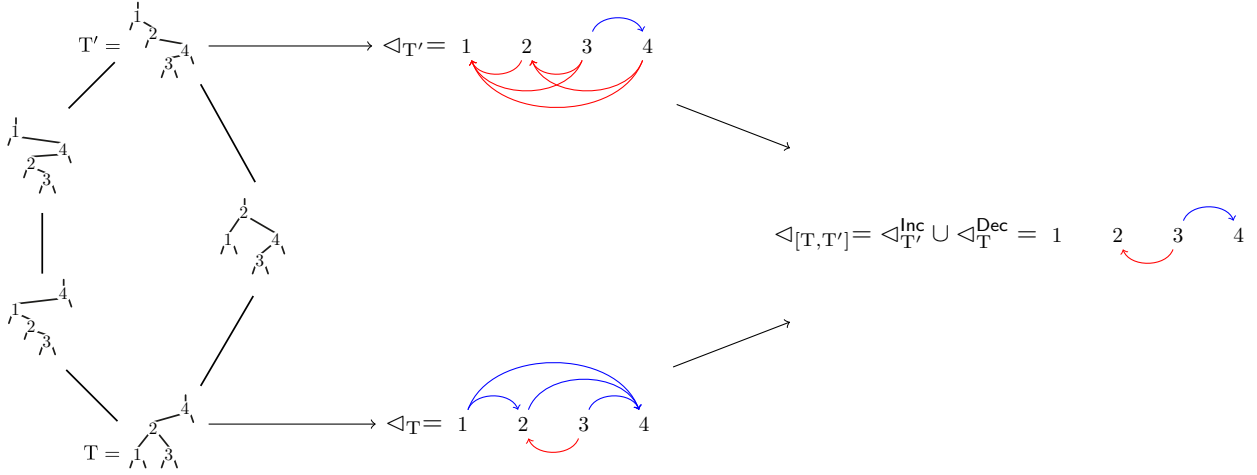


FIGURE 12. A Tamari Order Interval Poset (TOIP).

Proposition 44. *For any $S \preceq S'$ and $T \preceq T'$, we have $\triangleleft_{[S,S']} \preceq \triangleleft_{[T,T']} \iff S \preceq T$ and $S' \preceq T'$.*

Corollary 45. *The weak order $(\text{TOIP}(n), \preceq)$ is a lattice whose meet and join are given by*

$$\triangleleft_{[S,S']} \wedge_{\text{TOIP}} \triangleleft_{[T,T']} = \triangleleft_{[S \wedge_{\mathfrak{B}} T, S' \wedge_{\mathfrak{B}} T']} \quad \text{and} \quad \triangleleft_{[S,S']} \vee_{\text{TOIP}} \triangleleft_{[T,T']} = \triangleleft_{[S \vee_{\mathfrak{B}} T, S' \vee_{\mathfrak{B}} T']}.$$

Corollary 46. *The set $\text{TOEP}(n)$ induces a sublattice of the weak order $(\text{TOIP}, \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$.*

In fact, we will derive the following statement (see Corollary 85).

Proposition 47. *The set $\text{TOIP}(n)$ induces a sublattice of the weak order $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$.*

2.2.3. Tamari Order Face Posets. The binary trees of $\mathfrak{B}(n)$ correspond to the vertices of the associahedron $\text{Asso}(n)$ constructed e.g. by J.-L. Loday in [Lod04]. We now consider all the faces of the associahedron $\text{Asso}(n)$ which correspond to *Schröder trees*, i.e. planar rooted trees where each node has either none or at least two children. Given a Schröder tree S , we label the angles between consecutive children of the vertices of S in order, meaning that each angle is labeled after the angles in its left child and before the angles in its right child. Note that a binary tree T belongs to the face of the associahedron $\text{Asso}(n)$ corresponding to a Schröder tree S if and only if S is obtained by edge contractions in T . The set of such binary trees is an interval $[T^{\min}(S), T^{\max}(S)]$ in the Tamari lattice, where the minimal (resp. maximal) tree $T^{\min}(S)$ (resp. $T^{\max}(S)$) is obtained by replacing the nodes of S by left (resp. right) combs as illustrated in Figure 13.

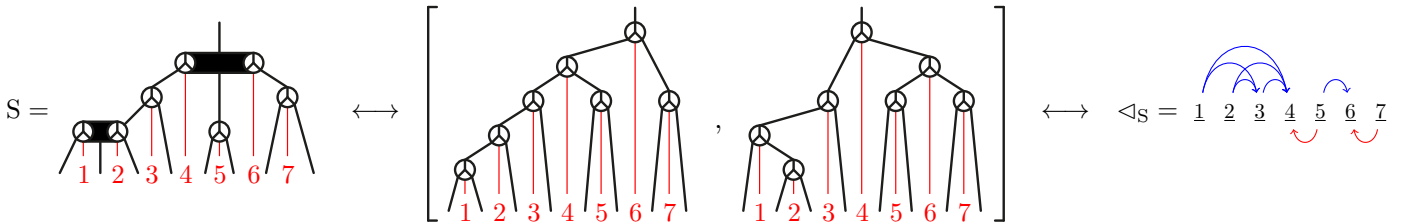


FIGURE 13. A Tamari Order Face Poset (TOFP).

We associate to a Schröder tree S the poset $\triangleleft_S := \triangleleft_{[T^{\min}(S), T^{\max}(S)]}$. Equivalently, $i \triangleleft_S j$ if and only if the angle i belongs to the left or the right child of the angle j . See Figure 13. Note that

- a binary tree T belongs to the face of the associahedron $\text{Asso}(n)$ corresponding to a Schröder tree S if and only if \triangleleft_T is an extension of \triangleleft_S , and

- the linear extensions of \triangleleft_S are precisely the linear extensions of \triangleleft_T for all binary trees T which belong to the face of the associahedron $\text{Asso}(n)$ corresponding to S .

We say that \triangleleft_S is a *Tamari order face poset*, and we denote by

$$\text{TOFP}(n) := \{ \triangleleft_S \mid S \text{ Schröder tree on } [n] \}$$

the set of all Tamari order face posets. We first characterize these posets (see Proposition 66).

Proposition 48. *A poset $\triangleleft \in \mathcal{P}(n)$ is in $\text{TOFP}(n)$ if and only if $\triangleleft \in \text{TOIP}(n)$ (see characterization in Corollary 43) and for all $a < c$ incomparable in \triangleleft , either there exists $a < b < c$ such that $a \not\prec b \not\prec c$, or for all $a < b < c$ we have $a \triangleright b \triangleleft c$.*

Consider now the weak order on $\text{TOFP}(n)$. It turns out (see Proposition 68) that this order on Schröder trees coincides with the *facial weak order* on the associahedron $\text{Asso}(n)$ studied in [PR06, NT06, DHP18]. This order is a quotient of the facial weak order on the permutahedron by the fibers of the Schröder tree insertion st . In particular, the weak order on $\text{TOFP}(n)$ is a lattice.

Remark 49. The example of Remark 32 shows that $(\text{TOFP}(n), \preceq, \wedge_{\text{TOFP}}, \vee_{\text{TOFP}})$ is not a sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$, nor a sublattice of $(\text{WOIP}(n), \preceq, \wedge_{\text{WOIP}}, \vee_{\text{WOIP}})$, nor a sublattice of $(\text{TOIP}(n), \preceq, \wedge_{\text{TOIP}}, \vee_{\text{TOIP}})$.

2.2.4. TOIP deletion. We finally define a projection from all posets of $\mathcal{P}(n)$ to $\text{TOIP}(n)$. We call *TOIP deletion* the map defined by

$$\triangleleft^{\text{TOIPd}} := \triangleleft \setminus (\{(a, c) \mid \exists a < b < c, b \not\prec c\} \cup \{(c, a) \mid \exists a < b < c, a \not\prec b\}).$$

Then $\triangleleft^{\text{TOIPd}} \in \text{TOIP}(n)$ for any poset $\triangleleft \in \mathcal{P}(n)$ (see Lemma 71). We compare this map with the binary search tree and Schröder tree insertions described earlier (see Proposition 73, Corollary 75 and Proposition 76).

Proposition 50. *For any permutation $\sigma \in \mathfrak{S}(n)$, for any permutations $\sigma, \sigma' \in \mathfrak{S}(n)$ with $\sigma \preceq \sigma'$, and for any ordered partition π of $[n]$, we have*

$$(\triangleleft_{\sigma})^{\text{TOIPd}} = \triangleleft_{\text{bt}(\sigma)}, \quad (\triangleleft_{[\sigma, \sigma']})^{\text{TOIPd}} = \triangleleft_{[\text{bt}(\sigma), \text{bt}(\sigma')]} \quad \text{and} \quad (\triangleleft_{\pi})^{\text{TOIPd}} = \triangleleft_{\text{st}(\pi)}.$$

Example 51. Compare Figures 6 and 11, Figures 7 and 12, and Figures 8 and 13.

2.3. FROM PERMUTREEHEDRA

Extending Sections 2.1 and 2.2, we describe further relevant families of posets corresponding to the elements, the faces, and the intervals in the permutreehedra introduced in [PP18]. This provides a wider range of examples and uniform proofs, at the cost of increasing the technicalities.

2.3.1. Permutree Element Posets. We first recall from [PP18] the definition of permutrees.

Definition 52 ([PP18]). *A permutree is a directed tree T with vertex set V endowed with a bijective vertex labeling $p : V \rightarrow [n]$ such that for each vertex $v \in V$,*

- (i) *v has one or two parents (outgoing neighbors), and one or two children (incoming neighbors);*
- (ii) *if v has two parents (resp. children), then all labels in the left ancestor (resp. descendant) subtree of v are smaller than $p(v)$ while all labels in the right ancestor (resp. descendant) subtree of v are larger than $p(v)$.*

The *orientation* of a permutree T is $\mathbb{O}(T) = (n, \mathbb{O}^+, \mathbb{O}^-)$ where \mathbb{O}^+ is the set of labels of the nodes with two parents while \mathbb{O}^- is the set of labels of the nodes with two children. Note that there is a priori no conditions on these sets \mathbb{O}^+ and \mathbb{O}^- : they can be empty, they can intersect, etc. For a given orientation $\mathbb{O} = (n, \mathbb{O}^+, \mathbb{O}^-)$, we denote by $\text{PT}(\mathbb{O})$ the set of permutrees with orientation \mathbb{O} .

Figure 14 gives five examples of permutrees. While the first is generic, the other four show that specific permutrees encode relevant combinatorial objects, depending on their orientations:

orientation $(n, \mathbb{O}^+, \mathbb{O}^-)$	$(n, \emptyset, \emptyset)$	$(n, \emptyset, [n])$	$\mathbb{O}^+ \sqcup \mathbb{O}^- = [n]$	$(n, [n], [n])$
combinatorial objects	permutations	binary trees	Cambrian trees [CP17]	binary sequences

See [PP18] for more details on the interpretation of these combinatorial objects as permutrees. We use drawing conventions from [PP18]: nodes are ordered by their labels from left to right, edges are oriented from bottom to top, and we draw a red wall separating the two parents or the two children of a node. Condition (ii) in Definition 52 says that no black edge can cross a red wall.

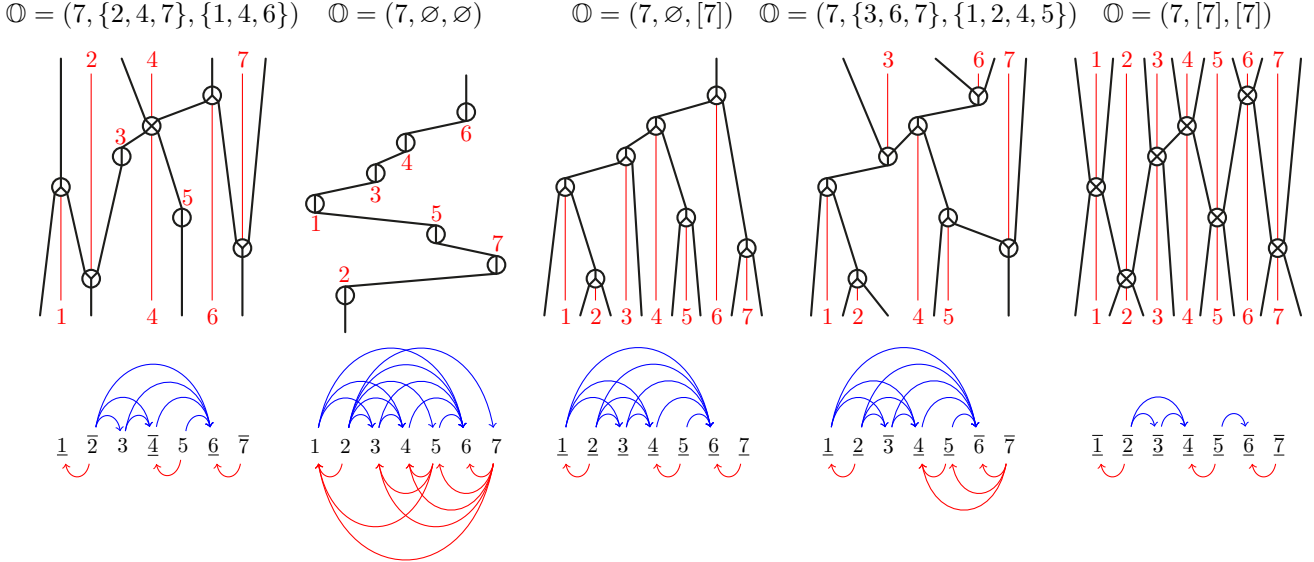


FIGURE 14. Five examples of permutrees T (top) with their posets \triangleleft_T (bottom). While the first is generic, the last four illustrate specific orientations corresponding to permutations, binary trees, Cambrian trees, and binary sequences.

For a permutree T , we denote by \triangleleft_T the transitive closure of T . That is to say, $i \triangleleft_T j$ if and only if there is an oriented path from i to j in T . See Figure 14 for illustrations. To visualize the orientation \mathbb{O} in the poset \triangleleft_T , we overline (resp. underline) the elements of \mathbb{O}^+ (resp. of \mathbb{O}^-).

We say that \triangleleft_T is a *permutree element poset* and we denote by

$$\text{PEP}(\mathbb{O}) := \{ \triangleleft_T \mid T \in \text{PT}(\mathbb{O}) \}$$

the set of all permutree element posets for a given orientation \mathbb{O} . These posets will be characterized in Proposition 63. For the moment, we need the following properties from [PP18].

Proposition 53 ([PP18]). *Fix an orientation $\mathbb{O} = (n, \mathbb{O}^+, \mathbb{O}^-)$ of $[n]$.*

- (1) *For a permutree $T \in \text{PT}(\mathbb{O})$, the set of linear extensions $\mathcal{L}(T)$ of \triangleleft_T is an interval in the weak order on $\mathfrak{S}(n)$ whose minimal element avoids the pattern $ca - b$ with $a < b < c$ and $b \in \mathbb{O}^-$ (denoted $3\bar{1}\underline{2}$) and the pattern $b - ca$ with $a < b < c$ and $b \in \mathbb{O}^+$ (denoted $\bar{2}3\bar{1}$), and whose maximal element avoids the pattern $ac - b$ with $a < b < c$ and $b \in \mathbb{O}^-$ (denoted $13\underline{2}$) and the pattern $b - ac$ with $a < b < c$ and $b \in \mathbb{O}^+$ (denoted $\bar{2}3\bar{1}$).*
- (2) *The collection of sets $\mathcal{L}(T) := \{\text{linear extensions of } \triangleleft_T\}$ for all permutrees $T \in \text{PT}(\mathbb{O})$ forms a partition of $\mathfrak{S}(n)$. This defines a surjection $\Psi_{\mathbb{O}}$ from $\mathfrak{S}(n)$ to $\text{PT}(\mathbb{O})$, which sends a permutation $\sigma \in \mathfrak{S}(n)$ to the unique permutree $T \in \text{PT}(\mathbb{O})$ such that $\sigma \in \mathcal{L}(T)$. This surjection can be described directly as an insertion algorithm (we skip this description and refer the interested reader to [PP18] as it is not needed for the purposes of this paper).*
- (3) *This partition defines a lattice congruence of the weak order (see [Rea04, Rea06, PP18] for details). Therefore, the set of permutrees $\text{PT}(\mathbb{O})$ is endowed with a lattice structure \preceq , called *permutree lattice*, defined by*

$$T \preceq T' \iff \exists \sigma, \sigma' \in \mathfrak{S}(n) \text{ such that } \Psi_{\mathbb{O}}(\sigma) = T, \Psi_{\mathbb{O}}(\sigma') = T' \text{ and } \sigma \preceq \sigma'$$

whose meet $\wedge_{\mathbb{O}}$ and join $\vee_{\mathbb{O}}$ are given by

$$T \wedge_{\mathbb{O}} T' = \Psi_{\mathbb{O}}(\sigma \wedge_{\mathfrak{S}} \sigma') \quad \text{and} \quad T \vee_{\mathbb{O}} T' = \Psi_{\mathbb{O}}(\sigma \vee_{\mathfrak{S}} \sigma')$$

for any representatives $\sigma, \sigma' \in \mathfrak{S}(n)$ such that $\Psi_{\mathbb{O}}(\sigma) = T$ and $\Psi_{\mathbb{O}}(\sigma') = T'$. In particular, $T \preceq T'$ if and only if $\sigma \preceq \sigma'$ where σ and σ' denote the minimal (resp. maximal) linear extensions of T and T' respectively.

- (4) This lattice structure can equivalently be described as the transitive closure of right rotations in permutrees as described in [PP18].
- (5) The minimal (resp. maximal) permutree in the permutree lattice is a left (resp. right) \mathbb{O} -comb: it is a chain where each vertex in \mathbb{O}^+ has an additional empty left (resp. right) parent, while each vertex in \mathbb{O}^- has an additional empty right (resp. left) child.

For example, we obtain well-known lattice structures for specific orientations:

orientation $(n, \mathbb{O}^+, \mathbb{O}^-)$	$(n, \emptyset, \emptyset)$	$(n, \emptyset, [n])$	$\mathbb{O}^+ \sqcup \mathbb{O}^- = [n]$	$(n, [n], [n])$
permutree lattice	classical weak order	Tamari lattice [MHPS12]	Cambrian lattices [Rea06]	boolean lattice

Now we establish the relationship between the permutree lattice on $\text{PT}(\mathbb{O})$ and the weak order on $\text{PEP}(\mathbb{O})$.

Proposition 54. *For any permutrees $T, T' \in \text{PT}(\mathbb{O})$, we have $T \preceq T'$ in the permutree lattice if and only if $\triangleleft_T \preceq \triangleleft_{T'}$ in the weak order on posets.*

Proof. By Proposition 53 (2), a permutree admits both a minimal and a maximal linear extensions. It follows that $\text{PEP}(\mathbb{O}) \subseteq \text{WOIP}(n)$ and the weak order on $\text{PEP}(\mathbb{O})$ is therefore given by

$$\triangleleft_T \preceq \triangleleft_{T'} \iff \triangleleft_T^{\text{minle}} \preceq \triangleleft_{T'}^{\text{minle}} \quad \text{and} \quad \triangleleft_T^{\text{maxle}} \preceq \triangleleft_{T'}^{\text{maxle}}$$

according to Proposition 27. However, we have already mentioned in Proposition 53 (3) that the two conditions on the right are both equivalent to $T \preceq T'$ in the permutree lattice. \square

Remark 55. In fact, we have that $T \preceq T' \iff \triangleleft_T \preceq \triangleleft_{T'} \iff \triangleleft_T^{\text{Inc}} \supseteq \triangleleft_{T'}^{\text{Inc}} \iff \triangleleft_T^{\text{Dec}} \subseteq \triangleleft_{T'}^{\text{Dec}}$.

Remark 56. Proposition 54 affirms that the subposet of the weak order (\mathcal{P}, \preceq) induced by the set $\text{PEP}(\mathbb{O})$ is isomorphic to the permutree lattice on $\text{PT}(\mathbb{O})$, and is thus a lattice. We will see in Remark 92 that the set $\text{PEP}(\mathbb{O})$ does not always induce a sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$. Theorem 90 will provide a sufficient condition on the orientation \mathbb{O} for this property. In contrast, we will see in Theorem 93 that $\text{PEP}(\mathbb{O})$ always induces a sublattice of $(\text{PIP}(\mathbb{O}), \preceq, \wedge_{\text{PIP}(\mathbb{O})}, \vee_{\text{PIP}(\mathbb{O})})$ and thus of $(\text{WOIP}(n), \preceq, \wedge_{\text{WOIP}}, \vee_{\text{WOIP}})$.

2.3.2. Permutree Interval Posets. For two permutrees $T, T' \in \text{PT}(\mathbb{O})$ with $T \preceq T'$, we denote by $[T, T'] := \{S \in \text{PT}(\mathbb{O}) \mid T \preceq S \preceq T'\}$ the permutree lattice interval between T and T' . As in Proposition 25, we can see this interval as the poset

$$\triangleleft_{[T, T']} := \bigcap_{T \preceq S \preceq T'} \triangleleft_S = \triangleleft_T \cap \triangleleft_{T'} = \triangleleft_{T'}^{\text{Inc}} \cap \triangleleft_T^{\text{Dec}}.$$

We say that $\triangleleft_{[T, T']}$ is a *permutree interval poset*, and we denote by

$$\text{PIP}(\mathbb{O}) := \{ \triangleleft_{[T, T']} \mid T, T' \in \text{PT}(\mathbb{O}), T \preceq T' \}$$

the set of all permutree interval posets for a given orientation \mathbb{O} .

We first aim at a concrete characterization of the posets of $\text{PIP}(\mathbb{O})$. Note that a poset is in $\text{PIP}(\mathbb{O})$ if and only if it admits a weak order minimal linear extension avoiding the patterns $31\bar{2}$ and $\bar{2}31$, and a weak order maximal linear extension avoiding the patterns $13\bar{2}$ and $\bar{2}13$. Similar to our study of $\text{IWOIP}(n)$ and $\text{DWOIP}(n)$ in Section 2.1.4, it is practical to consider these conditions separately. We thus define the set $\text{IPIP}(\mathbb{O})$ (resp. $\text{DPIP}(\mathbb{O})$) of posets which admit a maximal

(resp. minimal) linear extension that avoids the patterns $\bar{2}13$ and $13\bar{2}$ (resp. $\bar{2}31$ and $31\bar{2}$). In order to characterize these posets, we define

$$\begin{aligned} \text{IPIP}^+(\mathbb{O}) &:= \{\triangleleft \in \mathcal{P}(n) \mid \forall a < b < c \text{ with } b \in \mathbb{O}^+, a \triangleleft c \implies a \triangleleft b\}, \\ \text{IPIP}^-(\mathbb{O}) &:= \{\triangleleft \in \mathcal{P}(n) \mid \forall a < b < c \text{ with } b \in \mathbb{O}^-, a \triangleleft c \implies b \triangleleft c\}, \\ \text{IPIP}^\pm(\mathbb{O}) &:= \text{IPIP}^+(\mathbb{O}) \cap \text{IPIP}^-(\mathbb{O}), \end{aligned}$$

and similarly

$$\begin{aligned} \text{DPIP}^+(\mathbb{O}) &:= \{\triangleleft \in \mathcal{P}(n) \mid \forall a < b < c \text{ with } b \in \mathbb{O}^+, a \triangleright c \implies b \triangleright c\}, \\ \text{DPIP}^-(\mathbb{O}) &:= \{\triangleleft \in \mathcal{P}(n) \mid \forall a < b < c \text{ with } b \in \mathbb{O}^-, a \triangleright c \implies a \triangleright b\}, \\ \text{DPIP}^\pm(\mathbb{O}) &:= \text{IPIP}^+(\mathbb{O}) \cap \text{IPIP}^-(\mathbb{O}). \end{aligned}$$

Proposition 57. *For any orientation \mathbb{O} of $[n]$, we have*

$$\text{IPIP}(\mathbb{O}) = \text{IWOIP}(n) \cap \text{IPIP}^\pm(\mathbb{O}) \quad \text{and} \quad \text{DPIP}(\mathbb{O}) = \text{DWOIP}(n) \cap \text{DPIP}^\pm(\mathbb{O}).$$

Proof. Consider $\triangleleft \in \text{IWOIP}$ and let $\triangleleft^{\text{maxle}} = \triangleleft \cup \{(b, a) \mid a < b \text{ incomparable in } \triangleleft\}$ be its maximal linear extension (see the proof of Proposition 33). Assume first that there is $a < b < c$ with $b \in \mathbb{O}^+$ such that $a \triangleleft c$ while $a \not\triangleleft b$. Then we obtain $b \triangleleft^{\text{maxle}} a \triangleleft^{\text{maxle}} c$ which is a $\bar{2}13$ -pattern in $\triangleleft^{\text{maxle}}$. Reciprocally, if $\triangleleft^{\text{maxle}}$ contains a $\bar{2}13$ -pattern $b \triangleleft^{\text{maxle}} a \triangleleft^{\text{maxle}} c$ with $a < b < c$ and $b \in \mathbb{O}^+$, then $a \triangleleft c$ while $a \not\triangleleft b$ by definition of $\triangleleft^{\text{maxle}}$. We conclude that $\triangleleft^{\text{maxle}}$ avoids the pattern $\bar{2}13$ if and only if $\triangleleft \in \text{IPIP}^+(\mathbb{O})$. The proof for the other patterns is similar. \square

Corollary 58. *A poset \triangleleft is in $\text{PIP}(\mathbb{O})$ if and only if it is in $\text{WOIP}(n)$ (see characterization in Proposition 26) and satisfies the conditions of $\text{IPIP}^+(\mathbb{O})$, $\text{IPIP}^-(\mathbb{O})$, $\text{DPIP}^+(\mathbb{O})$ and $\text{DPIP}^-(\mathbb{O})$.*

Remark 59. Similarly to Remark 34, note that it suffices to check these conditions only for all cover relations $a \triangleleft c$ and $a \triangleright c$ in \triangleleft .

Some illustrations are given in Figure 15. The leftmost poset is not in $\text{PIP}(\mathbb{O})$: $\{1, 2, 3\}$ does not satisfy $\text{IPIP}^-(\mathbb{O})$, $\{2, 3, 5\}$ does not satisfy $\text{IPIP}^+(\mathbb{O})$, $\{3, 4, 6\}$ does not satisfy $\text{DWOIP}(6)$, and $\{3, 5, 6\}$ does not satisfy $\text{DPIP}^-(\mathbb{O})$. The other two posets of Figure 15 are both in $\text{PIP}(\mathbb{O})$.

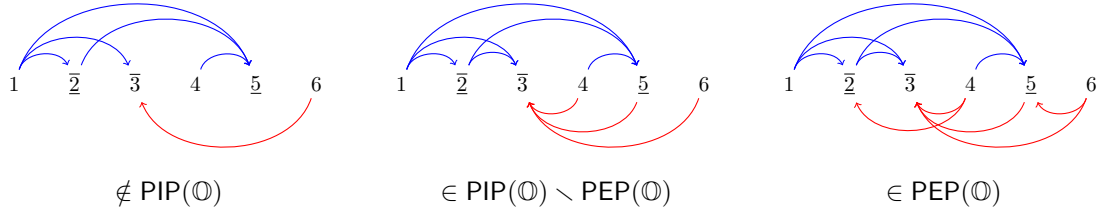


FIGURE 15. Examples and counterexamples of elements in $\text{PIP}(\mathbb{O})$ and $\text{PEP}(\mathbb{O})$, where $\mathbb{O} = (6, \{2, 3\}, \{2, 5\})$.

Now we describe the weak order on $\text{PIP}(\mathbb{O})$.

Proposition 60. *For any $S \preceq S'$ and $T \preceq T'$, we have $\triangleleft_{[S, S']} \preceq \triangleleft_{[T, T']} \iff S \preceq T$ and $S' \preceq T'$.*

Proof. The proof is similar to that of Proposition 27. \square

We immediately derive that $(\text{PIP}(\mathbb{O}), \preceq)$ has the lattice structure of a product.

Corollary 61. *The weak order $(\text{PIP}(\mathbb{O}), \preceq)$ is a lattice whose meet and join are given by*

$$\triangleleft_{[S, S']} \wedge_{\text{PIP}(\mathbb{O})} \triangleleft_{[T, T']} = \triangleleft_{[S \wedge_{\mathbb{O}} T, S' \wedge_{\mathbb{O}} T']} \quad \text{and} \quad \triangleleft_{[S, S']} \vee_{\text{PIP}(\mathbb{O})} \triangleleft_{[T, T']} = \triangleleft_{[S \vee_{\mathbb{O}} T, S' \vee_{\mathbb{O}} T']}.$$

Remark 62. As illustrated by $\text{WOIP}(n)$, the set $\text{PIP}(\mathbb{O})$ does not always induce a sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$. Theorem 84 will provide a sufficient condition on the orientation \mathbb{O} for this property. In contrast, we will see in Theorem 87 that $\text{PIP}(\mathbb{O})$ always induces a sublattice of $(\text{WOIP}(n), \preceq, \wedge_{\text{WOIP}}, \vee_{\text{WOIP}})$.

2.3.3. Characterization of $\text{PEP}(\mathbb{O})$. We are now ready to give a characterization of the posets of $\text{PEP}(\mathbb{O})$ left open in Section 2.3.1. We need one additional definition. For an orientation \mathbb{O} of $[n]$, an \mathbb{O} -snake in a poset \triangleleft is a sequence $x_0 < x_1 < \cdots < x_k < x_{k+1}$ such that

- either $x_0 \triangleleft x_1 \triangleright x_2 \triangleleft x_3 \triangleright \cdots$ with $\{x_i \mid i \in [k] \text{ odd}\} \subseteq \mathbb{O}^-$ and $\{x_i \mid i \in [k] \text{ even}\} \subseteq \mathbb{O}^+$,
- or $x_0 \triangleright x_1 \triangleleft x_2 \triangleright x_3 \triangleleft \cdots$ with $\{x_i \mid i \in [k] \text{ odd}\} \subseteq \mathbb{O}^+$ and $\{x_i \mid i \in [k] \text{ even}\} \subseteq \mathbb{O}^-$,

as illustrated in Figure 16.



FIGURE 16. Two \mathbb{O} -snakes joining x_0 to x_5 . The set \mathbb{O}^+ (resp. \mathbb{O}^-) must contain at least the overlined (resp. underlined) integers.

We say that the \mathbb{O} -snake $x_0 < x_1 < \cdots < x_k < x_{k+1}$ joins x_0 to x_{k+1} and has length k . Note that, by definition, we consider the relations $x \triangleleft y$ or $x \triangleright y$ themselves as (degenerate, length 0) \mathbb{O} -snakes between x and y .

Proposition 63. *A poset $\triangleleft \in \mathcal{P}(n)$ is in $\text{PEP}(\mathbb{O})$ if and only if it is in $\text{PIP}(\mathbb{O})$ (see characterization in Corollary 58) and it admits an \mathbb{O} -snake between any two values of $[n]$.*

Figure 15 illustrates this proposition: the middle poset is not in $\text{PEP}(\mathbb{O})$, since there is no \mathbb{O} -snake between 1 and 4 nor between 1 and 6. In contrast, the rightmost poset is in $\text{PEP}(\mathbb{O})$, as $1 \triangleleft \underline{2} \triangleright 4$ is an \mathbb{O} -snake between 1 and 4 and $1 \triangleleft \underline{5} \triangleright 6$ is an \mathbb{O} -snake between 1 and 6.

Proof or Proposition 63. Assume that $\triangleleft \in \text{PEP}(\mathbb{O})$, and let $T \in \text{PT}(\mathbb{O})$ be the permutree such that $\triangleleft = \triangleleft_T$. Then \triangleleft is certainly in $\text{PIP}(\mathbb{O})$. Now any two values $x, y \in [n]$ are connected by a (non-oriented) path in T , and recalling the local optima along this path provides an \mathbb{O} -snake joining x and y .

Reciprocally, consider $\triangleleft \in \text{PIP}(\mathbb{O})$ such that there is an \mathbb{O} -snake between any two values of $[n]$. We need the following two intermediate claims, proved in detail in Appendix A.2.

Claim C. For any $u, v, w \in [n]$ with $u < w$,

- if $u \triangleleft v$ and $v \triangleright w$ are cover relations of \triangleleft , then $u < v < w$ and $v \in \mathbb{O}^-$;
- if $u \triangleright v$ and $v \triangleleft w$ are cover relations of \triangleleft , then $u < v < w$ and $v \in \mathbb{O}^+$.

Claim D. Let $x_0, \dots, x_p \in [n]$ be a path in the Hasse diagram of \triangleleft (i.e. $x_{i-1} \triangleleft x_i$ or $x_{i-1} \triangleright x_i$ are cover relations in \triangleleft for any $i \in [p]$). Assume moreover that $x_0 \in \mathbb{O}^-$ and $x_0 \triangleright x_1$, or that $x_0 \in \mathbb{O}^+$ and $x_0 \triangleleft x_1$. Then all x_i are on the same side of x_0 , i.e. $x_0 < x_1 \iff x_0 < x_i$ for all $i \in [p]$.

Claims C and D show that the Hasse diagram of \triangleleft is a permutree:

- it is connected since any two values are connected by a snake,
- it cannot contain a cycle (otherwise, since this cycle cannot be oriented, there exist three distinct vertices u, v, w in this cycle with $u \triangleleft v \triangleright w$. Claim C ensures that $u < v < w$ and $v \in \mathbb{O}^-$. Since there is a path $v = x_0, w = x_1, x_2, \dots, x_p = u$ in the Hasse diagram of \triangleleft with $v \in \mathbb{O}^-$ and $v \triangleright w$, Claim D affirms that u and w are on the same side of v , a contradiction), and
- it fulfills the local conditions of Definition 52 to be a permutree (Claim C shows Condition (i) of Definition 52, and Claim D shows Condition (ii) of Definition 52). \square

For further purposes, we will need the following lemma to check the existence of \mathbb{O} -snakes.

Lemma 64. *Let $\triangleleft \in \mathcal{P}(n)$ and $a < c$ be incomparable in \triangleleft . The following are equivalent:*

- There is an \mathbb{O} -snake between a and c ,
- $\exists a < b < c$ such that there is an \mathbb{O} -snake between a and b , and either $b \in \mathbb{O}^-$ and $b \triangleright c$, or $b \in \mathbb{O}^+$ and $b \triangleleft c$,
- $\exists a < b < c$ such that there is an \mathbb{O} -snake between b and c , and either $b \in \mathbb{O}^-$ and $a \triangleleft b$, or $b \in \mathbb{O}^+$ and $a \triangleright b$.

Proof. The implication (i) \Rightarrow (ii) is immediate, considering $b = x_k$. Assume now that \triangleleft and $\{a, c\}$ satisfy (ii). Let b be given by (ii) and let $a < x_1 < \dots < x_k < b$ be an \mathbb{O} -snake between a and b . If $x_k \triangleleft b \triangleright c$ (or similarly if $x_k \triangleright b \triangleleft c$), then $a < x_1 < \dots < b < c$ is a \mathbb{O} -snake between a and c . In contrast, if $x_k \triangleleft b \triangleleft c$ (or similarly if $x_k \triangleright b \triangleright c$), then $x_k \triangleleft c$ (resp. $x_k \triangleright c$) by transitivity of \triangleleft , so that $a < x_1 < \dots < x_k < c$ is an \mathbb{O} -snake between a and c . Therefore, (i) \iff (ii). The proof of (i) \iff (iii) is identical. \square

2.3.4. Permutree Face Posets. The permutrees of $\text{PT}(\mathbb{O})$ correspond to the vertices of the \mathbb{O} -permutreehedron $\text{PT}(\mathbb{O})$ constructed in [PP18]. The precise definition of these polytopes is not needed here. Following Figure 14, we illustrate in Figure 17 classical polytopes that arise as permutreehedra for specific orientations:

orientation $(n, \mathbb{O}^+, \mathbb{O}^-)$	$(n, \emptyset, \emptyset)$	$(n, \emptyset, [n])$	$\mathbb{O}^+ \sqcup \mathbb{O}^- = [n]$	$(n, [n], [n])$
permutreehedron	permutahedron	Loday's associahedron [Lod04]	Hohlweg-Lange's associahedra [HL07]	parallelepiped

$\text{PT}(4, \emptyset, \{2\})$ $\text{PT}(4, \emptyset, \emptyset)$ $\text{PT}(4, \emptyset, [4])$ $\text{PT}(4, \{1, 3\}, \{2, 4\})$ $\text{PT}(4, [4], [4])$

FIGURE 17. Five examples of permutreehedra. While the first is generic, the last four are a permutahedron, some associahedra [Lod04, HL07], and a parallelepiped.

We now consider all the faces of the \mathbb{O} -permutreehedron. As shown in [PP18], they correspond to *Schröder \mathbb{O} -permutrees*, defined as follows.

Definition 65 ([PP18]). *For an orientation \mathbb{O} on $[n]$ and a subset $X \subseteq [n]$, we let $X^- := X \cap \mathbb{O}^-$ and $X^+ := X \cap \mathbb{O}^+$. A **Schröder \mathbb{O} -permutree** is a directed tree S with vertex set V endowed with a vertex labeling $p : V \rightarrow 2^{[n]} \setminus \emptyset$ such that*

- (i) *the labels of S partition $[n]$, i.e. $v \neq w \in V \implies p(v) \cap p(w) = \emptyset$ and $\bigcup_{v \in V} p(v) = [n]$;*
- (ii) *each vertex $v \in V$ has one incoming (resp. outgoing) subtree S_v^I (resp. S_v^O) for each interval I of $[n] \setminus p(v)^-$ (resp. of $[n] \setminus p(v)^+$) and all labels of S_v^I (resp. of S_v^O) are subsets of I .*

We denote by $\text{SchrPT}(\mathbb{O})$ the set of Schröder \mathbb{O} -permutrees.

For example, in the leftmost Schröder permutree of Figure 18, the vertices are labeled by the sets $\{1, 2\}$, $\{3\}$, $\{4, 6\}$, $\{5\}$, and $\{6, 7\}$. The vertex v labeled by $p(v) = \{4, 6\}$ has 3 incoming subtrees included in the 3 intervals of $[n] \setminus p(v)^- = [n] \setminus \{4, 6\} = \{1, 2, 3\} \sqcup \{5\} \sqcup \{7\}$ and 2 (empty) outgoing subtrees included in the 2 intervals of $[n] \setminus p(v)^+ = [n] \setminus \{4\} = \{1, 2, 3\} \sqcup \{5, 6, 7\}$.

Following Figure 14, we have represented in Figure 18 five Schröder permutrees, where the last four encode relevant combinatorial objects obtained for specific orientations:

orientation $(n, \mathbb{O}^+, \mathbb{O}^-)$	$(n, \emptyset, \emptyset)$	$(n, \emptyset, [n])$	$\mathbb{O}^+ \sqcup \mathbb{O}^- = [n]$	$(n, [n], [n])$
combinatorial objects	ordered partitions	Schröder trees	Schröder Cambrian trees [CP17]	ternary sequences

We refer again to [PP18] for more details on the interpretation of these combinatorial objects as permutrees, and we still use the drawing conventions of [PP18].

An \mathbb{O} -permutree T belongs to a face of the permutreehedron $\text{PT}(\mathbb{O})$ corresponding to a Schröder \mathbb{O} -permutree S if and only if S is obtained by edge contractions in T . The set of such \mathbb{O} -permutrees is the interval $[\text{T}^{\min}(S), \text{T}^{\max}(S)]$ of the \mathbb{O} -permutree lattice, where the minimal (resp. maximal) tree $\text{T}^{\min}(S)$ (resp. $\text{T}^{\max}(S)$) is obtained by replacing the nodes of S by left (resp. right) combs as illustrated in Figure 19. To be more precise, we need additional notations. For an interval $I = [i, j]$ of integers, define $\bar{I} := [i - 1, j + 1]$. For each edge $v \rightarrow w$ in S , we let I_v^w (resp. J_v^w) denote the

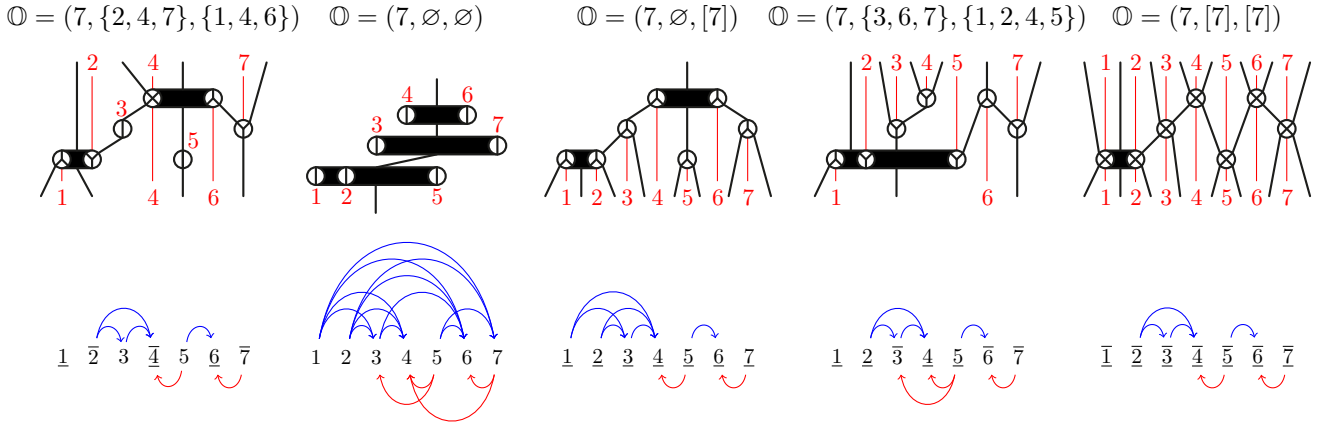


FIGURE 18. Five examples of Schröder permutrees S (top) with their posets \triangleleft_S (bottom). While the first is generic, the last four illustrate specific orientations corresponding to ordered partitions, Schröder trees, Schröder Cambrian trees, and ternary sequences.

interval of $[n] \setminus p(v)^+$ (resp. of $[n] \setminus p(w)^-$) such that $p(w) \subseteq I_v^w$ (resp. $p(v) \subseteq J_v^w$). The minimal and maximal permutrees in the face corresponding to the Schröder permutree S are then described as follows:

- (i) $T^{\min}(S)$ is the \mathbb{O} -permutree obtained from the left combs on the subsets $p(v)$ for v vertex of S by adding the edges connecting $\max(p(v) \cap \bar{I}_v^w)$ with $\min(p(w) \cap \bar{J}_v^w)$ for all edges $v \rightarrow w$ in S .
- (ii) $T^{\max}(S)$ is the \mathbb{O} -permutree obtained from the right combs on the subsets $p(v)$ for v vertex of S by adding the edges connecting $\min(p(v) \cap \bar{I}_v^w)$ with $\max(p(w) \cap \bar{J}_v^w)$ for all edges $v \rightarrow w$ in S .

For example, consider the edge $v = \{5\} \rightarrow \{4, 6\} = w$ in the Schröder permutree of Figure 19. We have $I_v^w = [n]$ and $J_v^w = \{5\}$. In $T^{\min}(S)$, we create the left comb $4 \rightarrow 6$ and we add the edge $5 = \max(\{5\} \cap [n]) \rightarrow \min(\{4, 6\} \cap \{4, 5, 6\}) = 4$. Similarly, in $T^{\max}(S)$, we create the right comb $6 \rightarrow 4$ and we add the edge $5 = \min(\{5\} \cap [n]) \rightarrow \max(\{4, 6\} \cap \{4, 5, 6\}) = 6$.

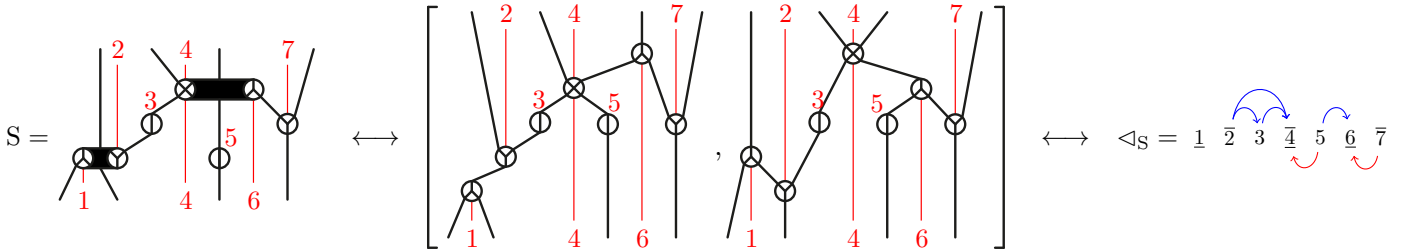


FIGURE 19. A Permutree Face Poset ($\text{PFP}(\mathbb{O})$).

For a Schröder permutree S , we define $\triangleleft_S = \triangleleft_{[T^{\min}(S), T^{\max}(S)]}$. Examples are given in Figure 18. One easily checks that \triangleleft_S could also be defined as the transitive closure of all relations $i \triangleleft_S j$ for all $i \in p(v) \cap \bar{I}_v^w$ and $j \in p(w) \cap \bar{J}_v^w$ for all edges $v \rightarrow w$ in S . For edge $\{5\} \rightarrow \{4, 6\}$ in the Schröder permutree of Figure 19, this corresponds to the relations $4 \triangleright 5 \triangleleft 6$ of the poset. Note that

- an \mathbb{O} -permutree T belongs to the face of the permutreehedron $\text{PT}(\mathbb{O})$ corresponding to a Schröder \mathbb{O} -permutree S if and only if \triangleleft_T is an extension of \triangleleft_S , and
- the linear extensions of \triangleleft_S are precisely the linear extensions of \triangleleft_T for all \mathbb{O} -permutrees T which belong to the face of the permutreehedron $\text{PT}(\mathbb{O})$ corresponding to S .

We say that \triangleleft_S is a *permutree face poset*, and we denote by

$$\text{PFP}(\mathbb{O}) := \{ \triangleleft_S \mid S \in \text{SchrPT}(\mathbb{O}) \}$$

the set of all permutree face posets. We now characterize these posets.

Proposition 66. *A poset $\triangleleft \in \mathcal{P}(n)$ is in $\text{PFP}(\mathbb{O})$ if and only if it is in $\text{PIP}(\mathbb{O})$ and for any $a < c$ incomparable in \triangleleft ,*

$$\begin{aligned} \exists a < b < c \text{ such that } b \in \mathbb{O}^+ \text{ and } a \not\triangleleft b \not\triangleright c \text{ or } b \in \mathbb{O}^- \text{ and } a \not\triangleright b \not\triangleleft c, & \quad (\spadesuit) \\ \text{or } \forall a < b < c \text{ we have } a \triangleleft b \iff b \triangleright c \text{ and } a \triangleright b \iff b \triangleleft c. & \quad (\clubsuit) \end{aligned}$$

This property is illustrated on the poset of Figure 19. For example, 1 and 2 are neighbors and thus satisfy (\clubsuit) , 1 and 3 satisfy (\spadesuit) with $b = 2$, 4 and 6 satisfy (\clubsuit) , etc.

Proof of Proposition 66. Assume that $\triangleleft \in \text{PFP}(\mathbb{O})$, and consider the Schröder permutree S such that $\triangleleft = \triangleleft_S$. Then $\triangleleft = \triangleleft_S = \triangleleft_{[\text{T}^{\min}(S), \text{T}^{\max}(S)]}$ belongs to $\text{PIP}(\mathbb{O})$. Moreover, any $a < c$ incomparable in \triangleleft_S

- (i) satisfy (\spadesuit) when either a, c belong to distinct vertices of S separated by a wall, or a, c belong to the same vertex v of S but there is another b in v with $b \in \mathbb{O}^+ \cup \mathbb{O}^-$ and $a < b < c$,
- (ii) satisfy (\clubsuit) when a, c belong to the same vertex v of S and $b \notin \mathbb{O}^+ \cup \mathbb{O}^-$ for any $a < b < c$ in v .

This shows one implication of the statement. Before proving the reciprocal implication, let us comment a little more to give some useful intuition. Note that two consecutive elements $a < c$ in a vertex v of S satisfy (\clubsuit) and not (\spadesuit) . In particular, if all $a < c$ incomparable in \triangleleft_S satisfy (\spadesuit) , then S is just a permutree. In general, the posets $\triangleleft_{\text{T}^{\min}(S)}$ and $\triangleleft_{\text{T}^{\max}(S)}$ corresponding to the minimal and maximal \mathbb{O} -permutrees in the face corresponding to S are given by

$$\begin{aligned} \triangleleft_{\text{T}^{\min}(S)} &= (\triangleleft \cup \{(a, c) \mid a < c \text{ incomparable in } \triangleleft_S \text{ and not satisfying } (\spadesuit)\})^{\text{tc}} \\ \text{and } \triangleleft_{\text{T}^{\max}(S)} &= (\triangleleft \cup \{(c, a) \mid a < c \text{ incomparable in } \triangleleft_S \text{ and not satisfying } (\spadesuit)\})^{\text{tc}}. \end{aligned}$$

Consider now an arbitrary poset $\triangleleft \in \text{PIP}(\mathbb{O})$ such that any $a < c$ incomparable in \triangleleft satisfy (\spadesuit) or (\clubsuit) . The previous observation motivates the following claim (see Appendix A.3 for the proof).

Claim E. If any $a < c$ incomparable in \triangleleft satisfy (\spadesuit) , then $\triangleleft \in \text{PEP}(\mathbb{O}) \subset \text{PFP}(\mathbb{O})$.

Suppose now that some $a < c$ incomparable in \triangleleft do not satisfy (\spadesuit) . The idea of our proof is to return to the previous claim by considering the auxiliary poset

$$\blacktriangleleft := (\triangleleft \cup \{(a, c) \mid a < c \text{ incomparable in } \triangleleft \text{ and not satisfying } (\spadesuit)\})^{\text{tc}}.$$

Claim F. We have $\triangleleft^{\text{Inc}} \subseteq \blacktriangleleft^{\text{Inc}}$ and $\triangleleft^{\text{Dec}} = \blacktriangleleft^{\text{Dec}}$.

Claim G. If $\triangleleft \in \text{PIP}(\mathbb{O})$ and any $a < c$ incomparable in \triangleleft satisfy (\spadesuit) or (\clubsuit) , then $\blacktriangleleft \in \text{PIP}(\mathbb{O})$ and any $a < c$ incomparable in \blacktriangleleft satisfy (\spadesuit) .

Combining Claims E and G, we obtain that there exists a permutree T such that $\blacktriangleleft = \triangleleft_T$. Intuitively, T is the minimal permutree in the face that will correspond to \triangleleft . To find the Schröder permutree of this face, we thus just need to contract some edges in T . We therefore consider the Schröder permutree S obtained from T by contracting all edges that appear in the Hasse diagram of \blacktriangleleft but are not in \triangleleft .

Claim H. We have $\triangleleft = \triangleleft_S$, so that $\triangleleft \in \text{PFP}(\mathbb{O})$.

The detailed proofs of Claims E to H are given in Appendix A.3. This concludes the proof of Proposition 66. \square

We now consider the weak order on $\text{PFP}(\mathbb{O})$. Let us first recall from [PP18] the definition of the Schröder permutree lattice.

Proposition 67 ([PP18]). *Fix an orientation $\mathbb{O} = (n, \mathbb{O}^+, \mathbb{O}^-)$ of $[n]$.*

- (1) *Each Schröder \mathbb{O} -permutree corresponds to a face of the permutreehedron $\text{PT}(\mathbb{O})$, and thus to a cone of its normal fan. Moreover, the normal fan of the permutreehedron $\text{Perm}(n)$ refines that of the permutreehedron $\text{PT}(\mathbb{O})$. This defines a surjection $\Psi_{\mathbb{O}}$ from the set of ordered partitions of $[n]$ to the set of Schröder permutrees of $\text{SchrPT}(\mathbb{O})$, which sends an ordered partition π to the unique Schröder permutree S satisfying that the interior of the normal cone of the face of $\text{PT}(\mathbb{O})$ corresponding to S contains the interior of the normal cone of the face of $\text{Perm}(n)$ corresponding to π .*

(2) The fibers of this surjection $\Psi_{\mathbb{O}}$ define a lattice congruence of the facial weak order discussed in Section 2.1.3 (see [PP18] for details). Therefore, the set of Schröder permutrees $\text{SchrPT}(\mathbb{O})$ is endowed with a lattice structure \preceq , called **Schröder permutree lattice**, defined by

$$S \preceq S' \iff \exists \pi, \pi' \text{ such that } \Psi_{\mathbb{O}}(\pi) = S, \Psi_{\mathbb{O}}(\pi') = S' \text{ and } \pi \preceq \pi'$$

(3) The contraction of an edge $e = v \rightarrow w$ in a Schröder permutree S is called **increasing** if $\max(p(v)) < \min(p(w))$ and **decreasing** if $\max(p(w)) < \min(p(v))$. The Schröder permutree lattice is the transitive closure of the relations $S \prec S/e$ (resp. $S/e \prec S$) for any Schröder permutree S and edge $e \in S$ defining an increasing (resp. decreasing) contraction.

Now we establish the relationship between the permutree order on $\text{PT}(\mathbb{O})$ and the weak order on $\text{PEP}(\mathbb{O})$.

Proposition 68. *For any Schröder permutrees $S, S' \in \text{SchrPT}(\mathbb{O})$, we have $S \preceq S'$ in the Schröder permutree lattice if and only if $\triangleleft_S \preceq \triangleleft_{S'}$ in the weak order on posets.*

Proof. We can identify the Schröder \mathbb{O} -permutree S with:

- (i) the interval $[\text{T}^{\min}(S), \text{T}^{\max}(S)]$ of \mathbb{O} -permutrees that belong to the face of $\text{PT}(\mathbb{O})$ given by S ,
- (ii) the interval $[\pi^{\min}(S), \pi^{\max}(S)]$ of ordered partitions π such that the interior of the normal cone of $\text{Perm}(n)$ corresponding to π is included in the interior of the normal cone of $\text{PT}(\mathbb{O})$ corresponding to S ,
- (iii) the interval $[\sigma^{\min}(S), \sigma^{\max}(S)]$ of $\mathfrak{S}(n)$ between the minimal and maximal extensions of \triangleleft_S .

It is immediate to check that $\sigma^{\min}(S)$ is the minimal linear extension of $\text{T}^{\min}(S)$ and of $\pi^{\min}(S)$ and that $\sigma^{\max}(S)$ is the maximal linear extension of $\text{T}^{\max}(S)$ and of $\pi^{\max}(S)$. We conclude that

$$\begin{aligned} S \preceq S' &\iff \pi^{\min}(S) \preceq \pi^{\min}(S') \text{ and } \pi^{\max}(S) \preceq \pi^{\max}(S') \\ &\iff \sigma^{\min}(S) \preceq \sigma^{\min}(S') \text{ and } \sigma^{\max}(S) \preceq \sigma^{\max}(S') \\ &\iff \text{T}^{\min}(S) \preceq \text{T}^{\min}(S') \text{ and } \text{T}^{\max}(S) \preceq \text{T}^{\max}(S') \iff \triangleleft_S \preceq \triangleleft_{S'}. \end{aligned}$$

The first line holds by definition of the Schröder permutree lattice (as $\Psi_{\mathbb{O}}^{-1}(S) = [\pi^{\min}(S), \pi^{\max}(S)]$), while the last holds by definition of the weak order on $\text{PFP}(\mathbb{O})$ (as $\triangleleft_S = \triangleleft_{[\text{T}^{\min}(S), \text{T}^{\max}(S)]}$). \square

Remark 69. Although the weak order on $\text{PFP}(\mathbb{O})$ is a lattice, the example of Remark 32 shows that it is not a sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$, nor a sublattice of $(\text{WOIP}(n), \preceq, \wedge_{\text{WOIP}}, \vee_{\text{WOIP}})$, nor a sublattice of $(\text{PIP}(\mathbb{O}), \preceq, \wedge_{\text{PIP}(\mathbb{O})}, \vee_{\text{PIP}(\mathbb{O})})$. We will discuss an alternative description of the meet and join in $\text{PFP}(\mathbb{O})$ in Section 2.4.4.

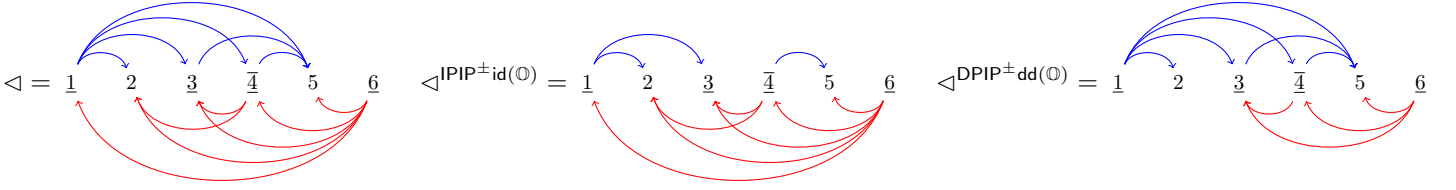
2.3.5. PIP(\mathbb{O}) deletion. Similar to the projection maps of Sections 2.1.4 and 2.2.4, we define the **IPIP $^+(\mathbb{O})$** (resp. **IPIP $^-(\mathbb{O})$** , **IPIP $^{\pm}(\mathbb{O})$** , **IPIP($\mathbb{O})$)** *increasing deletion* by

$$\begin{aligned} \triangleleft^{\text{IPIP}^+\text{id}(\mathbb{O})} &:= \triangleleft \setminus \{(a, c) \mid \exists a < b < c, b \in \mathbb{O}^+ \text{ and } a \not\prec b\}, \\ \triangleleft^{\text{IPIP}^-\text{id}(\mathbb{O})} &:= \triangleleft \setminus \{(a, c) \mid \exists a < b < c, b \in \mathbb{O}^- \text{ and } b \not\prec c\}, \\ \triangleleft^{\text{IPIP}^{\pm}\text{id}(\mathbb{O})} &:= \triangleleft \setminus \{(a, c) \mid \exists a \leq n < p \leq c \text{ with } n \in \{a\} \cup \mathbb{O}^- \text{ while } p \in \{c\} \cup \mathbb{O}^+ \text{ and } n \not\prec p\}, \\ \triangleleft^{\text{IPIPid}(\mathbb{O})} &:= (\triangleleft^{\text{IWOIPid}})^{\text{IPIP}^{\pm}\text{id}(\mathbb{O})}, \end{aligned}$$

and similarly the **DPIP $^+(\mathbb{O})$** (resp. **DPIP $^-(\mathbb{O})$** , **DPIP $^{\pm}(\mathbb{O})$** , **DPIP($\mathbb{O})$)** *decreasing deletion* by

$$\begin{aligned} \triangleleft^{\text{DPIP}^+\text{dd}(\mathbb{O})} &:= \triangleleft \setminus \{(c, a) \mid \exists a < b < c, b \in \mathbb{O}^+ \text{ and } b \not\prec c\}, \\ \triangleleft^{\text{DPIP}^-\text{dd}(\mathbb{O})} &:= \triangleleft \setminus \{(c, a) \mid \exists a < b < c, b \in \mathbb{O}^- \text{ and } a \not\prec b\}, \\ \triangleleft^{\text{DPIP}^{\pm}\text{dd}(\mathbb{O})} &:= \triangleleft \setminus \{(c, a) \mid \exists a \leq p < n \leq c \text{ with } p \in \{a\} \cup \mathbb{O}^+ \text{ while } n \in \{c\} \cup \mathbb{O}^- \text{ and } p \not\prec n\}, \\ \triangleleft^{\text{DPIPdd}(\mathbb{O})} &:= (\triangleleft^{\text{DWOIPdd}})^{\text{DPIP}^{\pm}\text{dd}(\mathbb{O})}. \end{aligned}$$

These operations are illustrated on Figure 20.

FIGURE 20. The $\text{IPIP}^\pm(\mathbb{O})$ increasing deletion and the $\text{DPIP}^\pm(\mathbb{O})$ decreasing deletion.

Remark 70. Similar to Remarks 11 and 35, for any $\varepsilon \in \{\emptyset, -, +, \pm\}$, the $\text{IPIP}^\varepsilon(\mathbb{O})$ increasing deletion (resp. $\text{DPIP}^\varepsilon(\mathbb{O})$ decreasing deletion) deletes at once all increasing relations which prevent the poset to be in $\text{IPIP}^\varepsilon(\mathbb{O})$ (resp. in $\text{DPIP}^\varepsilon(\mathbb{O})$). Note that we have

$$\triangleleft^{\text{IPIP}^+\text{id}(\mathbb{O})} = \triangleleft^{\text{IPIP}^\pm\text{id}(\mathbb{O}^+, \emptyset)} \subseteq \triangleleft^{\text{IPIP}^\pm\text{id}(\mathbb{O})} \quad \text{and} \quad \triangleleft^{\text{IPIP}^-\text{id}(\mathbb{O})} = \triangleleft^{\text{IPIP}^\pm\text{id}(\emptyset, \mathbb{O}^-)} \subseteq \triangleleft^{\text{IPIP}^\pm\text{id}(\mathbb{O})}.$$

However, we do not necessarily have $\triangleleft^{\text{IPIP}^\pm\text{id}(\mathbb{O})} = \triangleleft^{\text{IPIP}^+\text{id}(\mathbb{O})} \cap \triangleleft^{\text{IPIP}^-\text{id}(\mathbb{O})}$. Consider for example the poset $\triangleleft := \{(1, 3), (2, 4), (1, 4)\}$ and the orientation $(4, \{3\}, \{2\})$. Then $\triangleleft^{\text{IPIP}^+\text{id}(\mathbb{O})} = \{(2, 4), (1, 4)\}$, $\triangleleft^{\text{IPIP}^-\text{id}(\mathbb{O})} = \{(1, 3), (1, 4)\}$ so that $\triangleleft^{\text{IPIP}^+\text{id}(\mathbb{O})} \cap \triangleleft^{\text{IPIP}^-\text{id}(\mathbb{O})} = \{(1, 4)\} \neq \emptyset = \triangleleft^{\text{IPIP}^\pm\text{id}(\mathbb{O})}$. In other words, we might have to iterate several times the maps $\triangleleft \mapsto \triangleleft^{\text{IPIP}^+\text{id}(\mathbb{O})}$ and $\triangleleft \mapsto \triangleleft^{\text{IPIP}^-\text{id}(\mathbb{O})}$ to obtain the map $\triangleleft \mapsto \triangleleft^{\text{IPIP}^\pm\text{id}(\mathbb{O})}$. This explains the slightly more intricate definition of the map $\triangleleft \mapsto \triangleleft^{\text{IPIP}^\pm\text{id}(\mathbb{O})}$. The same remark holds for the map $\triangleleft \mapsto \triangleleft^{\text{DPIP}^\pm\text{dd}(\mathbb{O})}$.

Lemma 71. For any poset $\triangleleft \in \mathcal{P}(n)$ and any $\varepsilon \in \{\emptyset, -, +, \pm\}$, we have $\triangleleft^{\text{IPIP}^\varepsilon\text{id}(\mathbb{O})} \in \text{IPIP}^\varepsilon(\mathbb{O})$ and $\triangleleft^{\text{DPIP}^\varepsilon\text{dd}(\mathbb{O})} \in \text{DPIP}^\varepsilon(\mathbb{O})$.

Proof. We split the proof into three technical claims whose proofs are given in Appendix A.4.

Claim I. $\triangleleft^{\text{IPIP}^\pm\text{id}(\mathbb{O})}$ is a poset.

Claim J. $\triangleleft^{\text{IPIP}^\pm\text{id}(\mathbb{O})}$ is in $\text{IPIP}^\pm(\mathbb{O})$.

This proves the result for $\triangleleft^{\text{IPIP}^\pm\text{id}(\mathbb{O})}$. Note that it already contains the result for $\triangleleft^{\text{IPIP}^+\text{id}(\mathbb{O})}$, since $\triangleleft^{\text{IPIP}^+\text{id}(\mathbb{O})} = \triangleleft^{\text{IPIP}^\pm\text{id}(\mathbb{O}^+, \emptyset)} \in \text{IPIP}^\pm(\mathbb{O}^+, \emptyset) = \text{IPIP}^+(\mathbb{O})$, and similarly for $\triangleleft^{\text{IPIP}^-\text{id}(\mathbb{O})}$.

Claim K. $\triangleleft^{\text{IPIP}\text{id}(\mathbb{O})}$ is in $\text{IPIP}(\mathbb{O})$.

Finally, the result for $\triangleleft^{\text{DPIP}^\varepsilon\text{dd}(\mathbb{O})}$ with $\varepsilon \in \{\emptyset, -, +, \pm\}$ follows by symmetry. \square

Lemma 72. For any poset $\triangleleft \in \mathcal{P}(n)$ and any $\varepsilon \in \{\emptyset, -, +, \pm\}$, the poset $\triangleleft^{\text{IPIP}^\varepsilon\text{id}(\mathbb{O})}$ (resp. $\triangleleft^{\text{DPIP}^\varepsilon\text{dd}(\mathbb{O})}$) is the weak order minimal (resp. maximal) poset in $\text{IPIP}^\varepsilon(\mathbb{O})$ bigger than \triangleleft (resp. in $\text{DPIP}^\varepsilon(\mathbb{O})$ smaller than \triangleleft).

Proof. We prove the result for $\triangleleft^{\text{IPIP}^\varepsilon\text{id}(\mathbb{O})}$, the proof for $\triangleleft^{\text{DPIP}^\varepsilon\text{dd}(\mathbb{O})}$ being symmetric. Observe that $\triangleleft \preceq \triangleleft^{\text{IPIP}^\varepsilon\text{id}(\mathbb{O})}$ since $\triangleleft^{\text{IPIP}^\varepsilon\text{id}(\mathbb{O})}$ is obtained from \triangleleft by deleting increasing relations. Consider now $\blacktriangleleft \in \text{IPIP}^\varepsilon(\mathbb{O})$ such that $\triangleleft \preceq \blacktriangleleft$. The following claim is proved in Appendix A.4.

Claim L. $\blacktriangleleft^{\text{Inc}} \subseteq (\triangleleft^{\text{IPIP}^\varepsilon\text{id}(\mathbb{O})})^{\text{Inc}}$.

This concludes the proof since $(\triangleleft^{\text{IPIP}^\varepsilon\text{id}(\mathbb{O})})^{\text{Inc}} \supseteq \blacktriangleleft^{\text{Inc}}$ and $\blacktriangleleft^{\text{Dec}} \subseteq \triangleleft^{\text{Dec}} = (\triangleleft^{\text{IPIP}^\varepsilon\text{id}(\mathbb{O})})^{\text{Dec}}$ implies that $\triangleleft^{\text{IPIP}^\varepsilon\text{id}(\mathbb{O})} \preceq \blacktriangleleft$. \square

Consider now the **PIP(\mathbb{O}) deletion** defined by

$$\triangleleft^{\text{PIP}(\mathbb{O})} := (\triangleleft^{\text{DPIP}\text{dd}(\mathbb{O})})^{\text{IPIP}\text{id}(\mathbb{O})} = (\triangleleft^{\text{IPIP}\text{id}(\mathbb{O})})^{\text{DPIP}\text{dd}(\mathbb{O})}.$$

It follows from Lemma 71 that $\triangleleft^{\text{PIP}(\mathbb{O})} \in \text{PIP}(\mathbb{O})$ for any poset $\triangleleft \in \mathcal{P}(n)$. We now compare this map with the permutree insertion $\Psi_{\mathbb{O}}$ defined in Proposition 53.

Proposition 73. For any permutation $\sigma \in \mathfrak{S}(n)$, we have $\triangleleft_{\sigma}^{\text{PIP}(\mathbb{O})} = \triangleleft_{\Psi_{\mathbb{O}}(\sigma)}$.

Proof. Let σ be a permutation of $\mathfrak{S}(n)$ and let $\blacktriangleleft := \triangleleft_{\sigma}^{\text{PIP}(\mathbb{O})}$. We already know that $\blacktriangleleft \in \text{PIP}(\mathbb{O})$. The following claim is proved in Appendix A.4.

Claim M. \blacktriangleleft has an \mathbb{O} -snake between any two values of $[n]$.

By Proposition 63, we thus obtain that $\blacktriangleleft \in \text{PEP}(\mathbb{O})$. Since moreover \triangleleft_σ is a linear extension of \blacktriangleleft , we conclude that $\blacktriangleleft = \triangleleft_{\Psi_0(\sigma)}$. \square

To obtain a similar statement for $\text{WOIP}(n)$, we first need to observe that the map $\triangleleft \mapsto \triangleleft^{\text{PIPd}(\mathbb{O})}$ commutes with intersections. This straightforward proof is left to the reader.

Proposition 74. *For any posets $\triangleleft, \blacktriangleleft \in \mathcal{P}(n)$, we have $(\triangleleft \cap \blacktriangleleft)^{\text{PIPd}(\mathbb{O})} = \triangleleft^{\text{PIPd}(\mathbb{O})} \cap \blacktriangleleft^{\text{PIPd}(\mathbb{O})}$.*

Corollary 75. *For any permutations $\sigma \preceq \sigma'$, we have $\triangleleft_{[\sigma, \sigma']}^{\text{PIPd}(\mathbb{O})} = \triangleleft_{[\Psi_0(\sigma), \Psi_0(\sigma')]}$.*

Proof. Applying Propositions 73 and 74, we obtain

$$\triangleleft_{[\sigma, \sigma']}^{\text{PIPd}(\mathbb{O})} = (\triangleleft_\sigma \cap \triangleleft_{\sigma'})^{\text{PIPd}(\mathbb{O})} = \triangleleft_\sigma^{\text{PIPd}(\mathbb{O})} \cap \triangleleft_{\sigma'}^{\text{PIPd}(\mathbb{O})} = \triangleleft_{\Psi_0(\sigma)} \cap \triangleleft_{\Psi_0(\sigma')} = \triangleleft_{[\Psi_0(\sigma), \Psi_0(\sigma)]}. \quad \square$$

Finally, we compare the $\text{PIP}(\mathbb{O})$ deletion with the Schröder permutree insertion defined in Proposition 67.

Proposition 76. *For any ordered partition π of $[n]$, we have $\triangleleft_\pi^{\text{PIPd}(\mathbb{O})} = \triangleleft_{\Psi_0(\pi)}$.*

Proof. Let π be an ordered partition and let $\blacktriangleleft := \triangleleft_\pi^{\text{PIPd}(\mathbb{O})}$. We already know that $\blacktriangleleft \in \text{PIP}(\mathbb{O})$. The following claim is proved in Appendix A.4.

Claim N. Any $a < c$ incomparable in \blacktriangleleft satisfy at least one of the conditions (\spadesuit) and (\clubsuit) of Proposition 66.

By Proposition 66, we thus obtain that $\blacktriangleleft \in \text{PFP}(\mathbb{O})$. Since moreover any linear extension of \blacktriangleleft extends \triangleleft_π , we conclude that $\blacktriangleleft = \triangleleft_{\Psi_0(\pi)}$. \square

2.4. SUBLATTICES

The previous sections were dedicated to the characterization of various specific families of posets coming from permutreehedra and to the description of the weak order induced by these families. In this final section, we investigate which of these families induce sublattices of the weak order on posets $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$. We first introduce some additional notations based on conflict functions which will simplify later the presentation.

2.4.1. Conflict functions. A *conflict function* is a function cf which maps a poset $\triangleleft \in \mathcal{P}(n)$ to a conflict set $\text{cf}(\triangleleft) \subseteq \binom{[n]}{2}$. A poset \triangleleft is *cf-free* if $\text{cf}(\triangleleft) = \emptyset$, and we denote the set of cf-free posets on $[n]$ by $\mathcal{F}(\text{cf}, n) := \{\triangleleft \in \mathcal{P}(n) \mid \text{cf}(\triangleleft) = \emptyset\}$. Intuitively, the set $\text{cf}(\triangleleft)$ gathers the *conflicting pairs* that prevent \triangleleft to be a poset in the family $\mathcal{F}(\text{cf}, n)$.

Example 77. The characterizations of the families of posets discussed in Sections 2.1, 2.2 and 2.3 naturally translate to conflict functions. For example, the posets in $\text{IWOIP}(n)$ and in $\text{DWOIP}(n)$ are the conflict-free posets for the conflict functions respectively given by

$$\begin{aligned} \text{cf}_{\text{IWOIP}}(\triangleleft) &= \{ \{a, c\} \mid a \triangleleft c \text{ and } \exists a < b < c, a \not\triangleleft b \not\triangleleft c \}, \\ \text{cf}_{\text{DWOIP}}(\triangleleft) &= \{ \{a, c\} \mid a \triangleright c \text{ and } \exists a < b < c, a \not\triangleright b \not\triangleright c \}. \end{aligned}$$

The reader can derive from the characterizations of the previous sections other relevant conflict functions. In general, we denote by cf_X the conflict function defining a family X , *i.e.* such that $\mathcal{F}(\text{cf}_X, n) = X(n)$.

For a poset \triangleleft , we denote by $[\triangleleft] := \{\{i, j\} \mid i \triangleleft j\} \subseteq \binom{[n]}{2}$ the support of \triangleleft , *i.e.* the set of pairs of comparable elements in \triangleleft . We say that a conflict function cf is:

- (i) *local* if $\{a, b\} \in \text{cf}(\triangleleft) \iff \{a, b\} \in \text{cf}(\triangleleft \cap [a, b]^2)$ for any $a < b$ and any poset \triangleleft , *i.e.* a conflict $\{a, b\}$ only depends on the relations in the interval $[a, b]$,
- (ii) *increasing* if $\text{cf}(\triangleleft) \subseteq [\triangleleft]^{\text{Inc}}$ for any poset \triangleleft , *i.e.* only increasing relations are conflicting, *decreasing* if $\text{cf}(\triangleleft) \subseteq [\triangleleft]^{\text{Dec}}$ for any poset \triangleleft , *i.e.* only decreasing relations are conflicting, *incomparable* if $\text{cf}(\triangleleft) \subseteq \binom{[n]}{2} \setminus [\triangleleft]$ for any poset \triangleleft , *i.e.* only incomparable pairs are conflicting,
- (iii) *consistent* if $\text{cf}(\triangleleft) \cap [\triangleleft]^{\text{Inc}} = \text{cf}(\triangleleft^{\text{Inc}})$ and $\text{cf}(\triangleleft) \cap [\triangleleft]^{\text{Dec}} = \text{cf}(\triangleleft^{\text{Dec}})$ for any poset \triangleleft , *i.e.* increasing (resp. decreasing) conflicts only depends on increasing (resp. decreasing) relations,
- (iv) *monotone* if $\triangleleft \subseteq \blacktriangleleft \implies \triangleleft \setminus \text{cf}(\triangleleft) \subseteq \blacktriangleleft \setminus \text{cf}(\blacktriangleleft)$,

- (v) *semitransitive* if $\triangleleft \setminus \text{cf}(\triangleleft)$ is semitransitive, *i.e.* both increasing and decreasing subrelations of $\triangleleft \setminus \text{cf}(\triangleleft)$ are transitive. In other words, if $a < b < c$ are such that the relations $a \triangleleft b \triangleleft c$ are not conflicts for cf , then the relation $a \triangleleft c$ is not a conflict for cf (and similarly for \triangleright).

Example 78. The conflict functions cf_{IWOIP} and cf_{DWOIP} are both local, consistent, monotone and semitransitive. Moreover, cf_{IWOIP} is increasing while cf_{DWOIP} is decreasing. Indeed, all these properties but the semitransitivity follow directly from the definitions. For the semitransitivity, consider $a < b < c$ with $a \triangleleft b \triangleleft c$ and $\{a, c\} \in \text{cf}_{\text{IWOIP}}(\triangleleft)$. Then there is $a < d < c$ such that $a \not\triangleleft d \not\triangleleft c$. Assume for example that $a < d < b$. By transitivity of \triangleleft , we have $d \not\triangleleft b$, and thus $\{a, b\} \in \text{cf}_{\text{IWOIP}}(\triangleleft)$.

Remark 79. If cf and cf' are two conflict functions, then $\text{cf} \cup \text{cf}'$ is as well a conflict function with $\mathcal{F}(\text{cf} \cup \text{cf}', n) = \mathcal{F}(\text{cf}, n) \cap \mathcal{F}(\text{cf}', n)$. For example, $\text{cf}_{\text{WOIP}} = \text{cf}_{\text{IWOIP}} \cup \text{cf}_{\text{DWOIP}}$ is the conflict function for $\text{WOIP} = \text{IWOIP} \cap \text{DWOIP}$. Note that all the above conditions are stable by union.

The above conditions suffices to guaranty that cf -free posets induce semi-sublattices of $(\mathcal{P}(n), \preceq)$.

Proposition 80. *For any consistent monotone semitransitive increasing (resp. decreasing) conflict function cf , the set of cf -free posets induces a meet-semi-sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}})$ (resp. a join-semi-sublattice of $(\mathcal{P}(n), \preceq, \vee_{\mathcal{T}})$).*

Proof. We prove the result for increasing conflict functions, the proof being symmetric for decreasing ones. Let $\triangleleft, \blacktriangleleft$ be two cf -free posets and $\dashv := \triangleleft \wedge_{\mathcal{ST}} \blacktriangleleft = (\triangleleft^{\text{Inc}} \cup \blacktriangleleft^{\text{Inc}})^{\text{tc}} \cup (\triangleleft^{\text{Dec}} \cap \blacktriangleleft^{\text{Dec}})$, so that $\triangleleft \wedge_{\mathcal{T}} \blacktriangleleft = \dashv^{\text{tdd}}$. We want to prove that \dashv^{tdd} is also cf -free. Assume first that \dashv is not cf -free, and let $\{a, c\} \in \text{cf}(\dashv)$ with $a < c$ and $c - a$ minimal. Since cf is increasing, we have $(a, c) \in \dashv^{\text{Inc}} = (\triangleleft^{\text{Inc}} \cup \blacktriangleleft^{\text{Inc}})^{\text{tc}}$. If $(a, c) \notin (\triangleleft^{\text{Inc}} \cup \blacktriangleleft^{\text{Inc}})$, then there exists $a = b_1 < b_2 < \dots < b_k = c$ such that $a = b_1 \dashv b_2 \dashv \dots \dashv b_k = c$. By minimality of $c - a$, all (b_i, b_{i+1}) are in $\dashv \setminus \text{cf}(\dashv)$ while (a, c) is not, which contradicts the semitransitivity of cf . Therefore, $(a, c) \in (\triangleleft^{\text{Inc}} \cup \blacktriangleleft^{\text{Inc}})$ and we can assume without loss of generality that $(a, c) \in \triangleleft^{\text{Inc}}$. Since \triangleleft is cf -free and cf is consistent, we have $(a, c) \in \triangleleft^{\text{Inc}} \setminus \text{cf}(\triangleleft^{\text{Inc}})$. Thus, since cf is monotone and $\triangleleft^{\text{Inc}} \subseteq \dashv$, we obtain that $(a, c) \in \dashv \setminus \text{cf}(\dashv)$ which contradicts our assumption that $\{a, c\} \in \text{cf}(\dashv)$. We therefore obtained that \dashv is cf -free. Finally, since cf is monotone, consistent, and increasing, and since $\dashv^{\text{Inc}} = (\dashv^{\text{tdd}})^{\text{Inc}}$, we conclude that \dashv^{tdd} is cf -free. \square

Example 81. Applying Example 78 and Proposition 80, we obtain that the subposet of the weak order induced by $\text{IWOIP}(n)$ (resp. by $\text{DWOIP}(n)$) is a meet-semi-sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}})$ (resp. a join-semi-sublattice of $(\mathcal{P}(n), \preceq, \vee_{\mathcal{T}})$), as already proved in Proposition 38.

2.4.2. Intervals. We now consider lattice properties of the weak order on permutree interval posets $\text{PIP}(\mathbb{O})$. This section has two main goals:

- (i) provide a sufficient condition on \mathbb{O} for $\text{PIP}(\mathbb{O})$ to induce a sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$,
- (ii) show that $\text{PIP}(\mathbb{O})$ induces a sublattice of $(\text{WOIP}(n), \preceq, \wedge_{\text{WOIP}}, \vee_{\text{WOIP}})$ for any orientation \mathbb{O} .

Using the notations introduced in Section 2.4.1, we consider the conflict functions

$$\begin{aligned} \text{cf}_{\text{IPIP}^+(\mathbb{O})}(\triangleleft) &:= \{ \{a, c\} \mid a \triangleleft c \text{ and } \exists a < b < c, b \in \mathbb{O}^+ \text{ and } a \not\triangleleft b \}, \\ \text{cf}_{\text{IPIP}^-(\mathbb{O})}(\triangleleft) &:= \{ \{a, c\} \mid a \triangleleft c \text{ and } \exists a < b < c, b \in \mathbb{O}^- \text{ and } b \not\triangleleft c \}, \\ \text{cf}_{\text{IPIP}^\pm(\mathbb{O})}(\triangleleft) &:= \text{cf}_{\text{IPIP}^+(\mathbb{O})}(\triangleleft) \cup \text{cf}_{\text{IPIP}^-(\mathbb{O})}(\triangleleft), \\ \text{cf}_{\text{IPIP}(\mathbb{O})}(\triangleleft) &:= \text{cf}_{\text{IPIP}^\pm(\mathbb{O})}(\triangleleft) \cup \text{cf}_{\text{IWOIP}}(\triangleleft), \\ \text{cf}_{\text{DPIP}^+(\mathbb{O})}(\triangleleft) &:= \{ \{a, c\} \mid a \triangleright c \text{ and } \exists a < b < c, b \in \mathbb{O}^+ \text{ and } b \not\triangleright c \}, \\ \text{cf}_{\text{DPIP}^-(\mathbb{O})}(\triangleleft) &:= \{ \{a, c\} \mid a \triangleright c \text{ and } \exists a < b < c, b \in \mathbb{O}^- \text{ and } a \not\triangleright b \}, \\ \text{cf}_{\text{DPIP}^\pm(\mathbb{O})}(\triangleleft) &:= \text{cf}_{\text{DPIP}^+(\mathbb{O})}(\triangleleft) \cup \text{cf}_{\text{DPIP}^-(\mathbb{O})}(\triangleleft), \\ \text{cf}_{\text{DPIP}(\mathbb{O})}(\triangleleft) &:= \text{cf}_{\text{DPIP}^\pm(\mathbb{O})}(\triangleleft) \cup \text{cf}_{\text{DWOIP}}(\triangleleft), \end{aligned}$$

and finally

$$\text{cf}_{\text{PIP}(\mathbb{O})}(\triangleleft) := \text{cf}_{\text{IPIP}(\mathbb{O})}(\triangleleft) \cup \text{cf}_{\text{DPIP}(\mathbb{O})}(\triangleleft)$$

corresponding to the families studied in Section 2.3.2. As seen in Proposition 58, the $\text{cf}_{\text{PIP}(\mathbb{O})}$ -free posets are precisely that of $\text{PIP}(\mathbb{O})$.

COVERING ORIENTATIONS

In the next statements, we provide a sufficient condition on the orientation \mathbb{O} for $\text{PIP}(\mathbb{O})$ to induce a sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$. We first check the conditions of Proposition 80 to get semi-sublattices.

Lemma 82. *For any orientation \mathbb{O} and any $\varepsilon \in \{\emptyset, -, +, \pm\}$, the conflict functions $\text{cf}_{\text{IPIP}^\varepsilon(\mathbb{O})}$ and $\text{cf}_{\text{DPIP}^\varepsilon(\mathbb{O})}$ are local, consistent, monotone, and semitransitive. Moreover, $\text{cf}_{\text{IPIP}^\varepsilon(\mathbb{O})}$ is increasing while $\text{cf}_{\text{DPIP}^\varepsilon(\mathbb{O})}$ is decreasing.*

Proof. Since they are stable by union (Remark 79), and since they hold for the conflict functions cf_{WOIP} and cf_{DWOIP} (Example 78), it suffices to show these properties for the conflict functions $\text{cf}_{\text{IPIP}^+(\mathbb{O})}$, $\text{cf}_{\text{IPIP}^-(\mathbb{O})}$, $\text{cf}_{\text{DPIP}^+(\mathbb{O})}$ and $\text{cf}_{\text{DPIP}^-(\mathbb{O})}$. By symmetry, we only consider $\text{cf}_{\text{IPIP}^+(\mathbb{O})}$. We just need to prove the semitransitivity, the other properties being immediate from the definitions. Consider $a < b < c$ such that $a \triangleleft b \triangleleft c$ and $\{a, c\} \in \text{cf}_{\text{IPIP}^+(\mathbb{O})}(\triangleleft)$. Then there exists $a < d < c$ such that $d \in \mathbb{O}^+$ and $a \not\triangleleft d$. If $d < b$, then $\{a, b\} \in \text{cf}_{\text{IPIP}^+(\mathbb{O})}(\triangleleft)$. Otherwise, $b < d$ and the transitivity of \triangleleft ensures that $b \not\triangleleft d$, so that $\{b, c\} \in \text{cf}_{\text{IPIP}^+(\mathbb{O})}(\triangleleft)$. We conclude that $(\triangleleft \setminus \text{cf}_{\text{IPIP}^+(\mathbb{O})}(\triangleleft))^{\text{Inc}}$ is transitive. Since $(\triangleleft \setminus \text{cf}_{\text{IPIP}^+(\mathbb{O})}(\triangleleft))^{\text{Dec}} = \triangleleft^{\text{Dec}}$ is also transitive, we obtained that $\text{cf}_{\text{IPIP}^+(\mathbb{O})}$ is semitransitive. \square

Corollary 83. *For any orientation \mathbb{O} and any $\varepsilon \in \{\emptyset, -, +, \pm\}$, the set $\text{IPIP}^\varepsilon(\mathbb{O})$ (resp. $\text{DPIP}^\varepsilon(\mathbb{O})$) induces a meet-semi-sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}})$ (resp. a join-semi-sublattice of $(\mathcal{P}(n), \preceq, \vee_{\mathcal{T}})$).*

Proof. Direct application of Lemma 82 and Proposition 80. \square

To obtain sublattices, we need an additional condition on \mathbb{O} . Namely, we say that an orientation $\mathbb{O} = (n, \mathbb{O}^+, \mathbb{O}^-)$ is *covering* if $\{2, \dots, n-1\} \subseteq \mathbb{O}^+ \cup \mathbb{O}^-$. Note that we do not require a priori that $\mathbb{O}^+ \cap \mathbb{O}^- = \emptyset$ nor that $\{1, n\} \subseteq \mathbb{O}^+ \cup \mathbb{O}^-$. Observe also that when \mathbb{O} is covering, we have $\text{IPIP}^\pm(\mathbb{O}) = \text{IPIP}(\mathbb{O})$ and $\text{DPIP}^\pm(\mathbb{O}) = \text{DPIP}(\mathbb{O})$.

Theorem 84. *For any covering orientation \mathbb{O} , the sets $\text{IPIP}(\mathbb{O})$, $\text{DPIP}(\mathbb{O})$ and $\text{PIP}(\mathbb{O})$ all induce sublattices of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$.*

Proof. We only prove the result for $\text{DPIP}(\mathbb{O})$. It then follows by symmetry for $\text{IPIP}(\mathbb{O})$, which in turn implies the result for $\text{PIP}(\mathbb{O})$ since $\text{PIP}(\mathbb{O}) = \text{IPIP}(\mathbb{O}) \cap \text{DPIP}(\mathbb{O})$. We already know from Corollary 83 that $\text{DPIP}(\mathbb{O})$ is stable by $\vee_{\mathcal{T}}$ and it remains to show that it is stable by $\wedge_{\mathcal{T}}$. We thus consider two posets $\triangleleft, \blacktriangleleft \in \text{DPIP}(\mathbb{O})$ and let $\dashv := \triangleleft \wedge_{\mathcal{T}} \blacktriangleleft = (\triangleleft^{\text{Inc}} \cup \blacktriangleleft^{\text{Inc}})^{\text{tc}} \cup (\triangleleft^{\text{Dec}} \cap \blacktriangleleft^{\text{Dec}})$, so that $\triangleleft \wedge_{\mathcal{T}} \blacktriangleleft = \dashv^{\text{tdd}}$. We decompose the proof in two steps, whose detailed proofs are given in Appendix A.5.

Claim O. \dashv is in $\text{DPIP}(\mathbb{O})$.

Claim P. \dashv^{tdd} is in $\text{DPIP}(\mathbb{O})$. \square

Corollary 85. *The weak order on interval posets in the Tamari lattice, in any type A_n Cambrian lattice, and in the boolean lattice are all sublattices of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$.*

Proof. Apply Theorem 84 to the orientations illustrated in Figure 14: the Tamari lattice is the lattice $\text{PIP}(\emptyset, [n])$, the Cambrian lattices are the lattices $\text{PIP}(\mathbb{O}^+, \mathbb{O}^-)$ for all partitions $\mathbb{O}^+ \sqcup \mathbb{O}^- = [n]$, and the boolean lattice is the lattice $\text{PIP}([n], [n])$. \square

Remark 86. The covering condition is essential to the proof of Theorem 84. For example, Remark 62 shows that $\text{WOIP}(n) = \text{PIP}(\emptyset, \emptyset)$ does not induce a sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$.

PIP(\mathbb{O}) INDUCES A SUBLATTICE OF WOIP(n)

We now consider an arbitrary orientation \mathbb{O} , not necessarily covering. Although $\text{PIP}(\mathbb{O})$ does not always induce a sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$, we show that it always induces a sublattice of $(\text{WOIP}(n), \preceq, \wedge_{\text{WOIP}}, \vee_{\text{WOIP}})$.

Theorem 87. *For any orientation \mathbb{O} and any $\varepsilon \in \{\emptyset, -, +, \pm\}$, the set $\text{IPIP}^\varepsilon(\mathbb{O})$ (resp. $\text{DPIP}^\varepsilon(\mathbb{O})$) induces a sublattice of $(\text{WOIP}(n), \preceq, \wedge_{\text{WOIP}}, \vee_{\text{WOIP}})$.*

Proof. By symmetry, it suffices to prove the result for $\text{DPIP}^-(\mathbb{O})$. Let $\triangleleft, \blacktriangleleft \in \text{DPIP}^-(\mathbb{O})$. We already know from Corollary 83 that $\triangleleft \vee_{\mathcal{T}} \blacktriangleleft \in \text{DPIP}^-(\mathbb{O})$. Since $\text{cf}_{\text{DPIP}^-(\mathbb{O})}$ is a decreasing conflict function and since the IWOIP increasing deletion only deletes increasing relations, we thus obtain that

$$\triangleleft \vee_{\text{WOIP}} \blacktriangleleft = (\triangleleft \vee_{\mathcal{T}} \blacktriangleleft)^{\text{IWOIPid}} \in \text{DPIP}^-(\mathbb{O}).$$

It remains to prove that

$$\triangleleft \wedge_{\text{WOIP}} \blacktriangleleft = (\triangleleft \wedge_{\mathcal{T}} \blacktriangleleft)^{\text{DWOIPdd}} \in \text{DPIP}^-(\mathbb{O}).$$

For this, let us denote $\dashv := \triangleleft \wedge_{\mathcal{ST}} \blacktriangleleft = (\triangleleft^{\text{Inc}} \cup \blacktriangleleft^{\text{Inc}})^{\text{tc}} \cup (\triangleleft^{\text{Dec}} \cap \blacktriangleleft^{\text{Dec}})$ and $\dashv := \triangleleft \wedge_{\text{WOIP}} \blacktriangleleft$ so that $\dashv = (\dashv^{\text{tdd}})^{\text{DWOIPdd}}$. As in the proof of Theorem 84, we know that $\dashv \in \text{DPIP}^-(\mathbb{O})$. Assume now that $\dashv \notin \text{DPIP}^-(\mathbb{O})$. Consider $\{a, c\} \in \text{cf}_{\text{DPIP}(\mathbb{O})}(\dashv)$ with $a < c$ and $c - a$ minimal. We therefore have $a \Vdash c$ while there exists $a < b < c$ with $b \in \mathbb{O}^-$ and $a \not\Vdash b$. Note that since $\dashv \in \text{DPIP}^-(\mathbb{O})$, we have $a \vdash b$. We now distinguish two cases:

- If $a \Vdash^{\text{tdd}} b$, then there exists $i \leq b$ and $j \geq a$ such that $i \dashv b \dashv a \dashv j$ but $i \not\vdash j$. From Lemma 16, we know that there exists $a < k < b$ such that $a \Vdash^{\text{tdd}} k \not\Vdash^{\text{tdd}} b$.
- If $a \vdash^{\text{tdd}} b$, then there exists $a < k_1 < \dots < k_\ell < b$ such that $a \Vdash^{\text{tdd}} k_1 \not\Vdash^{\text{tdd}} \dots \not\Vdash^{\text{tdd}} k_\ell \not\Vdash^{\text{tdd}} b$.

In both cases, there exists $a < k < b$ such that $a \not\Vdash k \not\Vdash b$. Since $\dashv \in \text{IWOIP}$ and $a \Vdash c$ while $a \not\Vdash k$, we must have $k \Vdash c$. But since $k \not\Vdash b$, we then have $\{k, c\} \in \text{cf}_{\text{DPIP}(\mathbb{O})}(\dashv)$ contradicting the minimality of $c - a$ in our choice of $\{a, c\}$. \square

Corollary 88. *For any orientation \mathbb{O} , $\text{PIP}(\mathbb{O})$ induces a sublattice of $(\text{WOIP}(n), \preceq, \wedge_{\text{WOIP}}, \vee_{\text{WOIP}})$.*

Proof. Immediate consequence of Theorem 87 as $\text{PIP}(\mathbb{O}) = \text{IPIP}(\mathbb{O}) \cap \text{DPIP}(\mathbb{O})$. \square

2.4.3. Elements. We now consider lattice properties of the weak order on permutree element posets $\text{PEP}(\mathbb{O})$. Similarly to the previous section, the present section has two main goals:

- provide a sufficient condition on \mathbb{O} for $\text{PEP}(\mathbb{O})$ to induce a sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$,
- show that $\text{PEP}(\mathbb{O})$ induces a sublattice of $(\text{WOIP}(n), \preceq, \wedge_{\text{WOIP}}, \vee_{\text{WOIP}})$ for any orientation \mathbb{O} .

We start with a simple observation.

Proposition 89. *The set $\text{PEP}(\mathbb{O})$ induces a sublattice of $(\text{PIP}(\mathbb{O}), \preceq, \wedge_{\text{PIP}(\mathbb{O})}, \vee_{\text{PIP}(\mathbb{O})})$ for any orientation \mathbb{O} .*

Proof. We have seen in Corollary that the meet and join in $\text{PIP}(\mathbb{O})$ are given by

$$\triangleleft_{[S, S']} \wedge_{\text{PIP}(\mathbb{O})} \triangleleft_{[T, T']} = \triangleleft_{[S \wedge_{\mathbb{O}} T, S' \wedge_{\mathbb{O}} T']} \quad \text{and} \quad \triangleleft_{[S, S']} \vee_{\text{PIP}(\mathbb{O})} \triangleleft_{[T, T']} = \triangleleft_{[S \vee_{\mathbb{O}} T, S' \vee_{\mathbb{O}} T']}.$$

Therefore, for any $S, T \in \text{PEP}(\mathbb{O})$, we have

$$\triangleleft_S \wedge_{\text{PIP}(\mathbb{O})} \triangleleft_T = \triangleleft_{[S, S]} \wedge_{\text{PIP}(\mathbb{O})} \triangleleft_{[T, T]} = \triangleleft_{[S \wedge_{\mathbb{O}} T, S \wedge_{\mathbb{O}} T]} = \triangleleft_{S \wedge_{\mathbb{O}} T}. \quad \square$$

Proposition 89 enables to show Theorems 90 and 93 below.

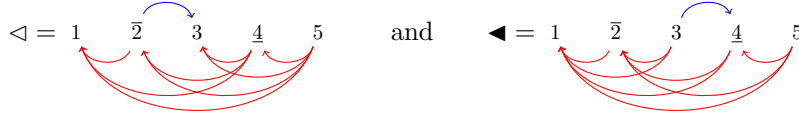
Theorem 90. *For any covering orientation \mathbb{O} , $\text{PEP}(\mathbb{O})$ induces a sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$.*

Proof. $\text{PEP}(\mathbb{O})$ induces a sublattice of $(\text{PIP}(\mathbb{O}), \preceq, \wedge_{\text{PIP}(\mathbb{O})}, \vee_{\text{PIP}(\mathbb{O})})$ (by Proposition 89), which in turn is a sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$ when \mathbb{O} is covering (by Theorem 84). \square

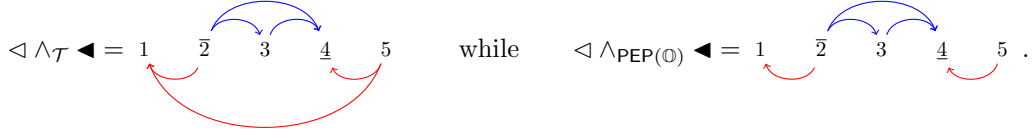
Corollary 91. *The Tamari lattice, any type A_n Cambrian lattice, and the boolean lattice are all sublattices of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$.*

Proof. Apply Theorem 90 to the orientations illustrated in Figure 14: the Tamari lattice is the lattice $\text{PIP}(\emptyset, [n])$, the Cambrian lattices are the lattices $\text{PIP}(\mathbb{O}^+, \mathbb{O}^-)$ for all partitions $\mathbb{O}^+ \sqcup \mathbb{O}^- = [n]$, and the boolean lattice is the lattice $\text{PIP}([n], [n])$. \square

Remark 92. Note that the covering condition in Theorem 90 is necessary in general. For example, for the orientation $\mathbb{O} = (5, \{2\}, \{4\})$ on $[5]$, the lattice $(\text{PEP}(\mathbb{O}), \preceq, \wedge_{\text{PEP}(\mathbb{O})}, \vee_{\text{PEP}(\mathbb{O})})$ is not a sublattice of $(\mathcal{P}(5), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$. For example, for



we have



However, for arbitrary orientation, we can still obtain the following weaker statement.

Theorem 93. *For any orientation \mathbb{O} , $\text{PEP}(\mathbb{O})$ induces a sublattice of $(\text{WOIP}(n), \preceq, \wedge_{\text{WOIP}}, \vee_{\text{WOIP}})$.*

Proof. $\text{PEP}(\mathbb{O})$ induces a sublattice of $(\text{PIP}(\mathbb{O}), \preceq, \wedge_{\text{PIP}(\mathbb{O})}, \vee_{\text{PIP}(\mathbb{O})})$ (by Proposition 89), which in turn induces a sublattice of $(\text{WOIP}(n), \preceq, \wedge_{\text{WOIP}}, \vee_{\text{WOIP}})$ (by Corollary 88). \square

Finally, let us give a poset proof of Proposition 89. Recall from Section 2.3.3 and Figure 16 that, for an orientation \mathbb{O} of $[n]$, an \mathbb{O} -snake in a poset \triangleleft is a sequence $x_0 < x_1 < \dots < x_k < x_{k+1}$ such that

- either $x_0 < x_1 \triangleright x_2 < x_3 \triangleright \dots$ with $\{x_i \mid i \in [k] \text{ odd}\} \subseteq \mathbb{O}^-$ and $\{x_i \mid i \in [k] \text{ even}\} \subseteq \mathbb{O}^+$,
- or $x_0 \triangleright x_1 < x_2 \triangleright x_3 < \dots$ with $\{x_i \mid i \in [k] \text{ odd}\} \subseteq \mathbb{O}^+$ and $\{x_i \mid i \in [k] \text{ even}\} \subseteq \mathbb{O}^-$.

Using the notations introduced in Section 2.4.1, we consider the two conflict functions

$$\begin{aligned} \text{cf}_{\text{sn}(\mathbb{O})}(\triangleleft) &:= \{ \{a, c\} \mid \text{there is no } \mathbb{O}\text{-snake joining } a \text{ to } c \}, \\ \text{cf}_{\text{PEP}(\mathbb{O})}(\triangleleft) &:= \text{cf}_{\text{PIP}(\mathbb{O})}(\triangleleft) \cup \text{cf}_{\text{sn}(\mathbb{O})}(\triangleleft). \end{aligned}$$

As seen in Proposition 63, $\text{cf}_{\text{sn}(\mathbb{O})}$ corresponds to the condition characterizing $\text{PEP}(\mathbb{O})$ in $\text{PIP}(\mathbb{O})$, so that the $\text{cf}_{\text{PEP}(\mathbb{O})}$ -free posets are precisely that of $\text{PEP}(\mathbb{O})$. We now prove that the conflict function $\text{cf}_{\text{sn}(\mathbb{O})}$ alone induces a sublattice of the weak order on posets.

Proposition 94. *For any orientation \mathbb{O} on $[n]$, the set of $\text{cf}_{\text{sn}(\mathbb{O})}$ -free posets induces a sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$.*

Proof. Consider two $\text{cf}_{\text{sn}(\mathbb{O})}$ -free posets $\triangleleft, \blacktriangleleft$ and let $\dashv := \triangleleft \wedge_{\mathcal{T}} \blacktriangleleft = (\triangleleft^{\text{Inc}} \cup \blacktriangleleft^{\text{Inc}})^{\text{tc}} \cup (\triangleleft^{\text{Dec}} \cap \blacktriangleleft^{\text{Dec}})$, so that $\triangleleft \wedge_{\mathcal{T}} \blacktriangleleft = \dashv^{\text{tdd}}$. As in the proof of Theorem 84, we decompose the proof in two steps, whose detailed proofs are given in Appendix A.6.

Claim Q. \dashv is $\text{cf}_{\text{sn}(\mathbb{O})}$ -free.

Claim R. \dashv^{tdd} is $\text{cf}_{\text{sn}(\mathbb{O})}$ -free. \square

Note that Proposition 94 provides a proof of Proposition 89 on posets. It also enables us to obtain further results for the specific orientation $(n, \emptyset, \emptyset)$. Indeed, Proposition 22 ensures that the $\text{cf}_{\text{sn}(\mathbb{O})}$ -free posets are precisely the posets of $\text{WOEP}(n)$. Proposition 94 therefore specializes to the following statement.

Corollary 95. *The set $\text{WOEP}(n)$ induces a sublattice of the weak order $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$.*

2.4.4. Faces. In this section, we study the lattice properties of the weak order on permutree face posets $\text{PFP}(\mathbb{O})$. We have seen in Propositions 67 and 68 that the weak order on $\text{PFP}(\mathbb{O})$ coincides with the Schröder permutree lattice, but we have observed in Remark 69 that it is not a sublattice of $(\mathcal{P}(n), \preceq, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}})$, nor a sublattice of $(\text{WOIP}(n), \preceq, \wedge_{\text{WOIP}}, \vee_{\text{WOIP}})$, nor a sublattice of $(\text{PIP}(\mathbb{O}), \preceq, \wedge_{\text{PIP}(\mathbb{O})}, \vee_{\text{PIP}(\mathbb{O})})$. For completeness, let us report on a method to compute the meet and join directly on the posets of $\text{PFP}(\mathbb{O})$. For that, define the $\text{PFP}(\mathbb{O})$ *increasing addition* and the $\text{PFP}(\mathbb{O})$ *decreasing addition* by

$$\triangleleft^{\text{PFPia}} = \begin{cases} \triangleleft & \text{if } \triangleleft \in \text{PFP}(\mathbb{O}) \\ \left(\triangleleft \cup \{ \{a, c\} \mid a < c \text{ not satisfying } (\spadesuit) \text{ nor } (\clubsuit) \} \right)^{\text{PFPia}} & \text{otherwise} \end{cases}$$

and

$$\triangleleft^{\text{PFPda}} = \begin{cases} \triangleleft & \text{if } \triangleleft \in \text{PFP}(\mathbb{O}) \\ (\triangleleft \cup \{(c, a) \mid a < c \text{ not satisfying } (\spadesuit) \text{ nor } (\clubsuit)\})^{\text{PFPda}} & \text{otherwise.} \end{cases}$$

Experimental observations indicate that for $S, S' \in \text{PFP}(\mathbb{O})$,

$$S \wedge_{\text{PFP}(\mathbb{O})} S' = (S \wedge_{\text{WOIP}} S')^{\text{PFPia}} \quad \text{and} \quad S \vee_{\text{PFP}(\mathbb{O})} S' = (S \vee_{\text{WOIP}} S')^{\text{PFPda}}.$$

A complete proof of this observation would however be quite technical. It would in particular require a converging argument to prove that the $\text{PFP}(\mathbb{O})$ increasing and decreasing additions are well defined.

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APPENDIX A. MISSING CLAIMS

This appendix gathers all missing proofs of the technical claims used in Part 2.

A.1. Proof of claims of Section 2.1.4.

Proof of Claim A. First, \triangleleft^{\maxle} is clearly antisymmetric since it is obtained from an antisymmetric relation by adding just decreasing relations between some incomparable elements. To prove that \triangleleft^{\maxle} is transitive, consider $u, v, w \in [n]$ be such that $u \triangleleft^{\maxle} v \triangleleft^{\maxle} w$. We distinguish four cases:

- (i) If $u \not\triangleleft v \not\triangleleft w$, then we have $w < v < u$ with $w \not\triangleleft v \not\triangleleft u$. Our assumption thus ensures that $w \not\triangleleft u$. Thus, either $u \triangleleft w$ or u and w are incomparable. In both cases, $u \triangleleft^{\maxle} w$.
- (ii) If $u \not\triangleleft v \triangleleft w$, then we have $v < u$ with $v \not\triangleleft u$. We then have two cases:
 - Assume that $u < w$. Since $v < u < w$ and $v \not\triangleleft u$ while $v \triangleleft w$, our assumption implies that $u \triangleleft w$, so that $u \triangleleft^{\maxle} w$.
 - Assume that $w < u$. Since $v \not\triangleleft u$ and $v \triangleleft w$, the transitivity of \triangleleft impose that $w \not\triangleleft u$. Thus either $u \triangleleft w$ or u and w are incomparable. In both cases, $u \triangleleft^{\maxle} w$.
- (iii) If $u \triangleleft v \not\triangleleft w$, then we have $w < v$ with $w \not\triangleleft v$. We then have two cases:
 - Assume that $u < w$. Since $u < w < v$ and $w \not\triangleleft v$ while $u \triangleleft v$, our assumption implies that $u \triangleleft w$ so that $u \triangleleft^{\maxle} w$.
 - Assume that $w < u$. Since $w \not\triangleleft v$ and $u \triangleleft v$, the transitivity of \triangleleft impose that $w \not\triangleleft u$. Thus, either $u \triangleleft w$ or u and w are incomparable. In both cases, $u \triangleleft^{\maxle} w$.
- (iv) If $u \triangleleft v \triangleleft w$, then $u \triangleleft w$ by transitivity of \triangleleft and thus $u \triangleleft^{\maxle} w$.

We proved in all cases that $u \triangleleft^{\maxle} w$, so that \triangleleft is transitive. Since all our relations are reflexive, we conclude that \triangleleft^{\maxle} is a poset. \square

Proof of Claim B. First, $\triangleleft^{\text{IWOIPid}}$ is clearly antisymmetric as it is contained in the antisymmetric relation \triangleleft . To prove that it is transitive, consider $u, v, w \in [n]$ such that $u \triangleleft^{\text{IWOIPid}} v \triangleleft^{\text{IWOIPid}} w$. Since $\triangleleft^{\text{IWOIPid}} \subseteq \triangleleft$, we have $u \triangleleft v \triangleleft w$, so that $u \triangleleft w$ by transitivity of \triangleleft . Assume by means of contradiction that $u \not\triangleleft^{\text{IWOIPid}} w$. Thus, $u < w$ and there exists $u < z_1 < \dots < z_k < w$ such that $u \not\triangleleft z_1 \not\triangleleft \dots \not\triangleleft z_k \not\triangleleft w$. We now distinguish three cases:

- (i) If $v < u$, then $v \not\triangleleft^{\text{IWOIPid}} w$ since $v < u < z_1 < \dots < z_k < w$ and $v \not\triangleleft u \not\triangleleft z_1 \not\triangleleft \dots \not\triangleleft z_k \not\triangleleft w$.
- (ii) If $u < v < w$, consider $\ell \in [k]$ such that $z_\ell \leq v < z_{\ell+1}$ (with $\ell = 0$ if $v < z_1$ and $\ell = k$ if $z_k \leq v$). Since $z_\ell \not\triangleleft z_{\ell+1}$ and \triangleleft is transitive, we have either $z_\ell \not\triangleleft v$ or $v \not\triangleleft z_{\ell+1}$. In the former case, we have $u \not\triangleleft^{\text{IWOIPid}} v$ since $u < z_1 < \dots < z_\ell \leq v$ and $u \not\triangleleft z_1 \not\triangleleft \dots \not\triangleleft z_\ell \not\triangleleft v$. In the latter case, we have $v \not\triangleleft^{\text{IWOIPid}} w$ since $v < z_{\ell+1} < \dots < z_k < w$ and $v \not\triangleleft z_{\ell+1} \not\triangleleft \dots \not\triangleleft z_k \not\triangleleft w$.
- (iii) If $w < v$, then $u \not\triangleleft^{\text{IWOIPid}} v$ since $u < z_1 < \dots < z_k < w < v$ and $u \not\triangleleft z_1 \not\triangleleft \dots \not\triangleleft z_k \not\triangleleft w \not\triangleleft v$.

As we obtained a contradiction in each case, we conclude that $\triangleleft^{\text{IWOIPid}}$ is transitive. Since all our relations are reflexive, we conclude that $\triangleleft^{\text{IWOIPid}}$ is a poset. \square

A.2. Proof of claims of Proposition 63.

Proof of Claim C. By symmetry, we only need to prove the first statement. Note that u and w are incomparable, otherwise $u \triangleleft v$ and $v \triangleright w$ could not both be cover relations. Therefore, there is a non-degenerate \mathbb{O} -snake $u = x_0 < x_1 < \dots < x_k < x_{k+1} = w$ from u to w . Assume first that $x_1 < v$. If $x_1 \in \mathbb{O}^+$ and $u \triangleright x_1$, then $u \triangleleft v$, $x_1 \in \mathbb{O}^+$ and $\triangleleft \in \text{IPIP}^+(\mathbb{O})$ implies that $u \triangleleft x_1$, a contradiction. If $x_1 \in \mathbb{O}^-$ and $u \triangleleft x_1$, then $u \triangleleft v$, $x_1 \in \mathbb{O}^-$ and $\triangleleft \in \text{IPIP}^+(\mathbb{O})$ implies that $x_1 \triangleleft v$ which together with $u \triangleleft x_1$ would contradict that $u \triangleleft v$ is a cover relation. As we reach a contradiction in both cases, we obtain that $v \leq x_1$, and by symmetry $x_k \leq v$. Therefore, we have $x_1 = v = x_k$, so that $u < v < w$ and $v \in \mathbb{O}^-$. \square

Proof of Claim D. We work by induction on p , the case $p = 1$ being immediate. By symmetry, we can assume that $x_0 \in \mathbb{O}^-$, $x_0 \triangleright x_1$ and $x_0 < x_1$. Let j be the first position in the path such that $x_{j-1} \triangleright x_j \triangleleft x_{j+1}$ (by convention $j = p$ if $x_0 \triangleright x_1 \triangleright \dots \triangleright x_p$). Assume that there is $i \in [j]$ such that $x_i \leq x_0$, and assume that i is the first such index. Since $x_i \leq x_0 < x_{i-1}$, $x_i \triangleleft x_{i-1}$, $x_0 \in \mathbb{O}^-$ and $\triangleleft \in \text{IPIP}^-(\mathbb{O})$, we obtain $x_0 \triangleleft x_{i-1}$, a contradiction. This shows that $x_0 < x_i$

for $i \in [j]$. If $j = p$, the statement is proved. Otherwise, we consider x_{j-1} , x_j and x_{j+1} . By Claim C, we have $x_j \in \mathbb{O}^+$ and either $x_{j-1} < x_j < x_{j+1}$ or $x_{j+1} < x_j < x_{j-1}$. In the latter case, $x_0 < x_j < x_{j-1}$, $x_0 \triangleright x_{j-1}$, $x_j \in \mathbb{O}^+$ and $\triangleleft \in \text{DPIP}^+(\mathbb{O})$ would imply $x_j \triangleright x_{j-1}$, a contradiction. We thus obtain that $x_j \in \mathbb{O}^+$, $x_j \triangleleft x_{j+1}$ and $x_j < x_{j+1}$. The induction hypothesis thus ensures that $x_j < x_i$ for all $j < i \leq p$. This concludes since $x_0 < x_j$. \square

A.3. Proof of claims of Proposition 66.

Proof of Claim E. By Proposition 63, we just need prove that there is an \mathbb{O} -snake between any two values of $[n]$. Otherwise, consider $a < c$ with $c - a$ minimal such that there is no \mathbb{O} -snake between a and c . In particular, a and c are incomparable. By (\spadesuit) , we can assume for instance that there is $a < b < c$ such that $b \in \mathbb{O}^-$ and $a \not\triangleright b \triangleleft c$. By minimality of $c - a$, there is an \mathbb{O} -snake $a = x_0 < x_1 < \dots < x_k < x_{k+1} = b$. Then we have either $x_1 \in \mathbb{O}^-$ and $a \triangleleft x_1$, or $x_1 \in \mathbb{O}^+$ and $a \triangleright x_1$ (note that this holds even when $x_1 = b$ since $a \not\triangleright b$ and $b \in \mathbb{O}^-$). Moreover, by minimality of $c - a$, there is an \mathbb{O} -snake between x_1 and c . Lemma 64 thus ensures that there is as well an \mathbb{O} -snake between a and c , contradicting our assumption. \square

Proof of Claim F. By definition, we have $\triangleleft \subseteq \blacktriangleleft$. Assume now that $\triangleleft^{\text{Dec}} \neq \blacktriangleleft^{\text{Dec}}$, and let $x < y$ be such that $x \not\triangleright y$ but $x \blacktriangleright y$. By definition of \blacktriangleleft , there exists a minimal path $y = z_0, z_1, \dots, z_k = x$ such that for all $i \in [k]$, either $z_{i-1} \triangleleft z_i$, or $z_{i-1} < z_i$ are incomparable in \triangleleft and do not satisfy (\spadesuit) . Since $x \not\triangleright y$ and $x < y$, we have $k \geq 2$ and there exists $i \in [k-1]$ such that $z_{i+1} < z_{i-1}$. We distinguish three cases:

- If $z_i < z_{i+1} < z_{i-1}$, then $z_i \triangleright z_{i-1}$ and $z_i \not\triangleright z_{i+1}$, and thus $z_{i+1} \triangleright z_{i-1}$ as $\triangleleft \in \text{DWOIP}(n)$.
- If $z_{i+1} < z_i < z_{i-1}$, then $z_{i+1} \triangleright z_i \triangleright z_{i-1}$ and thus $z_{i+1} \triangleright z_{i-1}$ by transitivity.
- If $z_{i+1} < z_{i-1} < z_i$, then $z_{i+1} \triangleright z_i$ and $z_{i-1} \not\triangleright z_i$, and thus $z_{i+1} \triangleright z_{i-1}$ as $\triangleleft \in \text{DWOIP}(n)$.

In all cases, $z_{i+1} \triangleright z_{i-1}$ contradicts the minimality of the path. \square

Proof of Claim G. We first show that $\blacktriangleleft \in \text{PIP}(\mathbb{O})$. Since $\triangleleft^{\text{Dec}} = \blacktriangleleft^{\text{Dec}}$ and $\triangleleft \in \text{DPIP}(\mathbb{O})$, we have $\blacktriangleleft \in \text{DPIP}(\mathbb{O})$ and we just need to show that $\blacktriangleleft \in \text{IPIP}(\mathbb{O})$. Consider thus $a < b < c$ such that $a \blacktriangleleft c$. By definition of \blacktriangleleft , there exists $a' < b < c'$ such that $a \blacktriangleleft a'$, $c' \blacktriangleleft c$, and either $a' \triangleleft c'$, or a' and c' are incomparable in \triangleleft and do not satisfy (\spadesuit) . We now proceed in two steps:

- (i) Our first goal is to show that either $a' \blacktriangleleft b$ or $b \blacktriangleleft c'$ which by transitivity shows that (a, c) satisfies the WOIP condition. Assume that $a' \not\blacktriangleleft b \not\blacktriangleleft c'$. The transitivity of \blacktriangleleft ensures that both pairs a', b and b, c' are incomparable in \blacktriangleleft . Therefore, they are incomparable in \triangleleft and satisfy (\spadesuit) . Let us focus on a', b . Assume first that there is $a' < d < b$ such that $d \in \mathbb{O}^+$ and $a' \not\blacktriangleleft d \not\blacktriangleleft b$. Since $\triangleleft \in \text{IPIP}(\mathbb{O})$, we cannot have $a' < d < c'$, $d \in \mathbb{O}^+$, $a' \not\blacktriangleleft d$ and $a' \triangleleft c'$. Therefore, a' and c' do not satisfy (\spadesuit) , which together with $a' \not\blacktriangleleft d$ implies that $d \triangleright c'$. We obtain $d < b < c'$ with $d \not\blacktriangleleft b \not\blacktriangleleft c'$ while $d \triangleright c'$ contradicting that $\triangleleft \in \text{DWOIP}(n)$. Assume now that there is $a' < d < b$ such that $d \in \mathbb{O}^-$ and $a' \not\blacktriangleleft d \not\blacktriangleleft b$. If $a' \triangleleft c'$, then $d \triangleleft c'$ since $a' < d < c'$, $d \in \mathbb{O}^-$ and $\triangleleft \in \text{IPIP}(\mathbb{O})$. If a' and c' do not satisfy (\spadesuit) , $d \in \mathbb{O}^-$ and $a' \not\blacktriangleleft d$ imply $d \triangleleft c'$. In both cases, we obtain $d < b < c'$ with $d \not\blacktriangleleft b \not\blacktriangleleft c'$ while $d \triangleleft c'$ contradicting that $\triangleleft \in \text{IWOIP}(n)$. Since we reach a contradiction in all cases, we conclude that $a' \blacktriangleleft b$ or $b \blacktriangleleft c'$.
- (ii) We now want to check that the orientation constraint on b is also satisfied. Assume $b \in \mathbb{O}^+$. If $a' \triangleleft c'$, we have $a' \triangleleft b$ since $\triangleleft \in \text{IPIP}(\mathbb{O})$, and thus $a' \blacktriangleleft b$ since $\triangleleft \subseteq \blacktriangleleft$. If a' and c' do not satisfy (\spadesuit) , then $a' \triangleleft b$ or $b \triangleright c'$, which implies $a' \triangleleft b \triangleright c'$ by (\clubsuit) , and thus $a' \blacktriangleleft b$ since $\triangleleft \subseteq \blacktriangleleft$. We conclude that $b \in \mathbb{O}^+ \Rightarrow a' \blacktriangleleft b$ and by symmetry that $b \in \mathbb{O}^- \Rightarrow b \blacktriangleleft c'$.

Since $a \blacktriangleleft a'$, $c' \blacktriangleleft c$ and \blacktriangleleft is transitive, we obtain that $a \blacktriangleleft b$ or $b \blacktriangleleft c$, and that $b \in \mathbb{O}^+ \Rightarrow a \blacktriangleleft b$ and $b \in \mathbb{O}^- \Rightarrow b \blacktriangleleft c$. We conclude that $\blacktriangleleft \in \text{PIP}(\mathbb{O})$.

There is left to prove that any $a < c$ incomparable in \blacktriangleleft satisfy (\spadesuit) . Assume the opposite and consider $a < c$ incomparable in \blacktriangleleft not satisfying (\spadesuit) with $c - a$ minimal. Since a and c are incomparable in \blacktriangleleft , they are also incomparable in \triangleleft and satisfy (\spadesuit) . Assume for example that there exists $b \in \mathbb{O}^+$ such that $a \not\blacktriangleleft b \not\blacktriangleleft c$ (the other case is symmetric). Since $\triangleleft^{\text{Dec}} = \blacktriangleleft^{\text{Dec}}$, we have $b \blacktriangleright c$. Since a and c do not satisfy (\spadesuit) in \blacktriangleleft , we obtain that $a \blacktriangleleft b$. We can assume that b

is the maximal integer such that $a < b < c$, $b \in \mathbb{O}^+$ and $a \blacktriangleleft b \blacktriangleright c$. Since $a \blacktriangleleft b$ but $a \not\blacktriangleleft c$, we have $b \blacktriangleleft c$, so that b and c are incomparable in \blacktriangleleft . By minimality of $c - a$, we obtain that b and c satisfy (\spadesuit) in \blacktriangleleft . We distinguish two cases:

- (i) Assume that there exists $b < d < c$ such that $d \in \mathbb{O}^+$ and $b \blacktriangleleft d \blacktriangleright c$. Since a and c do not satisfy (\spadesuit) in \blacktriangleleft , we have $a \blacktriangleleft d \blacktriangleright c$, contradicting the maximality of b .
- (ii) Assume that there exists $b < d < c$ such that $d \in \mathbb{O}^-$ and $b \blacktriangleright d \blacktriangleleft c$. Since a and c do not satisfy (\spadesuit) in \blacktriangleleft and $d \blacktriangleleft c$, we have $a \blacktriangleright d$. We thus obtained $a < b < d$ with $b \in \mathbb{O}^+$, and $a \blacktriangleright d$ while $b \blacktriangleright d$ contradicting that $\blacktriangleleft \in \text{PIP}(\mathbb{O})$.

Since we obtain a contradiction in both cases, we conclude that any $a < c$ incomparable in \blacktriangleleft satisfy (\spadesuit) . \square

Proof of Claim H. We first prove that $\blacktriangleleft \subseteq \blacktriangleleft_S$. Observe first that for a permutree T and a Schröder permutree S obtained from T by contracting a subset of edges E , the poset \blacktriangleleft_S is obtained from the poset \blacktriangleleft_T by deleting the sets

$$\{(a, d) \mid a < d, \exists a \leq b < c \leq d, b \in \{a\} \cup \mathbb{O}^-, c \in \{d\} \cup \mathbb{O}^+, a \blacktriangleleft_T b \blacktriangleleft_T c \blacktriangleleft_T d \text{ and } (b, c) \in E\},$$

$$\{(d, a) \mid a < d, \exists a \leq b < c \leq d, b \in \{a\} \cup \mathbb{O}^+, c \in \{d\} \cup \mathbb{O}^-, a \blacktriangleright_T b \blacktriangleright_T c \blacktriangleright_T d \text{ and } (c, b) \in E\}.$$

Assume now that we had $\blacktriangleleft \not\subseteq \blacktriangleleft_S$ and remember that $\blacktriangleleft \subseteq \blacktriangleleft_S$ by construction. Since we only contract increasing edges in \blacktriangleleft_T to obtain \blacktriangleleft_S , this would imply that there exists $a \leq b < c \leq d$ with $b \in \{a\} \cup \mathbb{O}^-, c \in \{d\} \cup \mathbb{O}^+$, and such that $a \blacktriangleleft d$ while $b \not\blacktriangleleft d$. This would contradict that $\blacktriangleleft \in \text{PIP}(\mathbb{O})$.

We now prove that $\blacktriangleleft_S \subseteq \blacktriangleleft$. Observe first that $\blacktriangleleft_S^{\text{Dec}} \subseteq \blacktriangleleft^{\text{Dec}} = \blacktriangleleft^{\text{Dec}}$. Assume now by contradiction that there exists $a < c$ such that $a \not\blacktriangleleft c$ and $a \blacktriangleleft_S c$, and choose such $a < c$ with $c - a$ minimal. Note that a and c are incomparable in \blacktriangleleft . We distinguish two cases:

- (i) If a and c satisfy (\spadesuit) , we can assume by symmetry that there exists $a < b < c$ such that $b \in \mathbb{O}^+$ and $a \not\blacktriangleleft b \not\blacktriangleright c$. Since $a < b < c$, $b \in \mathbb{O}^+$, $a \blacktriangleleft_S c$ and $\blacktriangleleft_S \in \text{PIP}(\mathbb{O})$, we obtain that $a \blacktriangleleft_S b$. Since $a \not\blacktriangleleft b$ and $a \blacktriangleleft_S b$, this contradicts the minimality of $c - a$.
- (ii) If a and c do not satisfy (\spadesuit) , then they satisfy (\clubsuit) , and $a \blacktriangleleft c$ is not a cover relation (otherwise, the relation $a \blacktriangleleft c$ would have been contracted in \blacktriangleleft_S). Let $b \in [n] \setminus \{a, c\}$ be such that $a \blacktriangleleft b \blacktriangleleft c$. If $a < b < c$, we have $a \not\blacktriangleright b \not\blacktriangleright c$, thus $a \not\blacktriangleleft b \not\blacktriangleleft c$ by (\clubsuit) , thus $a \not\blacktriangleleft_S b \not\blacktriangleleft_S c$ by minimality of $c - a$, contradicting that $a \blacktriangleleft_S c$ and $\blacktriangleleft_S \in \text{WOIP}(n)$. We can thus consider that $b < a < c$ (the case $a < c < b$ is symmetric). Note that we cannot have $a \in \mathbb{O}^+$ since $b \blacktriangleleft a$ and $b \blacktriangleleft c$ would contradict that $\blacktriangleleft \in \text{PIP}(\mathbb{O})$. We have $b \blacktriangleright a$ (since $b \blacktriangleright a$) and we can assume that b is maximal such that $b < a$ and $b \blacktriangleright a$. Observe that b and c are incomparable in \blacktriangleleft (indeed $b \not\blacktriangleleft c$ since $b \blacktriangleright a$ and $a \not\blacktriangleleft c$, and $b \not\blacktriangleright c$ since $b \blacktriangleleft c$). Since $b < a < c$ and $b \blacktriangleright a$ while $a \not\blacktriangleleft c$, b and c do not satisfy (\clubsuit) , thus they satisfy (\spadesuit) . We again have two cases:

- If there is $b < d < c$ with $d \in \mathbb{O}^+$ with $b \not\blacktriangleleft d \not\blacktriangleright c$. If $b < d < a$, we have $b \blacktriangleright a$ and $\blacktriangleleft \in \text{PIP}(\mathbb{O})$ implies $d \blacktriangleright a$ contradicting the maximality of b . Since $d \in \mathbb{O}^+$, we have $d \neq a$. Finally, if $a < d < c$, we have $a \not\blacktriangleleft d$ since $d \not\blacktriangleright c$ and a and c satisfy (\clubsuit) , and we obtain that $a \not\blacktriangleleft d \not\blacktriangleright c$ contradicting that a and c do not satisfy (\spadesuit) .
- If there is $b < d < c$ with $d \in \mathbb{O}^-$ with $b \not\blacktriangleright d \not\blacktriangleleft c$, then we have $a < d < c$ (because $b \not\blacktriangleright d$, $b \blacktriangleright a$ and $\blacktriangleleft \in \text{PIP}(\mathbb{O})$). Since $d \not\blacktriangleleft c$ and a and c satisfy (\clubsuit) , we obtain that $a \not\blacktriangleright d \not\blacktriangleleft c$, contradicting that a and c do not satisfy (\spadesuit) .

As we obtain a contradiction in all cases, we conclude that $\blacktriangleleft_S \subseteq \blacktriangleleft$. \square

A.4. Proof of claims of Section 2.3.5.

Proof of Claim I. First, $\blacktriangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})}$ is clearly antisymmetric as it is contained in the antisymmetric relation \blacktriangleleft . To prove that it is transitive, consider $u, v, w \in [n]$ such that $u \blacktriangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} v \blacktriangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} w$. Since $\blacktriangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} \subseteq \blacktriangleleft$, we have $u \blacktriangleleft v \blacktriangleleft w$, so that $u \blacktriangleleft w$ by transitivity of \blacktriangleleft . Assume by means of contradiction that $u \not\blacktriangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} w$. Thus, $u < w$ and there exists $u \leq n < p \leq w$ such that $n \in \{u\} \cup \mathbb{O}^-$ while $p \in \{w\} \cup \mathbb{O}^+$ and $n \not\blacktriangleleft p$. We now distinguish three cases:

- If $v \leq n$, then $n \not\blacktriangleleft p$ and $v \leq n < p \leq w$ contradicts our assumption that $v \blacktriangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} w$.

- If $p \leq v$, then $n \not\triangleleft p$ and $u \leq n < p \leq v$ contradicts our assumption that $u \triangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} v$.
- Finally, if $n < v < p$, then $u \triangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} v$ ensures that $n \triangleleft v$, and $v \triangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} w$ ensures that $v \triangleleft p$. Together with $n \not\triangleleft v$, this contradicts the transitivity of \triangleleft .

As we obtained a contradiction in each case, we conclude that $\triangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})}$ is transitive. Since all our relations are reflexive, we conclude that $\triangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})}$ is a poset. \square

Proof of Claim J. We prove that $\triangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})}$ is in $\text{IPIP}^+(\mathbb{O})$, the result follows by symmetry for $\text{IPIP}^-(\mathbb{O})$ and finally for $\text{IPIP}^\pm(\mathbb{O}) = \text{IPIP}^+(\mathbb{O}) \cap \text{IPIP}^-(\mathbb{O})$. Assume that there exists $a < b < c$ with $b \in \mathbb{O}^+$ and $a \not\triangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} b$. Then there are witnesses $a \leq n < p \leq b$ with $n \in \{a\} \cup \mathbb{O}^-$ while $p \in \{b\} \cup \mathbb{O}^+$ and $n \not\triangleleft p$. Since $b \in \mathbb{O}^+$, we have $p \in \mathbb{O}^+$. Therefore, n and p are also witnesses for $a \not\triangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} c$. This shows that $\triangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})}$ is in $\text{IPIP}^+(\mathbb{O})$. \square

Proof of Claim K. Let $\triangleleft \in \text{IWOIP}(n)$. Let $a < b < c$ be such that $a \not\triangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} b \not\triangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} c$ and $a \triangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} c$. Then there exist witnesses $a \leq m < p \leq b \leq n < q \leq c$ with $m \in \{a\} \cup \mathbb{O}^-$, $p \in \{b\} \cup \mathbb{O}^+$, $n \in \{b\} \cup \mathbb{O}^-$ and $q \in \{c\} \cup \mathbb{O}^+$, and such that $m \not\triangleleft p$ and $n \not\triangleleft q$. If $p \neq b$, then $p \in \mathbb{O}^+$ and $a \leq m < p < c$ are also witnesses for $a \not\triangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} c$. By symmetry, we can thus assume that $p = b = n$. Therefore, we have $m \not\triangleleft p = b = n \not\triangleleft q$, which implies that $m \not\triangleleft q$ since $\triangleleft \in \text{IWOIP}(n)$. Since $a \leq m < q \leq c$ with $m \in \{a\} \cup \mathbb{O}^-$, $q \in \{c\} \cup \mathbb{O}^+$ and $m \not\triangleleft q$, we obtain that $a \not\triangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} c$. We conclude that $\triangleleft \in \text{IWOIP}(n)$ implies $\triangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} \in \text{IWOIP}(n)$. In particular, we obtain that $\triangleleft^{\text{IPIP}^\pm \text{id}(\mathbb{O})} = (\triangleleft^{\text{IWOIPid}})^{\text{IPIP}^\pm \text{id}(\mathbb{O})} \in \text{IPIP}(\mathbb{O})$ since $\triangleleft^{\text{IWOIPid}} \in \text{IWOIP}(n)$ by Lemma 36. \square

Proof of Claim L. Claim L is immediate for $\varepsilon \in \{-, +\}$. For $\varepsilon = \pm$, assume that there exists $a \leq n < p \leq c$ with $n \in \{c\} \cup \mathbb{O}^-$ and $p \in \{b\} \cup \mathbb{O}^+$ such that $n \not\triangleleft p$. Then we also have $n \not\triangleleft p$ which implies $n \not\triangleleft c$ (since $\triangleleft \in \text{IPIP}^+(\mathbb{O})$) and $a \not\triangleleft p$ (since $\triangleleft \in \text{IPIP}^-(\mathbb{O})$), which in turn implies $a \not\triangleleft c$. Finally, this also implies the claim when $\varepsilon = \emptyset$ by applying first Lemma 37 and then the claim for $\varepsilon = \pm$. \square

Proof of Claim M. Otherwise, there exists $a < c$ such that there is no \mathbb{O} -snake from a to c in \triangleleft . Choose such a pair $a < c$ with $c - a$ minimal. Since \triangleleft_σ is a total order, we have either $a \triangleleft_\sigma c$ or $a \triangleright_\sigma c$. Assume for example that $a \triangleleft_\sigma c$, the other case being symmetric. Since $a \triangleleft_\sigma c$ while $a \not\triangleleft c$ (otherwise $a \triangleleft c$ is an \mathbb{O} -snake), there exists $a \leq n < p \leq c$ such that $n \in \{a\} \cup \mathbb{O}^-$ while $p \in \{c\} \cup \mathbb{O}^+$ and $n \not\triangleleft_\sigma p$. Since \triangleleft_σ is a total order, we get $n \triangleright_\sigma p$. Moreover, we have either $a \triangleleft_\sigma n$ or $a \triangleright_\sigma n$. In the latter case, we get by transitivity of \triangleleft_σ that $a \triangleright_\sigma p$. Therefore, up to forcing $a = n$, we can assume that $a \triangleleft_\sigma n$ and similarly up to forcing $p = c$, we can assume that $p \triangleleft_\sigma c$. It follows that $a \triangleleft_\sigma n \triangleright_\sigma p \triangleleft_\sigma c$ is an \mathbb{O} -snake from a to c in \triangleleft_σ , where either $a \neq n$ or $p \neq c$ (because $a \triangleleft_\sigma c$). By minimality of $c - a$ in our choice of $a < c$, there exists an \mathbb{O} -snake from a to n , from n to p , and from p to c in \triangleleft . Since $n \in \{a\} \cup \mathbb{O}^-$ and $p \in \{c\} \cup \mathbb{O}^+$, it is straightforward to construct from these snakes an \mathbb{O} -snake from a to c in \triangleleft , contradicting our assumption. \square

Proof of Claim N. Assume that there exists $a < c$ incomparable in \triangleleft that do not satisfy (\clubsuit) in \triangleleft . We choose such a pair $a < c$ with $c - a$ minimal. By symmetry, we can assume that there exists $a < b < c$ such that $a \triangleleft b \triangleright c$ and that b is maximal for this property. Since $\triangleleft \subseteq \triangleleft$, we have $a \triangleleft b$. We distinguish three cases:

- (i) Assume first that a and c are incomparable in \triangleleft . Since $\triangleleft \in \text{WOFP}(n)$, Proposition 31 and $a \triangleleft b$ imply that $a \triangleleft b \triangleright c$. Since $b \triangleright c$ while $b \triangleright c$, there is $b \leq p < n \leq c$ with $p \in \{b\} \cup \mathbb{O}^+$ while $n \in \{c\} \cup \mathbb{O}^-$ and $p \not\triangleright n$. We again have two cases:
 - If $b \neq p$, we have $p \triangleright c$ (since otherwise $p \not\triangleright n$ would contradict that $\triangleleft \in \text{DPIP}(\mathbb{O})$) and thus $a \not\triangleleft p$ (by maximality of b). We thus obtained $a < p < c$ with $p \in \mathbb{O}^+$ and $a \not\triangleleft p \triangleright c$, so that $a < c$ satisfy (\spadesuit) in \triangleleft .
 - If $b = p$, then $n \neq c$. We have $a \not\triangleright n$ (since otherwise $p \not\triangleright n$ would contradict that $\triangleleft \in \text{PIP}(\mathbb{O})$). Moreover, by minimality of $c - a$, we have $b = p$ and c satisfy (\clubsuit)

in \triangleleft , so that $p \blacktriangleright n$ implies that $n \blacktriangleleft c$. We obtained that $a < n < c$ with $n \in \mathbb{O}^-$ and $a \blacktriangleright n \blacktriangleleft c$, so that $a < c$ satisfy (\spadesuit) in \triangleleft .

(ii) Assume now that $a \triangleleft c$. Since $a \triangleleft c$ while $a \blacktriangleleft c$, there is $a \leq n < p \leq c$ with $n \in \{a\} \cup \mathbb{O}^-$ while $p \in \{c\} \cup \mathbb{O}^+$ and $n \not\triangleleft p$. Since $\triangleleft \in \text{DPIP}(\mathbb{O})$ and $n \blacktriangleleft p$, we must have $a \blacktriangleleft p$ and $n \blacktriangleleft c$. Assume that a and c do not satisfy (\spadesuit) in \triangleleft . This implies that $p \blacktriangleright c$ and $a \blacktriangleright n$. Since $a \blacktriangleright c$ and $p \blacktriangleright c$, we obtain by transitivity of \triangleleft that a and p are incomparable in \triangleleft . By minimality of $c - a$, we obtain that a and p satisfy (\clubsuit) . We now consider two cases:

- If $b < p$, then $a \triangleleft b$ implies that $b \blacktriangleright p$, which together with $p \blacktriangleright c$ and $b \blacktriangleright c$ contradicts the transitivity of \triangleleft .
- If $p \leq b$, then we have $a \leq n < p \leq b$ with $n \in \{a\} \cup \mathbb{O}^-$ while $p \in \{c\} \cup \mathbb{O}^+$ and $n \not\triangleleft p$, which contradicts that $a \triangleleft b$.

Since we obtained a contradiction in both cases, we conclude that a and c satisfy (\spadesuit) in \triangleleft .

(iii) Assume finally that $a \triangleright c$. Then $a \not\triangleright b$ and $\triangleleft \in \text{DWOIP}(n)$ implies that $a \triangleleft b \triangleright c$ and we are back to case (i). \square

A.5. Proof of claims of Theorem 84.

Proof of Claim O. As $\text{cf}_{\text{DPIP}(\mathbb{O})}$ is decreasing, we only consider $(a, c) \in \neg^{\text{Dec}}$. Since $\neg^{\text{Dec}} = \triangleleft^{\text{Dec}} \cap \blacktriangleleft^{\text{Dec}}$, we have $a \triangleright c$ and $a \blacktriangleright c$. Since both $\triangleleft, \blacktriangleleft \in \text{DPIP}(\mathbb{O})$, for any $a < b < c$, if $b \in \mathbb{O}^-$ then $a \triangleright b$ and $a \blacktriangleright b$ so that $a \vdash b$, while if $b \in \mathbb{O}^+$ then $b \triangleright c$ and $b \blacktriangleright c$ so that $b \vdash c$. Note that the important point here is that the behavior of b is the same in \triangleleft and \blacktriangleleft as it is dictated by the orientation of b . \square

Proof of Claim P. Assume now that $\neg^{\text{tdd}} \notin \text{DPIP}(\mathbb{O})$. Consider $\{a, c\} \in \text{cf}_{\text{DPIP}(\mathbb{O})}(\neg^{\text{tdd}})$ with $a < c$ and $c - a$ minimal. Since $\text{cf}_{\text{DPIP}(\mathbb{O})}(\neg^{\text{tdd}})$ is decreasing, we have $a \vdash^{\text{tdd}} c$. Assume for the moment that there exists $a < b < c$ such that $b \in \mathbb{O}^-$ and $a \not\vdash^{\text{tdd}} b$, and choose such b with $b - a$ minimal. Since $\neg \in \text{DPIP}(\mathbb{O})$, we have $a \vdash b$ while $a \not\vdash^{\text{tdd}} b$. By definition of \neg^{tdd} , there exists $i \leq b$ and $j \geq a$ such that $i \vdash b \neg a \vdash j$ but $i \not\vdash j$. From Lemma 16, we know that either $i \neq b$ or $j \neq a$. We thus distinguish two cases.

- Assume that $i \neq b$. Again by Lemma 16, there exists $a < k < b$ such that $i \vdash k \vdash b$. Thus, we have $k \not\vdash b$ (since \vdash is antisymmetric) while $a \vdash b$ and $a < k < b$, so $k \notin \mathbb{O}^+$ (since \neg if $\text{cf}_{\text{DPIP}(\mathbb{O})}$ -free). Since \mathbb{O} is covering, we therefore obtain that $k \in \mathbb{O}^-$. By minimality of $b - a$ in our choice of b , we obtain that $a \vdash^{\text{tdd}} k$. But $i \vdash k \vdash a \vdash j$ and $a \vdash^{\text{tdd}} k$ implies that $i \vdash j$, a contradiction to our assumption on i and j .
- Assume now that $j \neq a$. Again by Lemma 16, there exists $a < k < b$ such that $a \vdash k \vdash j$. Thus, we have $a \not\vdash k$ (since \vdash is antisymmetric) while $a \vdash b$ and $a < k < b$, so $k \notin \mathbb{O}^-$ (since \triangleleft if $\text{cf}_{\text{DPIP}(\mathbb{O})}$ -free). Since \mathbb{O} is covering, we therefore obtain that $k \in \mathbb{O}^+$. Since $\neg \in \text{DPIP}(\mathbb{O})$, $k \in \mathbb{O}^+$ and $a \vdash c$, we have $k \vdash c$. We claim that $k \vdash^{\text{tdd}} c$. Otherwise we could find $i' \leq c$ and $j' \geq k$ such that $i' \vdash c \vdash k \vdash j'$ while $i' \not\vdash j'$. Since $a \vdash k \vdash j'$ and \neg is semitransitive, we would also have $i' \vdash c \vdash a \vdash j'$ while $i' \not\vdash j'$, contradicting the fact that $a \vdash^{\text{tdd}} c$. Now by minimality of $c - a$ in our choice of (a, c) , we obtain that $(c, k) \in \neg^{\text{tdd}} \setminus \text{cf}_{\text{DPIP}(\mathbb{O})}(\neg^{\text{tdd}})$. Therefore, since $b \in \mathbb{O}^-$, we have $k \vdash^{\text{tdd}} b$. But $i \vdash b \vdash k \vdash j$ and $k \vdash^{\text{tdd}} b$ implies that $i \vdash j$, a contradiction to our assumption on i and j .

We therefore proved that $a \vdash^{\text{tdd}} b$ for all $a < b < c$ with $b \in \mathbb{O}^-$. The case of $b \in \mathbb{O}^+$ is symmetric and left to the reader. This concludes the proof. \square

A.6. Proof of claims of Proposition 94.

Proof of Claim Q. Assume that \neg is not $\text{cf}_{\text{sn}(\mathbb{O})}$ -free and let $\{a, c\} \in \text{cf}_{\text{sn}(\mathbb{O})}(\neg)$ with $a < c$ and $c - a$ minimal. Since $a \not\vdash c$, we have $a \not\triangleleft c$ and $a \blacktriangleleft c$. Since $a \not\vdash c$, we have $a \not\triangleright c$ or $a \blacktriangleright c$. We can thus assume without loss of generality that a and c are incomparable in \triangleleft . Since \triangleleft is $\text{cf}_{\text{sn}(\mathbb{O})}$ -free, there exists $a < b_1 < \dots < b_k < c$ such that either $b_{2i} \in \mathbb{O}^+$, $b_{2i+1} \in \mathbb{O}^-$ and $a \triangleleft b_1 \triangleright b_2 \triangleleft b_3 \triangleright \dots$, or $b_{2i} \in \mathbb{O}^-$, $b_{2i+1} \in \mathbb{O}^+$ and $a \triangleright b_1 \triangleleft b_2 \triangleright b_3 \triangleleft \dots$. We distinguish these two cases:

- In the former case, we obtain $b_1 \in \mathbb{O}^-$ and $a \vdash b_1$ (since $\triangleleft^{\text{Inc}} \subseteq \neg$).
- In the latter case, we distinguish three cases according to the order of a and b_1 in \triangleleft :

- if $a \blacktriangleleft b_1$, then $a \dashv b_1 \dashv b_2$ (since $\blacktriangleleft^{\text{Inc}} \cup \blacktriangleleft^{\text{Inc}} \subseteq \dashv$) so that we obtain $b_2 \in \mathbb{O}^-$ and $a \dashv b_2$.
- if $a \blacktriangleright b_1$, we obtain $b_1 \in \mathbb{O}^+$ and $a \vdash b_1$ (since $\blacktriangleleft^{\text{Dec}} \cap \blacktriangleleft^{\text{Dec}} \subseteq \dashv$).
- if a and b_1 are incomparable in \blacktriangleleft , then they are also incomparable in \dashv . By minimality of $c - a$ in our choice of (a, c) , we have $\{a, b_1\} \notin \text{cf}_{\text{sn}(\mathbb{O})}(\dashv)$. Lemma 64 (iii) thus ensures the existence of $a < b < b_1$ such that either $b \in \mathbb{O}^-$ and $a \dashv b$, or $b \in \mathbb{O}^+$ and $a \vdash b$.

In all situations, we have found $a < b < c$ such that either $b \in \mathbb{O}^-$ and $a \dashv b$, or $b \in \mathbb{O}^+$ and $a \vdash b$. Since $\{b, c\} \notin \text{cf}_{\text{sn}(\mathbb{O})}(\dashv)$ by minimality of $c - a$ in our choice of (a, c) , Lemma 64 (iii) thus contradicts that $\{a, c\} \in \text{cf}_{\text{sn}(\mathbb{O})}(\dashv)$. \square

Proof of Claim R. Assume that \dashv^{tdd} is not $\text{cf}_{\text{sn}(\mathbb{O})}$ -free and let $\{a, c\} \in \text{cf}_{\text{sn}(\mathbb{O})}(\dashv^{\text{tdd}})$ with $a < c$ and $c - a$ minimal. We distinguish two cases:

- If $a \not\vdash c$, since \dashv is $\text{cf}_{\text{sn}(\mathbb{O})}$ -free, Lemma 64 (iii) ensures that there exists $a < b < c$ such that $b \in \mathbb{O}^-$ and $a \dashv b$, or $b \in \mathbb{O}^+$ and $a \vdash b$. In the former case, we also have $a \dashv^{\text{tdd}} b$. In the latter case, we either have $a \vdash^{\text{tdd}} b$ or a and b are incomparable in \dashv^{tdd} . By minimality of $c - a$ in our choice of (a, c) , we have $\{a, b\} \notin \text{cf}_{\text{sn}(\mathbb{O})}(\dashv^{\text{tdd}})$. We thus obtain by Lemma 64 (iii) that there exists $a < b' < b$ such that $b' \in \mathbb{O}^-$ and $a \dashv^{\text{tdd}} b'$, or $b' \in \mathbb{O}^+$ and $a \vdash^{\text{tdd}} b'$.
- If $a \vdash c$, then there exists $i \leq c$ and $j \geq a$ such that $i \dashv c \dashv a \dashv j$ while $i \not\vdash j$. From Lemma 16, we can assume for example that $i \neq c$ so that there exists $a < k < c$ with $i \dashv k \dashv c$ (the proof when $j \neq a$ is similar). Note that $a \not\vdash^{\text{tdd}} k$ since otherwise $a \dashv^{\text{tdd}} k \dashv^{\text{tdd}} c$ and $a \dashv^{\text{tdd}} c$ would contradict the transitivity of \dashv^{tdd} . Moreover, $a \not\vdash^{\text{tdd}} k$ since either we already have $a \not\vdash k$, or $i \leq c$ and $j \geq a$ still satisfy $i \dashv k \dashv a \dashv j$ while $i \not\vdash j$. Therefore, a and k are incomparable in \dashv . By minimality of $c - a$ in our choice of (a, c) , we have $\{a, k\} \notin \text{cf}_{\text{sn}(\mathbb{O})}(\dashv^{\text{tdd}})$. We thus obtain by Lemma 64 (iii) that there exists $a < b < k$ such that $b \in \mathbb{O}^-$ and $a \dashv^{\text{tdd}} b$, or $b \in \mathbb{O}^+$ and $a \vdash^{\text{tdd}} b$.

In all situations, we have found $a < b < c$ such that either $b \in \mathbb{O}^-$ and $a \dashv^{\text{tdd}} b$, or $b \in \mathbb{O}^+$ and $a \vdash^{\text{tdd}} b$. Since $\{b, c\} \notin \text{cf}_{\text{sn}(\mathbb{O})}(\dashv^{\text{tdd}})$ by minimality of $c - a$ in our choice of (a, c) , Lemma 64 (iii) thus contradicts that $\{a, c\} \in \text{cf}_{\text{sn}(\mathbb{O})}(\dashv^{\text{tdd}})$. \square

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