DENOMINATOR VECTORS AND COMPATIBILITY DEGREES IN CLUSTER ALGEBRAS OF FINITE TYPE

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Abstract. We present two simple descriptions of the denominator vectors of the cluster variables of a cluster algebra of finite type, with respect to any initial cluster seed: one in terms of the compatibility degrees between almost positive roots defined by S. Fomin and A. Zelevinsky, and the other in terms of the root function of a certain subword complex. These descriptions only rely on linear algebra. They provide two simple proofs of the known fact that the d-vector of any non-initial cluster variable with respect to any initial cluster seed has non-negative entries and is different from zero.

1. Introduction

Cluster algebras were introduced by S. Fomin and A. Zelevinsky in the series of papers [FZ02, FZ03a, FZ05, FZ07]. They are commutative rings generated by a (possibly infinite) set of *cluster variables*, which are grouped into overlapping *clusters*. The clusters can be obtained from any *initial cluster seed* $X = \{x_1, \dots, x_n\}$ by a mutation process. Each mutation exchanges a single variable y to a new variable y' satisfying a relation of the form $yy' = M_+ + M_-$, where M_+ and M_- are monomials in the variables involved in the current cluster and distinct from y and y'. The precise content of these monomials M_+ and M_- is controlled by a combinatorial object (a skew-symmetrizable matrix, or equivalently a weighted quiver [Kel12]) which is attached to each cluster and is also transformed during the mutation. We refer to [FZ02] for the precise definition of these joint dynamics. In [FZ02, Theorem 3.1], S. Fomin and A. Zelevinsky proved that given any initial cluster seed $X = \{x_1, \dots, x_n\}$, the cluster variables obtained during this mutation process are *Laurent polynomials* in the variables $x_1, \dots, x_n$. That is to say, every non-initial cluster variable y can be written in the form

$$y = \frac{F(x_1, \dots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}}$$

where $F(x_1, \dots, x_n)$ is a polynomial which is not divisible by any variable x_i for $i \in [n]$. This intriguing property is called *Laurent Phenomenon* in cluster algebras [FZ02]. The *denominator vector* (or *d-vector* for short) of the cluster variable y with respect to the initial cluster seed X is the vector $\mathbf{d}(X, y) := (d_1, \dots, d_n)$. The

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d-vector of the initial cluster variable x_i is $\mathbf{d}(X, x_i) := -e_i := (0, \dots, -1, \dots, 0)$ by definition.

Note that we think of the cluster variables as a set of variables satisfying some algebraic relations. These variables can be expressed in terms of the variables in any initial cluster seed $X = \{x_1, \dots, x_n\}$ of the cluster algebra. Starting from a different cluster seed $X' = \{x'_1, \dots, x'_n\}$ would give rise to an isomorphic cluster algebra, expressed in terms of the variables $x'_1, \dots, x'_n$ of this seed. Therefore, the d-vectors of the cluster variables depend on the choice of the initial cluster seed X in which the Laurent polynomials are expressed. This dependence is explicit in the notation $\mathbf{d}(X, y)$. Note also that since the denominator vectors do not depend on coefficients, we restrict our attention to coefficient-free cluster algebras.

In this paper, we only consider finite type cluster algebras, *i.e.* cluster algebras whose mutation graph is finite. They were classified in [FZ03a, Theorem 1.4] using the Cartan-Killing classification for finite crystallographic root systems. In [FZ03a, Theorem 1.9], S. Fomin and A. Zelevinsky proved that in the cluster algebra of any given finite type, with a bipartite quiver as initial cluster seed,

(i) there is a bijection ϕ from almost positive roots to cluster variables, which sends the negative simple roots to the initial cluster variables;
(ii) the d-vector of the cluster variable $\phi(\beta)$ corresponding to an almost positive root β is given by the vector $(b_1, \dots, b_n)$ of coefficients of the root $\beta = \sum b_i \alpha_i$ on the linear basis Δ formed by the simple roots $\alpha_1, \dots, \alpha_n$; and
(iii) these coefficients coincide with the *compatibility degrees* $(\alpha_i \parallel \beta)$ defined in [FZ03b, Section 3.1].

These results were extended to all cluster seeds corresponding to Coxeter elements of the Coxeter group (see *e.g.* [Kel12, Theorem 3.1 and Section 3.3]). More precisely, assume that the initial seed is the cluster X_c corresponding to a Coxeter element c (its associated quiver is the Coxeter graph oriented according to c). Then one can define a bijection ϕ_c from almost positive roots to cluster variables such that the d-vector of the cluster variable $\phi_c(\beta)$ corresponding to β, with respect to the initial cluster seed X_c, is still given by the vector $(b_1, \dots, b_n)$ of coordinates of $\beta = \sum b_i \alpha_i$ in the basis Δ of simple roots. Under this bijection, the collections of almost positive roots corresponding to clusters are called *c-clusters* and were studied by N. Reading [Rea07, Section 7].

In this paper, we provide similar interpretations for the denominators of the cluster variables of any finite type cluster algebra with respect to *any initial cluster seed* (acyclic or not):

(i) Our first description (Corollary 3.2) uses compatibility degrees: if $\{\beta_1, \dots, \beta_n\}$ is the set of almost positive roots corresponding to the cluster variables in any initial cluster seed $X = \{\phi(\beta_1), \dots, \phi(\beta_n)\}$, then the d-vector of the cluster variable $\phi(\beta)$ corresponding to an almost positive root β, with respect to the initial cluster seed X, is still given by the vector of compatibility degrees $((\beta_1 \parallel \beta), \dots, (\beta_n \parallel \beta))$ of [FZ03b, Section 3.1]. We also provide a refinement of this result parametrized by a Coxeter element c, using the bijection ϕ_c together with the notion of c-compatibility degrees (Corollary 3.3).
(ii) Our second description (Corollary 3.4) uses the recent connection [CLS13] between the theory of cluster algebras of finite type and the theory of subword complexes, initiated by A. Knutson and E. Miller [KM04]. We describe the

entries of the d-vector in terms of certain coefficients given by the root function of a subword complex associated to a certain word.

Using these results, we provide two alternative proofs of the known fact that, in a cluster algebra of finite type, the d-vector of any non-initial cluster variable with respect to any initial cluster seed is non-negative and not equal to zero (Corollary 3.5).

Even if we restrict here to crystallographic finite types since we deal with cluster variables of the associated cluster algebras, all the results not involving cluster variables remain valid for any arbitrary finite type. This includes in particular the results about almost positive roots, c-clusters, c-compatibility degrees, rotation maps, and their counterparts in subword complexes. We also highlight that subword complexes played a fundamental role in the results of this paper. Even if the main result describing denominator vectors in terms of compatibility degrees can be proved independently, we would not have been able to find it without using the subword complex approach.

Finally, we also provide explicit geometric interpretations for all the concepts and results in this paper for the classical types A, B, C and D in Section 7. Our interpretation in type D is new and differs from known interpretations in the literature. It simplifies certain combinatorial and algebraic aspects and makes an additional link between the theory of cluster algebras and pseudotriangulations.

2. Preliminaries

Let (W, S) be a finite crystallographic Coxeter system of rank n. We consider a root system Φ, with simple roots $\Delta := \{\alpha_1, \dots, \alpha_n\}$, positive roots Φ^+, and almost positive roots $\Phi_{\geq -1} := \Phi^+ \cup -\Delta$. We refer to [Hum90] for a reference on Coxeter groups and root systems.

Let $\mathcal{A}(W)$ denote the cluster algebra associated to type W, as defined in [FZ03a]. Each cluster is formed by n cluster variables, and is endowed with a weighted quiver (an oriented and weighted graph on S) which controls the cluster dynamics. Since we will not make extensive use of it, we believe that it is unnecessary to recall here the precise definition of the quiver and cluster dynamics, and we refer to [FZ02, Kel12] for details. For illustrations, we recall geometric descriptions of these dynamics in types A, B, C, and D in Section 7.

Let c be a Coxeter element of W, and $\mathrm{c} := (c_1, \cdots, c_n)$ be a reduced expression of c. The element c defines a particular weighted quiver $\mathcal{Q}_c$: the Coxeter graph of the Coxeter system (W, S) directed according to the order of appearance of the simple reflections in c. We denote by X_c the cluster seed whose associated quiver is $\mathcal{Q}_c$. Let $\mathrm{w}_\circ(\mathrm{c}) := (w_1, \cdots, w_N)$ denote the c-*sorting word* for $w_\circ$, *i.e.* the lexicographically first subword of the infinite word c^∞ which represents a reduced expression for the longest element $w_\circ \in W$. We consider the word $\mathrm{Q_c} := \mathrm{c}\mathrm{w}_\circ(\mathrm{c})$ and denote by $m := n + N$ the length of this word.

2.1. Cluster variables, almost positive roots, and positions in the word $\mathrm{Q_c}$.

We recall here the above-mentioned bijections between cluster variables, almost positive roots and positions in the word $\mathrm{Q_c}$. We will see in the next sections that both the clusters and the d-vectors (expressed on any initial cluster seed X) can also be read off in these different contexts. Figure 1 summarizes these different notions and the corresponding notations. We insist that the choice of the Coxeter

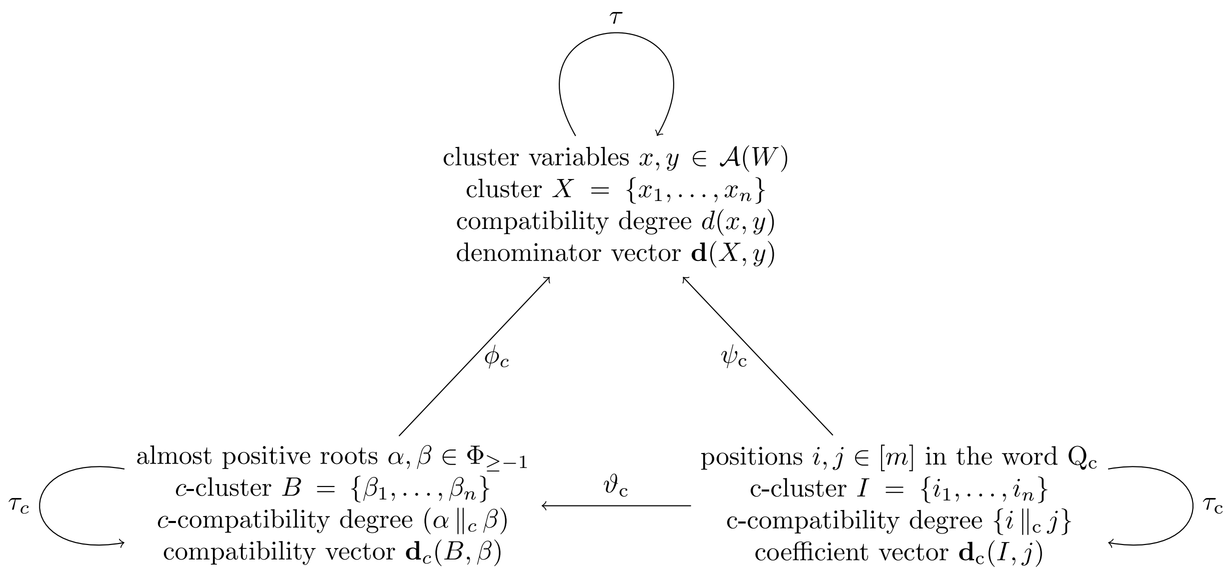


Figure 1. Three different contexts for cluster algebras of finite type, their different notions of compatibility degrees, and the bijections between them. See Sections 2.1, 2.2 and 2.3 for definitions.

element c and the choice of the initial cluster X are not related. The former provides a labeling of the cluster variables by the almost positive roots or by the positions in Q_{c}, while the latter gives an algebraic basis to express the cluster variables and to assign them d-vectors.

First, there is a natural bijection between cluster variables and almost positive roots, which can be parametrized by the Coxeter element c. Start from the initial cluster seed X_c associated to the weighted quiver $\mathcal{Q}_c$ corresponding to the Coxeter element c. Then the d-vectors of the cluster variables of $\mathcal{A}(W)$ with respect to the initial seed X_c are given by the almost positive roots $\Phi_{\geq -1}$. This defines a bijection

$$\phi_c : \Phi_{\geq -1} \longrightarrow \{\text{cluster variables of } \mathcal{A}(W)\}$$

from almost positive roots to cluster variables. Notice that this bijection depends on the choice of the Coxeter element c. When c is a bipartite Coxeter element, it is the bijection ϕ of S. Fomin and A. Zelevinsky [FZ03a, Theorem 1.9] mentioned above. Transporting the structure of the cluster algebra $\mathcal{A}(W)$ through the bijection ϕ_c, we say that a subset B of almost positive roots forms a *c-cluster* iff the corresponding subset of cluster variables $\phi_c(B)$ forms a cluster of $\mathcal{A}(W)$. The collection of c-clusters forms a simplicial complex on the set $\Phi_{\geq -1}$ of almost positive roots called the *c-cluster complex*. This complex was described in purely combinatorial terms by N. Reading in [Rea07, Section 7]. Given an initial c-cluster seed $B := \{\beta_1, \ldots, \beta_n\}$ in $\Phi_{\geq -1}$ and an almost positive root β, we define the d-vector of β with respect to B as

$$\mathbf{d}_c(B, \beta) := \mathbf{d}\big(\phi_c(B), \phi_c(\beta)\big).$$

If c is a bipartite Coxeter element, then we speak about classical clusters and omit c in the previous notation to write $\mathbf{d}(B, \beta)$.

Second, there is a bijection

$$\vartheta_{\mathrm{c}} : [m] \longrightarrow \Phi_{\geq -1}$$

from the positions in the word $\mathrm{Q_c} = \mathrm{cw_\circ(c)}$ to the almost positive roots as follows. The letter c_i of c is sent to the negative root $-\alpha_{c_i}$, while the letter w_i of $\mathrm{w_\circ(c)}$ is sent to the positive root $w_1 \cdots w_{i-1}(\alpha_{w_i})$. To be precise, note that this bijection depends not only on the Coxeter element c, but also on its reduced expression c. This bijection was defined by C. Ceballos, J.-P. Labbé and C. Stump in [CLS13, Theorem 2.2].

Composing the two maps described above provides a bijection

$$\psi_{\mathrm{c}} : [m] \longrightarrow \{\text{cluster variables of } \mathcal{A}(W)\}$$

from positions in the word $\mathrm{Q_c}$ to cluster variables (precisely defined by $\psi_{\mathrm{c}} := \phi_c \circ \vartheta_{\mathrm{c}}$). Transporting the structure of $\mathcal{A}(W)$ through the bijection ψ_{c}, we say that a subset I of positions in $\mathrm{Q_c}$ forms a c-*cluster* iff the corresponding cluster variables $\psi_{\mathrm{c}}(I)$ form a cluster of $\mathcal{A}(W)$. Moreover, given an initial c-cluster seed $I \subseteq [m]$ in $\mathrm{Q_c}$ and a position $j \in [m]$ in $\mathrm{Q_c}$, we define the d-vector of j with respect to I as

$$\mathbf{d}_{\mathrm{c}}(I, j) := \mathbf{d}\big(\psi_{\mathrm{c}}(I), \psi_{\mathrm{c}}(j)\big).$$

It turns out that the c-clusters can be read off directly in the word $\mathrm{Q_c}$ as follows.

Theorem 2.1 ([CLS13, Theorem 2.2 and Corollary 2.3])**.** *A subset I of positions in $\mathrm{Q_c}$ forms a* c*-cluster in $\mathrm{Q_c}$ if and only if the subword of $\mathrm{Q_c}$ formed by the complement of I is a reduced expression for $\mathrm{w_\circ}$.*

Remark 2.2. The previous theorem relates c-cluster complexes to subword complexes as defined by A. Knutson and E. Miller [KM04]. Given a word Q on the generators S of W and an element $\pi \in W$, the *subword complex* $\mathcal{SC}(\mathrm{Q}, \pi)$ is the simplicial complex whose faces are subwords P of Q such that the complement $\mathrm{Q} \smallsetminus \mathrm{P}$ contains a reduced expression of π. See [CLS13] for more details on this connection.

2.2. **The rotation map.** In this section we introduce a rotation map τ_{c} on the positions in the word $\mathrm{Q_c}$, and naturally extend it to a map on almost positive roots and cluster variables using the bijections of Section 2.1 (see Figure 1). The rotation map plays the same role for arbitrary finite type as the rotation of the polygons associated to the classical types A, B, C and D, see e.g. [CLS13, Theorem 8.10] and Section 7.

Definition 2.3 (Rotation maps)**.** The rotation

$$\tau_{\mathrm{c}} : [m] \longrightarrow [m]$$

is the map on the positions in the word $\mathrm{Q_c}$ defined as follows. If $q_i = s$, then $\tau_{\mathrm{c}}(i)$ is defined as the position in $\mathrm{Q_c}$ of the next occurrence of s if possible, and as the first occurrence of $w_\circ s w_\circ$ otherwise.

Using the bijection ϑ_{c} from the positions in the word $\mathrm{Q_c}$ to almost positive roots, this rotation can also be regarded as a map from almost positive roots to almost positive roots. For simplicity, we abuse of notation and also write

$$\tau_c : \Phi_{\geq -1} \longrightarrow \Phi_{\geq -1}$$

for the composition $\vartheta_{\mathrm{c}} \circ \tau_{\mathrm{c}} \circ \vartheta_{\mathrm{c}}^{-1}$. This composition can be expressed purely in terms of roots as

$$\tau_c(\alpha) = \begin{cases} c_1 \cdots c_{i-1}(\alpha_{c_i}) & \text{if } \alpha = -\alpha_{c_i}, \\ -\alpha_{c_i} & \text{if } \alpha = c_n \cdots c_{i+1}(\alpha_{c_i}), \\ c(\alpha) & \text{otherwise.} \end{cases}$$

The first and the third lines of this equation are given by the root corresponding to the next occurrence of the letter associated to α in $\mathrm{Q_c}$. The second line corresponds to the case when the letter associated to α in $\mathrm{Q_c}$ is the last occurrence of this letter in $\mathrm{Q_c}$. This case can be easily explained as follows. Let $\eta : S \to S$ be the involution $\eta(s) = w_\circ s w_\circ$. The last occurrence of $\eta(c_i)$ in $\mathrm{Q_c}$ is the position which is mapped under τ_c to the first occurrence of c_i in $\mathrm{Q_c}$. In other words, if we denote by α the root associated to the last occurrence of $\eta(c_i)$ in $\mathrm{Q_c}$, then $\tau_c(\alpha) = -\alpha_{c_i}$. In addition, the word $\mathrm{w_\circ(c)}$ is, up to commutations, equal to a word with suffix $(\eta(c_1), \dots, \eta(c_n))$ [CLS13, Proposition 7.1]. From this we conclude that $\alpha = c_n \cdots c_{i+1}(\alpha_{c_i})$ as desired.

Using the bijection ψ_{c} from the positions in the word $\mathrm{Q_c}$ to cluster variables, the rotation can also be regarded as a map on the set of cluster variables. Again for simplicity, we also write

$$\tau : \{\text{cluster variables of } \mathcal{A}(W)\} \longrightarrow \{\text{cluster variables of } \mathcal{A}(W)\}$$

for the composition $\psi_{\mathrm{c}} \circ \tau_{\mathrm{c}} \circ \psi_{\mathrm{c}}^{-1}$. This composition can be expressed purely in terms of cluster variables as follows. Consider the cluster variables expressed in terms of the initial cluster seed X_c associated to the weighted quiver $\mathcal{Q}_c$ (recall that this quiver is by definition the Coxeter graph of the Coxeter system (W, S) directed according to the order of appearance of the simple reflections in c). If y is the cluster variable at vertex i of a quiver obtained from $\mathcal{Q}_c$ after a sequence of mutations $\mu_{i_1} \to \cdots \to \mu_{i_r}$, then the rotation $\tau(y)$ is the cluster variable at vertex i of the quiver obtained from $\mathcal{Q}_c$ after the sequence of mutations $\mu_{c_1} \to \cdots \to \mu_{c_n} \to \mu_{i_1} \to \cdots \to \mu_{i_r}$. Although defined using a Coxeter element c, this rotation map is independent of the choice of c and we denote it by τ.

The following lemma is implicit in [CLS13, Proposition 8.6], see Corollary 6.5.

Lemma 2.4. *The rotation map preserves clusters:*

(i) a subset $I \subset [m]$ of positions in the word $\mathrm{Q_c}$ is a c-cluster if and only if $\tau_{\mathrm{c}}(I)$ is a c-cluster;
(ii) a subset $B \subset \Phi_{\geq -1}$ of almost positive roots is a c-cluster if and only if $\tau_{\mathrm{c}}(B)$ is a c-cluster; and
(iii) a subset X of cluster variables is a cluster if and only if $\tau(X)$ is a cluster.

We present a specific example for the rotation map on the positions in the word $\mathrm{Q_c}$, almost positive roots, and cluster variables below. For the computation of cluster variables in terms of a weighted quiver we refer the reader to [Kel12].

Example 2.5. Consider the Coxeter group $A_2 = \mathfrak{S}_3$, generated by the simple transpositions s_1, s_2 for $s_i := (i\ \ i+1)$, and the associated root system with simple roots α_1, α_2. Let $c = s_1 s_2$ be a Coxeter element and $\mathrm{Q_c} = (s_1, s_2, s_1, s_2, s_1)$ be the associated word. The rotation map on the positions in the word $\mathrm{Q_c}$, almost positive roots, and cluster variables is given by

$$
\begin{array}{ccc@{\qquad}ccc@{\qquad}ccc}
\tau_c : [m] & \longrightarrow & [m] & \tau_c : \Phi_{\geq -1} & \longrightarrow & \Phi_{\geq -1} & \tau : \{\text{var}\} & \longrightarrow & \{\text{var}\} \\
1 & \longmapsto & 3 & -\alpha_1 & \longmapsto & \alpha_1 & x_1 & \longmapsto & \frac{1+x_2}{x_1} \\
2 & \longmapsto & 4 & -\alpha_2 & \longmapsto & \alpha_1+\alpha_2 & x_2 & \longmapsto & \frac{1+x_1+x_2}{x_1x_2} \\
3 & \longmapsto & 5 & \alpha_1 & \longmapsto & \alpha_2 & \frac{1+x_2}{x_1} & \longmapsto & \frac{1+x_1}{x_2} \\
4 & \longmapsto & 1 & \alpha_1+\alpha_2 & \longmapsto & -\alpha_1 & \frac{1+x_1+x_2}{x_1x_2} & \longmapsto & x_1 \\
5 & \longmapsto & 2 & \alpha_2 & \longmapsto & -\alpha_2 & \frac{1+x_1}{x_2} & \longmapsto & x_2
\end{array}
$$

Remark 2.6. Let c be a bipartite Coxeter element, with sources corresponding to the positive vertices $(+)$ and sinks corresponding to the negative vertices $(-)$. Then, the rotation τ_c on the set of almost positive roots is the product of the maps $\tau_+, \tau_- : \Phi_{\geq -1} \to \Phi_{\geq -1}$ defined in [FZ03b, Section 2.2]. We refer the interested reader to that paper for the definitions of τ_+ and τ_-.

2.3. **Three descriptions of c-compatibility degrees.** In this section we introduce three notions of compatibility degrees on the set of cluster variables, almost positive roots, and positions in the word Q_c. We will see in Section 3 that these three notions coincide under the bijections of Section 2.1, and will use it to describe three different ways to compute d-vectors for cluster algebras of finite type. We refer again to Figure 1 for a summary of our notations in these three situations.

2.3.1. *On cluster variables.* Let $X = \{x_1, \dots, x_n\}$ be a set of cluster variables of $\mathcal{A}(W)$ forming a cluster, and let

$$y = \frac{F(x_1, \dots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}} \tag{2.1}$$

be a cluster variable of $\mathcal{A}(W)$ expressed in terms of the variables $\{x_1, \dots, x_n\}$ such that $F(x_1, \dots, x_n)$ is a polynomial which is not divisible by any variable x_j for $j \in [n]$. Recall that the d-vector of y with respect to X is $\mathbf{d}(X, y) = (d_1, \dots, d_n)$.

Lemma 2.7. *For cluster algebras of finite type, the i-th component of the d-vector $\mathbf{d}(X, y)$ is independent of the cluster X containing the cluster variable x_i.*

In view of this lemma, which is proven in Section 5.1, one can define a compatibility degree between two cluster variables as follows.

Definition 2.8 (Compatibility degree on cluster variables)**.** For any two cluster variables x and y, we denote by $d(x, y)$ the x-component of the d-vector $\mathbf{d}(X, y)$ for any cluster X containing the variable x. We refer to $d(x, y)$ as the *compatibility degree* of y with respect to x.

Observe that this compatibility degree is well defined for any pair of cluster variables x and y, since any cluster variable x of $\mathcal{A}(W)$ is contained in at least one cluster X of $\mathcal{A}(W)$.

2.3.2. *On almost positive roots.*

Definition 2.9 (c-compatibility degree on almost positive roots)**.** The *c-compatibility degree* on the set of almost positive roots is the unique function

$$\begin{array}{ccc} \Phi_{\geq -1} \times \Phi_{\geq -1} & \longrightarrow & \mathbb{Z} \\ (\alpha, \beta) & \longmapsto & (\alpha \parallel_c \beta) \end{array}$$

characterized by the following two properties:

$$(-\alpha_i \parallel_c \beta) = b_i, \qquad \text{for all } i \in [n] \text{ and } \beta = \sum b_i \alpha_i \in \Phi_{\geq -1}, \tag{2.2}$$

$$(\alpha \parallel_c \beta) = (\tau_c \alpha \parallel_c \tau_c \beta), \quad \text{for all } \alpha, \beta \in \Phi_{\geq -1}. \tag{2.3}$$

Remark 2.10. This definition is motivated by the classical compatibility degree defined by S. Fomin and A. Zelevinsky in [FZ03b, Section 3.1]. Namely, if c is a bipartite Coxeter element, then the c-compatibility degree $(\cdot \parallel_c \cdot)$ coincides with the compatibility degree $(\cdot \parallel \cdot)$ of [FZ03b, Section 3.1] except that $(\alpha \parallel_c \alpha) = -1$ while $(\alpha \parallel \alpha) = 0$ for any $\alpha \in \Phi_{\geq -1}$. Throughout this paper, we ignore this difference: we still call *classical compatibility degree*, and denote by $(\cdot \parallel \cdot)$, the c-compatibility degree for a bipartite Coxeter element c.

Remark 2.11. In [MRZ03], R. Marsh, M. Reineke, and A. Zelevinsky defined the c-compatibility degree for simply-laced types in a representation theoretic way, and extended this definition for arbitrary finite type by "folding" techniques. In [Rea07], N. Reading also used the similar notion of c-compatibility between almost positive roots. Namely, α and β are c-compatible when their compatibility degree vanishes. Here, we really need to know the value of the c-compatibility degree, and not only whether or not it vanishes.

Remark 2.12. Note that it is not immediately clear from the conditions in Definition 2.9 that the c-compatibility degree is well-defined. Uniqueness follows from the fact that the orbits of the negative roots under τ_c cover all almost positive roots. Existence is more involved and can be proved by representation theoretic arguments. Our interpretation in Theorem 3.1 below gives alternative direct definitions of c-compatibility, and in particular proves directly existence and uniqueness.

Remark 2.13. As observed by S. Fomin and A. Zelevinsky in [FZ03b, Proposition 3.3], if α, β are two almost positive roots and $\alpha^\vee, \beta^\vee$ are their dual roots in the dual root system, then $(\alpha \parallel \beta) = (\beta^\vee \parallel \alpha^\vee)$. Although not needed in this paper, we remark that this property also holds for c-compatibility degrees. This property is for example illustrated by the fact that the c-compatibility tables 4 and 6 of respective types B_2 and C_2 are transpose to each other.

2.3.3. *On positions in the word* $\mathrm{Q_c}$. In this section, we recall the notion of root functions associated to c-clusters in $\mathrm{Q_c}$, and use them in order to define a c-compatibility degree on the set of positions in $\mathrm{Q_c}$. This description relies only on linear algebra and is one of the main contributions of this paper. The root function was defined by C. Ceballos, J.-P. Labbé, and C. Stump in [CLS13, Definition 3.2] and was extensively used by V. Pilaud and C. Stump in the construction of Coxeter brick polytopes [PS11].

Definition 2.14 ([CLS13])**.** The *root function*

$$\mathsf{r}(I, \cdot) : [m] \longrightarrow \Phi$$

associated to a c-cluster $I \subseteq [m]$ in $\mathrm{Q_c}$ is defined by

$$\mathsf{r}(I,j) := \sigma^{\mathrm{c}}_{[j-1]\smallsetminus I}(\alpha_{q_j}),$$

where σ^{c}_X denotes the product of the reflections $q_x \in \mathrm{Q_c}$ for $x \in X$ in this order. The *root configuration* of I is the set $\mathsf{R}(I) := \{\mathsf{r}(I,i) \mid i \in I\}$ (Although the root configuration is a priori a multi-set for general subword complexes, it is indeed a set with no repeated elements for the particular choice of word $\mathrm{Q_c}$.)

As proved in [CLS13, Section 3.1], the root function $\mathsf{r}(I,\cdot)$ encodes exchanges in the c-cluster I. Namely, any $i \in I$ can be exchanged with the unique $j \notin I$ such that $\mathsf{r}(I,j) = \pm\mathsf{r}(I,j)$ (see Lemma 5.1), and the root function can be updated during this exchange (see Lemma 5.2). It was moreover shown in [PS11, Section 6] that the root configuration $\mathsf{R}(I)$ forms a basis for $\mathbb{R}^n$ for any given initial c-cluster I in $\mathrm{Q_c}$. It enables us to decompose any other root on this basis to get the following coefficients, which will play a central role in the remainder of the paper.

Definition 2.15 (c-compatibility degree on positions in $\mathrm{Q_c}$)**.** Fix any initial c-cluster $I \subseteq [m]$ of $\mathrm{Q_c}$. For any position $j \in [m]$, we decompose the root $\mathsf{r}(I,j)$ on the basis $\mathsf{R}(I)$ as follows:

$$\mathsf{r}(I,j) = \sum_{i \in I} \rho_i(j)\mathsf{r}(I,i).$$

For $i \in I$ and $j \in [m]$, we define the c-*compatibility degree* as the coefficient

$$\{i \,\|_{\mathrm{c}}\, j\} = \begin{cases} \rho_i(j) & \text{if } j > i, \\ -\rho_i(j) & \text{if } j \le i. \end{cases}$$

According to the following lemma, it is indeed valid to omit to mention the specific c-cluster I in which these coefficients are computed. We refer to Section 5.2 for the proof of Lemma 2.16.

Lemma 2.16. *The coefficients $\{i \,\|_{\mathrm{c}}\, j\}$ are independent of the choice of the* c*-cluster $I \subseteq [m]$ of $\mathrm{Q_c}$ containing i.*

Moreover, this compatibility degree is well defined for any pair of positions $i, j \in [m]$ of the word $\mathrm{Q_c}$, since for any i there is always a c-cluster I containing i.

3. Main results: Three descriptions of d-vectors

In this section we present the main results of this paper. We refer to Section 7.4 for explicit examples.

Theorem 3.1. *The three notions of compatibility degrees on the set of cluster variables, almost positive roots, and positions in the word $\mathrm{Q_c}$ coincide under the bijections of Section 2.1. More precisely, for every pair of positions i, j in the word $\mathrm{Q_c}$ we have*

$$d(\psi_{\mathrm{c}}(i), \psi_{\mathrm{c}}(j)) = (\vartheta_{\mathrm{c}}(i) \,\|_{\mathrm{c}}\, \vartheta_{\mathrm{c}}(j)) = \{i \,\|_{\mathrm{c}}\, j\}.$$

In particular, if c is a bipartite Coxeter element, then these coefficients coincide with the classical compatibility degrees of S. Fomin and A. Zelevinsky [FZ03b, Section 3.1] *(except for Remark 2.10).*

The proof of this theorem can be found in Section 6. The following three statements are the main results of this paper and are direct consequences of Theorem 3.1. The first statement describes the denominator vectors in terms of the compatibility degrees of [FZ03b, Section 3.1].

Corollary 3.2. *Let* $B \coloneqq \{\beta_1, \dots, \beta_n\} \subseteq \Phi_{\geq -1}$ *be a (classical) cluster in the sense of S. Fomin and A. Zelevinsky* [FZ03a, Theorem 1.9]*, and let* $\beta \in \Phi_{\geq -1}$ *be an almost positive root. Then the d-vector* $\mathbf{d}(B, \beta)$ *of the cluster variable* $\phi(\beta)$ *with respect to the initial cluster seed* $\phi(B) = \{\phi(\beta_1), \dots, \phi(\beta_n)\}$ *is given by*

$$\mathbf{d}(B, \beta) = \big((\beta_1 \parallel \beta), \dots, (\beta_n \parallel \beta) \big),$$

where $(\beta_i \parallel \beta)$ *is the compatibility degree of* β *with respect to* β_i *as defined by S. Fomin and A. Zelevinsky* [FZ03b, Section 3.1] *(except for Remark 2.10).*

The next statement extends this result to any Coxeter element c of W.

Corollary 3.3. *Let* $B \coloneqq \{\beta_1, \dots, \beta_n\} \subseteq \Phi_{\geq -1}$ *be a c-cluster in the sense of N. Reading* [Rea07, Section 7]*, and let* $\beta \in \Phi_{\geq -1}$ *be an almost positive root. Then the d-vector* $\mathbf{d}_c(B, \beta)$ *of the cluster variable* $\phi_c(\beta)$ *with respect to the initial cluster seed* $\phi_c(B) = \{\phi_c(\beta_1), \dots, \phi_c(\beta_n)\}$ *is given by*

$$\mathbf{d}_c(B, \beta) = \big((\beta_1 \parallel_c \beta), \dots, (\beta_n \parallel_c \beta) \big),$$

where $(\beta_i \parallel_c \beta)$ *is the c-compatibility degree of* β *with respect to* β_i *as defined in Definition 2.9.*

Finally, the third statement describes the denominator vectors in terms of the coefficients $\{i \parallel_{\mathrm{c}} j\}$ obtained from the word Q_{c}.

Corollary 3.4. *Let* $I \subseteq [m]$ *be a* c*-cluster and* $j \in [m]$ *be a position in* Q_{c}*. Then the d-vector* $\mathbf{d}_{\mathrm{c}}(I, j)$ *of the cluster variable* $\psi_{\mathrm{c}}(j)$ *with respect to the initial cluster seed* $\psi_{\mathrm{c}}(I) = \{\psi_{\mathrm{c}}(i) \mid i \in I\}$ *is given by*

$$\mathbf{d}_{\mathrm{c}}(I, j) = \big(\{i \parallel_{\mathrm{c}} j\} \big)_{i \in I}.$$

As a consequence, we obtain the following result which is proven in Section 6.4.

Corollary 3.5. *For cluster algebras of finite type, the d-vector of a cluster variable that is not in the initial seed is non-negative and not equal to zero.*

This corollary was conjectured by S. Fomin and A. Zelevinsky for arbitrary cluster algebras [FZ07, Conjecture 7.4]. In the case of cluster algebras of finite type, this conjecture also follows from [CCS06, Theorem 4.4 and Remark 4.5] and from [BMR07, Theorem 2.2], where the authors show that the d-vectors can be computed as the dimension vectors of certain indecomposable modules.

4. Cluster variables and orbits of positions in the bi-infinite word $\widetilde{\mathrm{Q}}$

We now want to present the results of this paper in terms of orbits of letters of the bi-infinite word $\widetilde{\mathrm{Q}} \coloneqq (\widetilde{\mathrm{q}}_i)_{i \in \mathbb{Z}}$ obtained by infinitely many repetitions of the product of all generators in S. Up to commutations of consecutive commuting letters, this word does not depend on the order of the elements of S in this product. Our motivation is to avoid the dependence in the Coxeter element c, which is just a technical tool to deal with clusters in terms of almost positive roots or positions in the word Q_{c}. This point of view was already considered in type A by V. Pilaud and M. Pocchiola in [PP12] in their study of pseudoline arrangements on the Möbius strip. For arbitrary finite Coxeter groups, the good formalism is given by the Auslander-Reiten quiver (see *e.g.* [CLS13, Section 8]). However, we do not really need this formalism here and skip its presentation. The proofs of this section are omitted but can be easily deduced from the results proved in Section 6 (more precisely from Lemmas 6.3, 6.4 and 6.6) and Corollary 3.4.

We denote by $\eta : S \to S$ the involution $\eta(s) = w_\circ s w_\circ$ which conjugates a simple reflection by the longest element $w_\circ$ of W, and by $\eta(\mathrm{Q_c})$ the word obtained by conjugating each letter of $\mathrm{Q_c}$ by $w_\circ$. As observed in [CLS13], the bi-infinite word $\widetilde{\mathrm{Q}}$ coincides, up to commutations of consecutive commuting letters, with the bi-infinite word $\cdots \mathrm{Q_c}\,\eta(\mathrm{Q_c})\,\mathrm{Q_c}\,\eta(\mathrm{Q_c})\,\mathrm{Q_c} \cdots$ obtained by repeating infinitely many copies of $\mathrm{Q_c}\,\eta(\mathrm{Q_c})$. We can therefore consider the word $\mathrm{Q_c}$ as a fundamental domain in the bi-infinite word $\widetilde{\mathrm{Q}}$, and a position i in $\mathrm{Q_c}$ as a representative of an orbit $\tilde{i}$ of positions in $\widetilde{\mathrm{Q}}$ under the translation map $\tau : i \mapsto i + m$ (note that this translation maps the letter q_i to the conjugate letter $q_{i+m} = w_\circ q_i w_\circ$). For a subset I of positions in $\mathrm{Q_c}$, we denote by $\widetilde{I} := \{\tilde{i} \mid i \in I\}$ its corresponding orbit in $\widetilde{\mathrm{Q}}$. It turns out that the orbits of c-clusters are now independent of c.

Proposition 4.1. *Let* c *and* c′ *be reduced expressions of two Coxeter elements. Let I and I' be subsets of positions in $\mathrm{Q_c}$ and $\mathrm{Q_{c'}}$ respectively such that their orbits $\widetilde{I}$ and $\widetilde{I'}$ in $\widetilde{\mathrm{Q}}$ coincide. Then I is a* c*-cluster in $\mathrm{Q_c}$ if and only if I' is a* c′*-cluster of $\mathrm{Q_{c'}}$. We then say that $\widetilde{I} = \widetilde{I'}$ forms a cluster in $\widetilde{\mathrm{Q}}$.*

In other words, we obtain a bijection Ψ from the orbits of positions in $\widetilde{\mathrm{Q}}$ (under the translation map $\tau : i \mapsto i + m$) to the cluster variables of $\mathcal{A}(W)$. A collection of orbits forms a cluster if and only if their representatives in any (or equivalently all) fundamental domain $\mathrm{Q_c}$ for τ form the complement of a reduced expression for $w_\circ$. Choosing a particular Coxeter element c defines a specific fundamental domain $\mathrm{Q_c}$ in $\widetilde{\mathrm{Q}}$, which provides specific bijections ϑ_c and ϕ_c with almost positive roots and cluster variables. We insist on the fact that Ψ does not depend on the choice of a Coxeter element, while ϑ_c and ϕ_c do.

We now want to describe the results of this paper directly on the bi-infinite word $\widetilde{\mathrm{Q}}$. We first transport the d-vectors through the bijection Ψ: for a given initial cluster seed $\widetilde{I}$ in $\widetilde{\mathrm{Q}}$, and an orbit $\tilde{j}$ of positions in $\widetilde{\mathrm{Q}}$, we define the d-vector

$$\mathbf{d}(\widetilde{I}, \tilde{j}) := \mathbf{d}\big(\Psi(\widetilde{I}), \Psi(\tilde{j})\big).$$

We want to express these d-vectors in terms of the coefficients $\{i \,\|_\mathrm{c}\, j\}$ from Definition 2.15. For this, we first check that these coefficients are independent of the fundamental domain $\mathrm{Q_c}$ on which they are computed.

Proposition 4.2. *Let* c *and* c′ *be reduced expressions of two Coxeter elements. Let i, j be positions in $\mathrm{Q_c}$ and i', j' be positions in $\mathrm{Q_{c'}}$ be such $\tilde{i} = \tilde{i}'$ and $\tilde{j} = \tilde{j}'$. Then the coefficients $\{i \,\|_\mathrm{c}\, j\}$ and $\{i' \,\|_{\mathrm{c'}}\, j'\}$, computed in $\mathrm{Q_c}$ and $\mathrm{Q_{c'}}$ respectively, coincide.*

For any orbits $\tilde{i}$ and $\tilde{j}$ of $\widetilde{\mathrm{Q}}$, we can therefore define with no ambiguity the coefficient $\{\tilde{i} \,\|\, \tilde{j}\}$ to be the coefficient $\{i \,\|_\mathrm{c}\, j\}$ for i and j representatives of $\tilde{i}$ and $\tilde{j}$ in any arbitrary fundamental domain $\mathrm{Q_c}$. The d-vectors of the cluster algebra can then be expressed from these coefficients.

Theorem 4.3. *Let $\widetilde{I}$ be a collection of orbits of positions in $\widetilde{\mathrm{Q}}$ forming a cluster in $\widetilde{\mathrm{Q}}$, and let $\tilde{j}$ be an orbit of positions in $\widetilde{\mathrm{Q}}$. Then the d-vector $\mathbf{d}(\widetilde{I}, \tilde{j})$ of the cluster* *variable $\Psi(\tilde{j})$ with respect to the initial cluster seed $\Psi(\widetilde{I})$ is given by*

$$\mathbf{d}(\widetilde{I}, \tilde{j}) = (\{\tilde{i} \parallel \tilde{j}\})_{\tilde{i} \in \widetilde{I}}.$$

5. Proofs of Lemma 2.7 and Lemma 2.16

5.1. **Proof of Lemma 2.7.** For cluster algebras of finite type all clusters containing the cluster variable x_i are connected under mutations. Therefore, it is enough to prove the lemma for the cluster $X' = \{x_1, \dots, x_{j-1}, x'_j, x_{j+1}, \dots, x_n\}$ obtained from $X = \{x_1, \dots, x_n\}$ by mutating a variable x_j with $j \neq i$. The variables x_j and x'_j satisfy a relation

$$x_j = \frac{P(x_1, \dots, \widehat{x}_j, \dots, x_n)}{x'_j}$$

for a polynomial P in the variables $\{x_1, \dots, \widehat{x}_j, \dots, x_n\}$, where $\widehat{x}_j$ means that we skip variable x_j. Replacing x_j in equation (2.1) we obtain

$$y = \frac{\widetilde{P}(x_1, \dots, x'_j, \dots, , x_n)/{x'_j}^{m_j}}{x_1^{d_1} \cdots \widehat{x_j^{d_j}} \cdots x_n^{d_n} P(x_1, \dots, \widehat{x}_j, \dots, x_n)/{x'_j}^{d_j}}$$

where m_j is a non-negative integer and $\widetilde{P}$ is a polynomial which is not divisible by x'_j and by any x_ℓ with $\ell \neq j$. The Laurent phenomenon implies that P divides $\widetilde{P}$. Thus , we obtain

$$y = \frac{\widetilde{F}(x_1, \dots, x'_j, \dots, x_n)}{x_1^{d_1} \cdots \widehat{x_j^{d_j}} \cdots x_n^{d_n}} \cdot \frac{{x'_j}^{d_j}}{{x'_j}^{m_j}}$$

is the rational expression of the cluster variable y expressed in terms of the variables of X'. As a consequence, the d-vectors $\mathbf{d}(X, y)$ and $\mathbf{d}(X', y)$ differ only in the j-th coordinate. In particular, the i-th coordinate remains constant after mutation of any $j \neq i$ as desired.

5.2. **Proof of Lemma 2.16.** Before proving this lemma we need some preliminaries on subword complexes. Recall that $I \subset [m]$ is a c-cluster of $\mathsf{Q_c}$ if and only if the subword of $\mathsf{Q_c}$ with positions at the complement of I is a reduced expression of $w_\circ$ (see Theorem 2.1).

Lemma 5.1 ([CLS13, Lemma 3.3])**.** *Let $I \subset [m]$ be a* c*-cluster of $\mathsf{Q_c}$. Then,*

(i) For every $i \in I$ there exist a unique $j \notin I$ such that $I \triangle \{i, j\}$ is again a c*-cluster, where $A \triangle B := (A \cup B) \smallsetminus (A \cap B)$ denotes the symmetric difference.*
(ii) This j is the unique $j \notin I$ satisfying $\mathsf{r}(I, j) = \pm \mathsf{r}(I, i)$.

This exchange operation between c-clusters is called *flip*. It correspond to mutations between clusters in the cluster algebra. During the flip, the root function is updated as follows.

Lemma 5.2 ([CLS13, Lemma 3.6])**.** *Let I and J be two adjacent* c*-clusters of $\mathsf{Q_c}$ with $I \smallsetminus i = J \smallsetminus j$, and assume that $i < j$. Then, for every $k \in [m]$,*

$$\mathsf{r}(I', k) = \begin{cases} t_i(\mathsf{r}(I, k)) & \text{if } i < k \leq j, \\ \mathsf{r}(I, k) & \text{otherwise} \end{cases}$$

Here, $t_i = w q_i w^{-1}$ where w is the product of the reflections $q_x \in \mathsf{Q_c}$ for $x \in [i-1] \smallsetminus I$. By construction, t_i is the reflection in W orthogonal to the root $\mathsf{r}(I, i) = w(\alpha_{q_i})$.

This result implies the following Lemma.

Lemma 5.3. *Let I and J be two adjacent* c*-clusters of* $\mathrm{Q_c}$ *with* $I \smallsetminus i = J \smallsetminus j$*. Then, for every* $k \in [m]$*,*

$$\mathsf{r}(J,k) = \mathsf{r}(I,k) + a_k \mathsf{r}(I,i)$$

for some constant $a_k \in \mathbb{R}$*.*

Using the previous lemma, we now derive Lemma 2.16.

Proof of Lemma 2.16. This result is equivalent to prove that the coefficients $\rho_i(j)$ from Definition 2.15 are independent of the c-cluster $I \subset [m]$ of $\mathrm{Q_c}$ containing i. Since all the c-clusters containing i are connected by flips, it is enough to prove that $\rho_i(j)$ is preserved by any flip not involving i. Let $i \in I$ and let $i' \in I \smallsetminus i$. Then, i' can be exchanged with a unique $j' \notin I$ such that $I' = I \triangle \{i', j'\}$ is again a c-cluster. By Lemma 5.3 and part (ii) of Lemma 5.1,

$$\mathsf{r}(I,j) = \sum_{\ell \in I} \rho_\ell(j) \mathsf{r}(I,\ell).$$

implies

$$\mathsf{r}(I',j) = \sum_{\ell \in I \smallsetminus i'} \rho_\ell(j) \mathsf{r}(I',\ell) + a\, \mathsf{r}(I,j')$$

for some constant $a \in \mathbb{R}$. In particular, this implies that the coefficients $\rho_i(j)$ are the same for I and I'. □

6. Proof of Theorem 3.1

Our proof of Theorem 3.1 is based in Proposition 6.2 and Proposition 6.7. These propositions are stated and proved in Section 6.1 and Section 6.2 respectively.

6.1. The map $d(\cdot,\cdot)$ satisfies relations (2.2) and (2.3).

In this section we show that the map $d(\cdot,\cdot)$ induces a map on the set of almost positive roots which satisfy the properties (2.2) and (2.3) in the definition of the c-compatibility degree among almost positive roots (Definition 2.9). Before stating this result in Proposition 6.2 we need the following lemma.

Lemma 6.1. *Let $X = \{x_1, \dots, x_n\}$ be a cluster and let y be a cluster variable of $\mathcal{A}$. Then the rational function of y expressed in terms of the variables in X is exactly the same as the rational function of τy in terms of the variables in τX. In particular,*

$$\mathbf{d}(X,y) = \mathbf{d}(\tau X, \tau y).$$

Proof. We prove this proposition in two parts: first in the case where the cluster seed X corresponds to a Coxeter element c, and then for any arbitrary cluster seed.

Let X_c be the cluster corresponding to a Coxeter element c, *i.e.* the set of variables on the vertices of the weighted quiver $\mathcal{Q}_c$ corresponding to c. By definition of the rotation map on the set of cluster variables, the rotated cluster τX_c consists of the variables on the vertices of the quiver obtained from $\mathcal{Q}_c$ by consecutively applying the mutations $\mu_{c_1} \to \dots \to \mu_{c_n}$. The resulting underlying quiver $\mathcal{Q}'$ after these mutations is exactly equal to $\mathcal{Q}_c$. Moreover, every sequence of mutations giving rise to a variable y, which starts at the cluster seed $(\mathcal{Q}_c, X_c)$ can be viewed as a sequence of mutations giving rise to τy starting at the rotated seed cluster $(\mathcal{Q}', \tau X_c)$.

Since the quivers $\mathcal{Q}_c$ and $\mathcal{Q}'$ coincide, the rational functions for y and τy in terms of the variables of X_c and τX_c respectively are exactly the same as desired.

In the general case where X is an arbitrary cluster seed we proceed as follows. For any cluster variable y and any clusters X and Y, denote by $y(X)$ the rational function of y in terms of the variables of the cluster X, and by $Y(X)$ the rational functions of the variables of Y with respect to the variables of the cluster X. Then, using the fact $y(X) = y(Y) \circ Y(X)$ and the first part of this proof we obtain

$$y(X) = y(X_c) \circ X_c(X) = \tau y(\tau X_c) \circ \tau X_c(\tau X) = \tau y(\tau X)$$

as desired. □

As a direct consequence of this lemma we obtain the following key result.

Proposition 6.2. *Let c be a Coxeter element. For every position i in the word $\mathrm{Q_c}$ denote by $x_i = \psi_{\mathrm{c}}(i)$ the associated cluster variable. Then, the values $d(x_i, x_j)$ satisfy the following two properties:*

$$d(x_i, x_j) = b_i, \qquad \textit{for all } i \in [n] \textit{ and } \vartheta_{\mathrm{c}}(j) = \sum b_i \alpha_i \in \Phi_{\geq -1}, \tag{6.1}$$

$$d(x_i, x_j) = d(\tau x_i, \tau x_j), \quad \textit{for all } i, j \in [m]. \tag{6.2}$$

Proof. Let X_c be the set of cluster variables corresponding to the first n positions of the word $\mathrm{Q_c}$, or equivalently, the cluster associated to the weighted quiver $\mathcal{Q}_c$ corresponding to the Coxeter element c. As discussed in Section 2.1, the d-vector of the variable $x_j = \phi_c(\vartheta_{\mathrm{c}}(j))$ in terms of this initial quiver $\mathcal{Q}_c$ is given by the almost positive root $\vartheta_{\mathrm{c}}(j)$. More precisely, we have

$$\mathbf{d}(X_c, x_j) = (b_1, \dots, b_n).$$

This implies relation (6.1) of the proposition. Relation (6.2) follows directly from Lemma 6.1. □

6.2. **The map $\{\cdot \,\|_{\mathrm{c}}\, \cdot\}$ satisfies relations (2.2) and (2.3).** In this section we show that the map $\{\cdot \,\|_{\mathrm{c}}\, \cdot\}$ induces a map on the set of almost positive roots which satisfy the properties (2.2) and (2.3) in the definition of the c-compatibility degree among almost positive roots (Definition 2.9). Before stating this result in Proposition 6.7 we need some preliminaries concerning the coefficients $\{i \,\|_{\mathrm{c}}\, j\}$.

For this, we use an operation of jumping letters between the words $\mathrm{Q_c}$ defined in [CLS13, Section 3.2]. We denote by $\eta : S \to S$ the involution $\eta(s) = w_\circ s w_\circ$ which conjugates a simple reflection by the longest element $w_\circ$ of W. Given a word $\mathrm{Q} := (q_1, q_2, \cdots, q_r)$, we say that the word $(q_2, \cdots, q_r, \eta(q_1))$ is the *jumping word* of Q, or is obtained by *jumping* the first letter in Q. In the following three lemmas, we consider a reduced expression $\mathrm{c} := (s, c_2, \dots, c_n)$ of a Coxeter element c, and the reduced expression $\mathrm{c}' := (c_2, \dots, c_n, s)$ of the Coxeter element $c' := scs$ obtained by deleting the letter s in c and putting it at the end.

Lemma 6.3 ([CLS13, Proposition 4.3])**.** *The jumping word of $\mathrm{Q_c}$ coincides with the word $\mathrm{Q_{c'}}$ up to commutations of consecutive commuting letters.*

Let σ denote the map from positions in $\mathrm{Q_c}$ to positions in $\mathrm{Q_{c'}}$ which jumps the first letter and reorders the letters (by commutations).

Lemma 6.4 ([CLS13, Proposition 3.9])**.** *A subset $I \subset [m]$ of positions in the word $\mathrm{Q_c}$ is a c-cluster if and only if $\sigma(I)$ is a c'-cluster in $\mathrm{Q_{c'}}$.*

We include a short proof of this lemma here for convenience of the reader.

Proof. Let P be the subword of $\mathrm{Q_c}$ with positions at the complement of I in $[m]$, and let P' be the subword of $\mathrm{Q_{c'}}$ with positions at the complement of $\sigma(I)$ in $[m]$. Recall from Theorem 2.1 that I is a c-cluster in $\mathrm{Q_c}$ if and only if P is a reduced expression of $w_\circ$. If $1 \in I$, then P and P' are the same. If $1 \notin I$, then P' is the jumping word of P. In both cases P is a reduced expression of $w_\circ$ if and only if P' is a reduced expression of $w_\circ$. Therefore, I is a c-cluster in $\mathrm{Q_c}$ if and only if $\sigma(I)$ is a c′-cluster in $\mathrm{Q_{c'}}$. □

Observe now that we obtain again c when we jump repeatedly all its letters. However, a position i in $\mathrm{Q_c}$ is rotated to the position $\tau_{\mathrm{c}}^{-1}(i)$ by these operations. This implies the following statement.

Corollary 6.5. *A subset $I \subset [m]$ of positions in the word $\mathrm{Q_c}$ is a* c*-cluster if and only if $\tau_{\mathrm{c}}(I)$ is a* c*-cluster.*

Using the jumping operation studied above, we now derive the following results concerning the coefficients $\{i \,\|_{\mathrm{c}}\, j\}$.

Lemma 6.6. *If $i' := \sigma(i)$ and $j' := \sigma(j)$ denote the positions in $\mathrm{Q_{c'}}$ corresponding to positions i and j in $\mathrm{Q_c}$ after jumping the letter s, then*

$$\{i' \,\|_{\mathrm{c'}}\, j'\} = \{i \,\|_{\mathrm{c}}\, j\}.$$

Proof. Let I be a c-cluster and k be a position in $\mathrm{Q_c}$. We denote by $I' := \sigma(I)$ the c′-cluster corresponding to I and by $k' := \sigma(k)$ the position in $\mathrm{Q_{c'}}$ corresponding to k after jumping the letter s. By the definition of the root function we obtain that

$$\mathsf{r}(I', k') = \begin{cases} -\mathsf{r}(I,k) & \text{if } k = 1 \in I, \\ \mathsf{r}(I,k) & \text{if } k = 1 \notin I, \\ \mathsf{r}(I,k) & \text{if } k \neq 1 \in I, \\ s\mathsf{r}(I,k) & \text{if } k \neq 1 \notin I, \end{cases}$$

Applying this relation to a c-cluster I containing i and a position $j \neq i$, we derive that

$$\rho_{i'}(j') = \begin{cases} -\rho_i(j) & \text{if } i = 1 \text{ or } j = 1, \\ \rho_i(j) & \text{otherwise,} \end{cases}$$

where $\rho_i(j)$ denotes the ith coordinate of $\mathsf{r}(I,j)$ in the linear basis $\mathsf{R}(I)$ and $\rho_{i'}(j')$ denotes the i'th coordinate of $\mathsf{r}(I',j')$ in the linear basis $\mathsf{R}(I')$. This implies that $\{i' \,\|_{\mathrm{c'}}\, j'\} = \{i \,\|_{\mathrm{c}}\, j\}$ by definition of $\{\cdot \,\|_{\mathrm{c}}\, \cdot\}$. □

Proposition 6.7. *Let c be a Coxeter element and i and j be positions in the word $\mathrm{Q_c}$. Then, the coefficients $\{i \,\|_{\mathrm{c}}\, j\}$ satisfy the following two properties:*

$$\{i \,\|_{\mathrm{c}}\, j\} = b_i, \qquad \text{for all } i \in [n] \text{ and } \vartheta_{\mathrm{c}}(j) = \sum b_i \alpha_i \in \Phi_{\geq -1}, \tag{6.3}$$

$$\{i \,\|_{\mathrm{c}}\, j\} = \{\tau_{\mathrm{c}} i \,\|_{\mathrm{c}}\, \tau_{\mathrm{c}} j\}, \quad \text{for all } i,j \in [m]. \tag{6.4}$$

Proof. Let $I_{\mathrm{c}} = [n]$ be the c-cluster given by the first n positions in the word $\mathrm{Q_c}$. Then

$$\mathsf{r}(I_{\mathrm{c}}, j) = \begin{cases} \alpha_{c_j} & \text{if } 1 \leq j \leq n, \\ \vartheta_{\mathrm{c}}(j) & \text{if } n < j \leq m. \end{cases}$$

Therefore, by definition of the coefficients $\{i \,\|_{\mathrm{c}}\, j\}$ we have that the vector $(\{i \,\|_{\mathrm{c}}\, j\})_{i \in I_{\mathrm{c}}}$ is given by the almost positive root $\vartheta_{\mathrm{c}}(j)$. More precisely,

$$(\{i \,\|_{\mathrm{c}}\, j\})_{i \in I_{\mathrm{c}}} = (b_1, \dots, b_n).$$

This implies relation (6.3) of the proposition. Relation (6.4) follows from jumping repeatedly all the letters of c in $\mathrm{Q_c}$. Thus, position i in $\mathrm{Q_c}$ is rotated to position $\tau_{\mathrm{c}}^{-1}(i)$ by these operations, and the result follows from n applications of Lemma 6.6. □

6.3. **Proof of Theorem 3.1.** By Definition 2.9, Proposition 6.2 and Proposition 6.7, the three maps $d(\psi_{\mathrm{c}}(\cdot), \psi_{\mathrm{c}}(\cdot))$, $(\vartheta_{\mathrm{c}}(\cdot) \,\|_{c}\, \vartheta_{\mathrm{c}}(\cdot))$ and $\{\cdot \,\|_{\mathrm{c}}\, \cdot\}$ on the set of positions in the word $\mathrm{Q_c}$ satisfy the two properties (6.3) and (6.4) of Proposition 6.7. Since these properties uniquely determine a map on the positions in $\mathrm{Q_c}$ the result follows.

6.4. **Proof of Corollary 3.5.** In this section we present two independent proofs of Corollary 3.5. The first proof follows from the description of d-vectors in terms of c-compatibility degrees $(\alpha \,\|_{c}\, \beta)$ between almost positive roots as presented in Corollaries 3.2 and 3.3. The second proof is based on the description of d-vectors in terms of the coefficients $\{i \,\|_{\mathrm{c}}\, j\}$ obtained from the word $\mathrm{Q_c}$ as presented in Corollary 3.4. Our motivation for including this second proof here is to extend several results of this paper to the family of "root-independent subword complexes". This family of simplicial complexes, defined in [PS11], contains all cluster complexes of finite type. Using the brick polytope approach [PS12, PS11], V. Pilaud and C. Stump constructed polytopal realizations of these simplicial complexes. Extending the results of this paper to root-independent subword complexes might lead to different polytopal realizations of these simplicial complexes.

6.4.1. *First proof.* Corollary 3.5 follows from the following known fact.

Lemma 6.8 ([FZ03b, MRZ03, Rea07])**.** *The compatibility degree $(\alpha \,\|_{c}\, \beta)$ is non-negative for any pair of almost positive roots $\alpha \neq \beta$. Moreover, $(\alpha \,\|_{c}\, \beta) = 0$ if and only if α and β are c-compatible, i.e. if they belong to some c-cluster.*

Let $B = \{\beta_1, \dots, \beta_n\} \subset \Phi_{\geq -1}$ be an initial c-cluster seed and $\beta \in \Phi_{\geq -1}$ be an almost positive root which is not in B. Then, by Lemma 6.8 there is at least one $i \in [n]$ such that $(\beta_i \,\|_{c}\, \beta) > 0$, otherwise β would be c-compatible to with all β_i which is a contradiction. Corollary 3.5 thus follows from Corollary 3.3 and Lemma 6.8.

6.4.2. *Second proof.* Corollary 3.5 follows from the next statement.

Lemma 6.9. *The coefficient $\{i \,\|_{\mathrm{c}}\, j\}$ is non-negative for any pair of positions $i \neq j$ in the word $\mathrm{Q_c}$. Moreover, $\{i \,\|_{\mathrm{c}}\, j\} = 0$ if and only if i and j are compatible, i.e. if they belong to some* c*-cluster.*

Proof. The non-negativity is clear if i is one of the first n letters. Indeed, computing in the c-cluster I given by the initial prefix c of the word $\mathrm{Q_c}$, the root configuration $\mathsf{R}(I)$ is the linear basis of simple roots, and the coefficients $\{i \,\|_{\mathrm{c}}\, j\}$ are the coefficients of $\vartheta_{\mathrm{c}}(j)$, which is an almost positive root. The non-negativity for an arbitrary position i thus follows from Corollary 6.7 since the orbit of the initial prefix c under the rotation τ_{c} cover all positions in $\mathrm{Q_c}$.

For the second part of the lemma observe that if i and j belong to some c-cluster I then the coefficient $\{i \,\|_{\mathrm{c}}\, j\}$ computed in terms of this cluster is clearly equal to zero. Moreover, by Proposition 6.7 the coefficient $\{i \,\|_{\mathrm{c}}\, j\} = 0$ if and only if $\{\tau_{\mathrm{c}} i \,\|_{\mathrm{c}}\, \tau_{\mathrm{c}} j\} = 0$, and by Corollary 6.5 i and j belong to some c-cluster if and only if $\tau_{\mathrm{c}} i$ and $\tau_{\mathrm{c}} j$ belong to some c-cluster. Therefore, it is enough to prove the result in the case when $i \in [n]$ belongs to the first n positions in $\mathrm{Q_c}$. The result in this case can be deduced from [CLS13, Theorem 5.1]. This theorem states that if $i \in [n]$, then i and j belong to some c-cluster if and only if $\vartheta_{\mathrm{c}}(j) \in (\Phi_{\langle c_i \rangle})_{\geq -1}$ is an almost positive root of the parabolic root system that does not contain the root α_{c_i}. Since for $i \in [n]$ the coefficient $\{i \,\|_{\mathrm{c}}\, j\}$ is the coefficient of the root α_{c_i} in the almost positive root $\vartheta_{\mathrm{c}}(j)$, the result immediately follows. □

7. Geometric interpretations in types A, B, C and D

In this section, we present geometric interpretations for the classical types A, B, C, and D of the objects discussed in this paper: cluster variables, clusters, mutations, compatibility degrees, and d-vectors. These interpretations are classical in types A, B and C when the initial cluster seed corresponds to a bipartite Coxeter element, and can already be found in [FZ03b, Section 3.5] and [FZ03a, Section 12]. In contrast, our interpretation in type D slightly differs from that of S. Fomin and A. Zelevinsky since we prefer to use pseudotriangulations (we motivate this choice in Remark 7.1). Moreover, these interpretations are extended here to any initial cluster seed, acyclic or not.

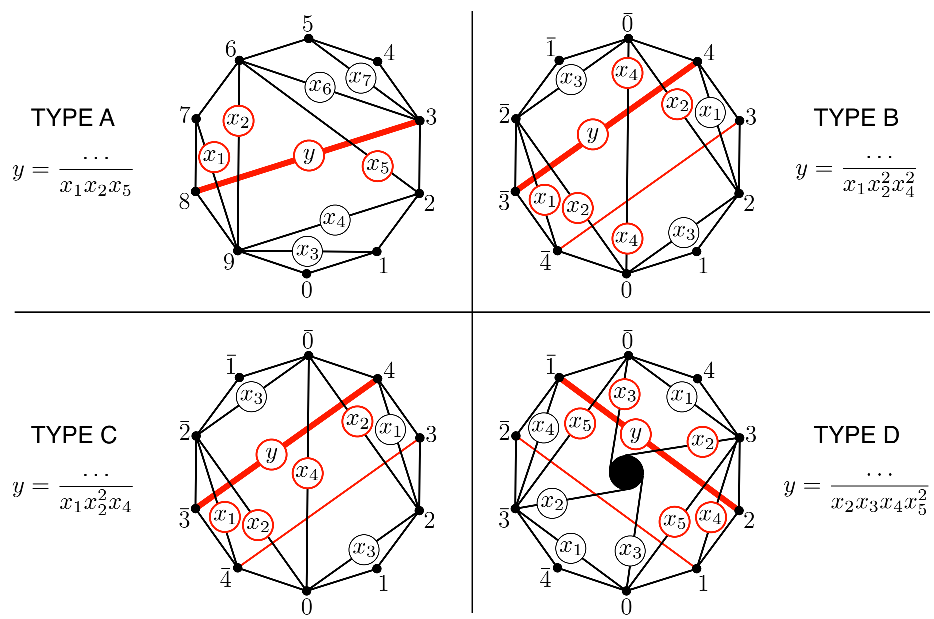


Figure 2. Illustration of the geometric interpretations of denominator vectors in cluster algebras of classical types A, B, C and D.

In Section 7.1, we associate to each classical finite type a geometric configuration and observe a correspondence between:

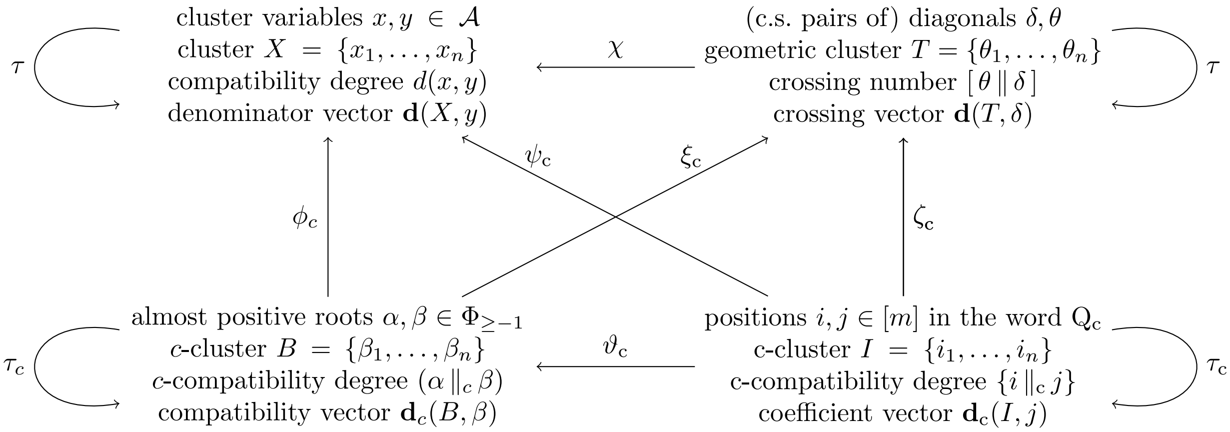


FIGURE 3. Four different contexts for cluster algebras of classical finite types, their different notions of compatibility degrees, and the bijections between them.

(i) cluster variables and *diagonals* (or centrally symmetric pairs of diagonals) in the geometric picture;
(ii) clusters and *geometric clusters*: triangulations in type A, centrally symmetric triangulations in types B and C, and centrally symmetric pseudotriangulations in type D (*i.e.* maximal crossing-free sets of centrally symmetric pairs of chords in the geometric picture);
(iii) cluster mutations and *geometric flips* (we can also express geometrically the exchange relations on cluster variables);
(iv) compatibility degrees and *crossing numbers* of (c.s. pairs of) diagonals;
(v) d-vectors and *crossing vectors* of (c.s. pairs of) diagonals.

These interpretations can also be extended to read quivers in the geometric pictures and to compute cluster variables in terms of perfect matchings of weighted graphs. We focus in this paper on the d-vectors, and will study these extensions in a forthcoming paper.

We denote by χ the bijection from (c.s. pairs of) diagonals to cluster variables. Composing this bijection χ with the the bijections from Section 2.1 gives bijections between the four contexts studied in this paper: cluster variables, (c.s. pairs of) diagonals, almost positive roots, and positions in $\mathrm{Q_c}$. See Figure 3. All these bijections preserve clusters, mutations, compatiblity degrees, and d-vectors. Observe again that the two combinatorial descriptions of clusters and compatibility degrees in terms of almost positive roots and positions in $\mathrm{Q_c}$ both depend on the choice of a Coxeter element c, while the algebraic and geometric descriptions do not.

In Section 7.2, we give a direct description of the map $\zeta_{\mathrm{c}} := \chi^{-1} \circ \psi_{\mathrm{c}}$ from positions in the word $\mathrm{Q_c}$ to (c.s. pairs of) diagonals in the geometric picture. This yields a simple geometric characterization of the reduced expressions of $w_\circ$ in the word $\mathrm{Q_c}$. Using this bijection, we can also explicitly describe the bijection $\xi_{\mathrm{c}} = \chi^{-1} \circ \phi_c = \zeta_{\mathrm{c}} \circ \vartheta_{\mathrm{c}}^{-1}$ from almost positive roots to (c.s. pairs of) diagonals in the geometric picture. This yields a simple geometric characterization of c-cluster complexes for any Coxeter element c, which we present in Section 7.3.

Explicit examples, which illustrate how to compute all d-vectors with respect to a non-acyclic cluster seed, are given in Section 7.4.

7.1. **Denominator vectors and crossing vectors.**

7.1.1. *Type A_n.* Consider the Coxeter group $A_n = \mathfrak{S}_{n+1}$, generated by the simple transpositions $\tau_i := (i\ i+1)$, for $i \in [n]$. The corresponding geometric picture is a convex regular $(n+3)$-gon. Cluster variables, clusters, exchange relations, compatiblity degrees, and d-vectors in the cluster algebra $\mathcal{A}(A_n)$ can be interpreted geometrically as follows:

(i) Cluster variables correspond to *(internal) diagonals* of the $(n+3)$-gon. We denote by $\chi(\delta)$ the cluster variable corresponding to a diagonal δ.
(ii) Clusters correspond to *triangulations* of the $(n+3)$-gon.
(iii) Cluster mutations correspond to *flips* between triangulations. See Figure 4.

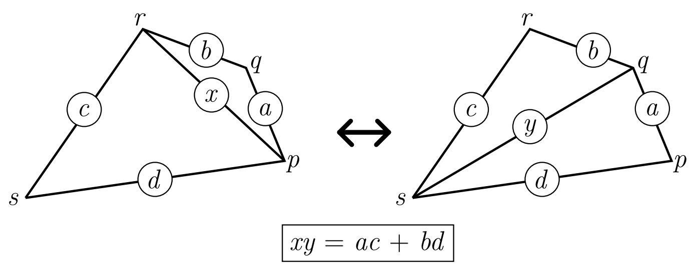


FIGURE 4. Flip in type A.

Moreover, flipping diagonal $[p, r]$ to diagonal $[q, s]$ in a quadrilateral $\{p, q, r, s\}$ results in the exchange relation

$$\chi([p, r]) \cdot \chi([q, s]) = \chi([p, q]) \cdot \chi([r, s]) + \chi([p, s]) \cdot \chi([q, r]).$$

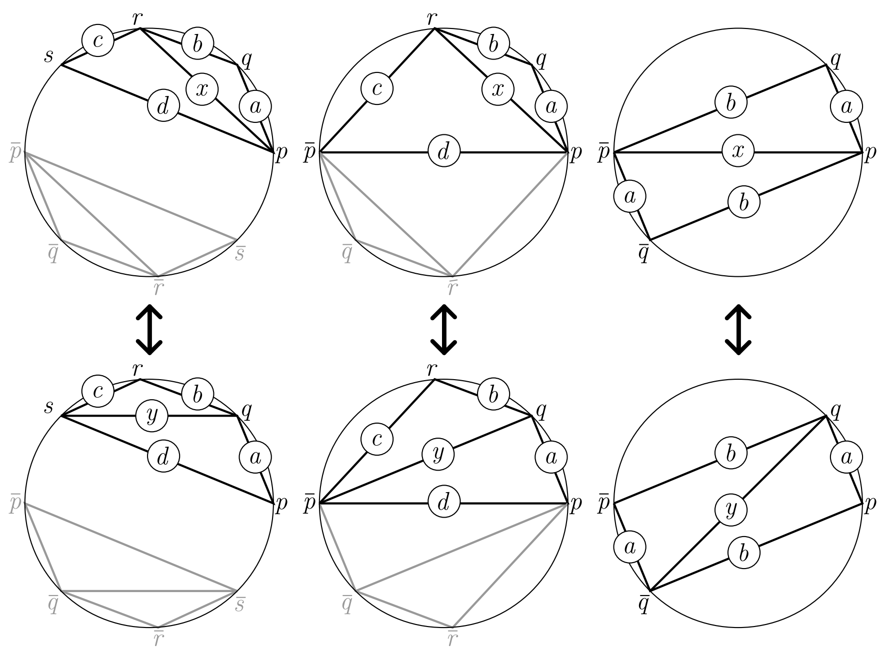
In this relation, if δ is a boundary edge of the $(n+3)$-gon, we can either set $\chi(\delta) = 1$, or work with a frozen but arbitrary variable x_δ.

(iv) Given any two diagonals θ, δ, the compatibility degree $d(\chi(\theta), \chi(\delta))$ between the corresponding cluster variables $\chi(\theta)$ and $\chi(\delta)$ is given by the *crossing number* $[\,\theta \,\|\, \delta\,]$ of the diagonals θ and δ. By definition, $[\,\theta \,\|\, \delta\,]$ is equal to -1 if $\theta = \delta$, to 1 if the diagonals $\theta \neq \delta$ cross, and to 0 otherwise.
(v) Given any initial seed triangulation $T := \{\theta_1, \ldots, \theta_n\}$ and any diagonal δ, the d-vector of the cluster variable $\chi(\delta)$ with respect to the initial cluster seed $\chi(T)$ is the *crossing vector* $\mathbf{d}(T, \delta) := ([\,\theta_1 \,\|\, \delta\,], \ldots, [\,\theta_n \,\|\, \delta\,])$ of δ with respect to T.
(vi) The rotation map τ on the cluster variables corresponds to the rotation of angle $\frac{2\pi}{n+3}$ in the geometric picture.

See Figure 9 and Tables 1–2 in Section 7.4 for illustrations.

7.1.2. *Types B_n and C_n.* Consider the Coxeter group B_n (or C_n) of signed permutations of $[n]$, generated by the simple transpositions $\tau_i := (i\ i+1)$, for $i \in [n-1]$, and the sign change τ_0. The corresponding geometric picture is a convex regular $(2n+2)$-gon. We label its vertices counter-clockwise from 0 to $2n+1$, and we define $\bar{p} := p + n + 1 \pmod{2n+2}$, for $p \in \{0, \ldots, 2n+1\}$. Cluster variables, clusters, exchange relations, compatiblity degrees, and d-vectors in the cluster algebras $\mathcal{A}(B_n)$ and $\mathcal{A}(C_n)$ can be interpreted geometrically as follows:

(i) Cluster variables correspond to *centrally symmetric pairs of (internal) diagonals* or to *long diagonals* of the $(2n+2)$-gon (for us, a long diagonal is a diameter of the polygon, and we also consider it as a pair of diagonals). To simplify notations, we identify a diagonal δ, its centrally symmetric copy $\bar{\delta}$, and the pair $\{\delta, \bar{\delta}\}$. We denote by $\chi(\delta) = \chi(\bar{\delta})$ the cluster variable corresponding to the pair $\{\delta, \bar{\delta}\}$.
(ii) Clusters correspond to *centrally symmetric triangulations* of the $(2n+2)$-gon.
(iii) Cluster mutations correspond to *flips* of centrally symmetric pairs of diagonals between centrally symmetric triangulations. See Figure 5.



Type B	$xy = ac + bd$	$xy = ac + bd^2$	$xy = a + b$
Type C	$xy = ac + bd$	$xy = ac + bd$	$xy = a^2 + b^2$

FIGURE 5. Three possible flips in types B and C: either the quadrilateral involves no long diagonal (left), or one long diagonal is an edge of the quadrilateral (middle), or the two diagonals of the quadrilateral are long diagonals (right).

As in type A, the exchange relations between cluster variables during a cluster mutation can be understood in the geometric picture. For this, define $\kappa = 1$ in type B and $\kappa = 2$ in type C. Then, flipping the diagonal $[p, r]$ to the diagonal $[q, s]$ in the quadrilateral $\{p, q, r, s\}$ (and simultaneously $[\bar{p}, \bar{r}]$ to $[\bar{q}, \bar{s}]$ in the centrally symmetric quadrilateral $\{\bar{p}, \bar{q}, \bar{r}, \bar{s}\}$) results in the following exchange relations:

- if $p < q < r < s < \bar{p}$ as in Figure 5 (left), then

$$\chi([p, r]) \cdot \chi([q, s]) = \chi([p, q]) \cdot \chi([r, s]) + \chi([p, s]) \cdot \chi([q, r]);$$

- if $p < q < r < s = \bar{p}$ as in Figure 5 (middle), then
$$\chi([p,r]) \cdot \chi([\bar{p},q]) = \chi([p,q]) \cdot \chi([\bar{p},r]) + \chi([q,r]) \cdot \chi([p,\bar{p}])^{2/\kappa};$$
- if $p < q < r = \bar{p} < s = \bar{q}$ as in Figure 5 (right), then
$$\chi([p,\bar{p}]) \cdot \chi([q,\bar{q}]) = \chi([p,q])^{\kappa} + \chi([p,\bar{q}])^{\kappa}.$$

In these relations, if δ is a boundary edge of the $(2n+2)$-gon, we can either set $\chi(\delta) = 1$, or work with a frozen but arbitrary variable x_δ.

(iv) Given any two centrally symmetric pairs of diagonals θ, δ, the compatibility degree $d(\chi(\theta), \chi(\delta))$ between the corresponding cluster variables $\chi(\theta)$ and $\chi(\delta)$ is given by the *crossing number* $[\,\theta \,\|\, \delta\,]$ of the pairs of diagonals θ and δ, defined as follows. First, $[\,\delta \,\|\, \delta\,] = -1$. If now $\theta \neq \delta$, then
- In type B_n, we represent long diagonals by doubled long diagonals. If δ is not a long diagonal then $[\,\theta \,\|\, \delta\,]$ is the number of times that a representative diagonal of the pair δ crosses the pair θ. If δ is a long diagonal then $[\,\theta \,\|\, \delta\,]$ is 1 if θ and δ cross, and 0 otherwise.
- In type C_n, the long diagonals remain as single long diagonals. The crossing number $[\,\theta \,\|\, \delta\,]$ is the number of times that a representative diagonal of the pair δ crosses θ.

(v) Given any initial centrally symmetric seed triangulation $T := \{\theta_1, \dots, \theta_n\}$ and any centrally symmetric pair of diagonals δ, the d-vector of the cluster variable $\chi(\delta)$ with respect to the initial cluster seed $\chi(T)$ is the *crossing vector* $\mathbf{d}(T, \delta) := ([\,\theta_1 \,\|\, \delta\,], \dots, [\,\theta_n \,\|\, \delta\,])$ of δ with respect to T.

(vi) The rotation map τ on the cluster variables corresponds to the rotation of angle $\frac{2\pi}{2n+2}$ in the geometric picture.

See Figure 10 and Tables 3–4–5–6 in Section 7.4 for illustrations.

7.1.3. *Type D_n.* Consider the Coxeter group D_n of even signed permutations of $[n]$, generated by the simple transpositions $\tau_i := (i \;\; i+1)$ for $i \in [n-1]$ and by the operator τ_0 which exchanges 1 and 2 and invert their signs. Note that τ_0 and τ_1 play symmetric roles in D_n (they both commute with all the other simple generators except with τ_2). This Coxeter group can be folded in type C_{n-1}, which provides a geometric interpretation of the cluster algebra $\mathcal{A}(D_n)$ on a $2n$-gon with bicolored long diagonals [FZ03b, Section 3.5][FZ03a, Section 12.4]. In this section, we present a new interpretation of the cluster algebra $\mathcal{A}(D_n)$ in terms of pseudotriangulations. Our motivations for this interpretation are given in Remark 7.1.

We consider a regular convex $2n$-gon, together with a disk D (placed at the center of the $2n$-gon), whose radius is small enough such that D only intersects the long diagonals of the $2n$-gon. We denote by $\mathcal{D}_n$ the resulting configuration, see Figure 6. The *chords* of $\mathcal{D}_n$ are all the diagonals of the $2n$-gon, except the long ones, plus all the segments tangent to the disk D and with one endpoint among the vertices of the $2n$-gon. Note that each vertex p is adjacent to two of the latter chords; we denote by p^{L} (resp. by p^{R}) the chord emanating from p and tangent on the left (resp. right) to the disk D. Cluster variables, clusters, exchange relations, compatiblity degrees, and d-vectors in the cluster algebras $\mathcal{A}(D_n)$ can be interpreted geometrically as follows:

(i) Cluster variables correspond to *centrally symmetric pairs of (internal) chords* of the geometric configuration $\mathcal{D}_n$. See Figure 6 (left). To simplify notations, we identify a chord δ, its centrally symmetric copy $\bar{\delta}$, and the pair $\{\delta, \bar{\delta}\}$.

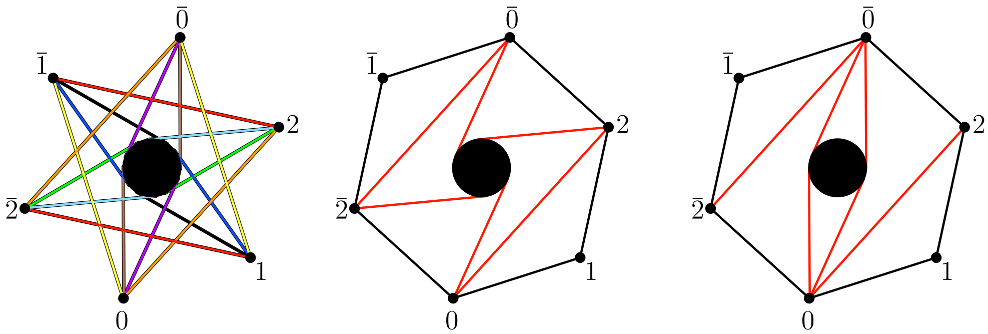


FIGURE 6. The configuration $\mathcal{D}_3$ with its 9 centrally symmetric pairs of chords (left). A centrally symmetric pseudotriangulation T of $\mathcal{D}_3$ (middle). The centrally symmetric pseudotriangulation of $\mathcal{D}_3$ obtained from T by flipping the chords 2^{R} and $\bar{2}^{\mathrm{R}}$.

We denote by $\chi(\delta) = \chi(\bar{\delta})$ the cluster variable corresponding to the pair of chords $\{\delta, \bar{\delta}\}$.

(ii) Clusters of $\mathcal{A}(D_n)$ correspond to *centrally symmetric pseudotriangulations* of $\mathcal{D}_n$ (*i.e.* maximal centrally symmetric crossing-free sets of chords of $\mathcal{D}_n$). Each pseudotriangulation of $\mathcal{D}_n$ contains exactly $2n$ chords, and partitions $\mathrm{conv}(\mathcal{D}_n) \smallsetminus D$ into *pseudotriangles* (*i.e.* interiors of simple closed curves with three convex corners related by three concave chains). See Figure 6 (middle) and (right). We refer to [RSS08] for a complete survey on pseudotriangulations, including their history, motivations, and applications.

(iii) Cluster mutations correspond to *flips* of centrally symmetric pairs of chords between centrally symmetric pseudotriangulations of $\mathcal{D}_n$. A flip in a pseudotriangulation T replaces an internal chord e by the unique other internal chord f such that $(T \smallsetminus e) \cup f$ is again a pseudotriangulation of T. To be more precise, deleting e in T merges the two pseudotriangles of T incident to e into a pseudoquadrangle $\square$ (*i.e.* the interior of a simple closed curve with four convex corners related by four concave chains), and adding f splits the pseudoquadrangle $\square$ into two new pseudotriangles. The chords e and f are the two unique chords which lie both in the interior of $\square$ and on a geodesic between two opposite corners of $\square$. We refer again to [RSS08] for more details.

For example, the two pseudotriangulations of Figure 6 (middle) and (right) are related by a centrally symmetric pair of flips. We have represented different types of flips between centrally symmetric pseudotriangulations of the configuration $\mathcal{D}_n$ in Figure 8. Finally, Figure 7 and Figures 12 – 13 – 14 show the flip graph on centrally symmetric pseudotriangulations of $\mathcal{D}_3$ and $\mathcal{D}_4$, respectively.

As in types A, B, and C, the exchange relations between cluster variables during a cluster mutation can be understood in the geometric picture. More precisely, flipping e to f in the pseudoquadrangle $\square$ with convex corners $\{p, q, r, s\}$ (and simultaneously $\bar{e}$ to $\bar{f}$ in the centrally symmetric pseudoquadrangle $\bar{\square}$) results in the exchange relation

$$\Pi(\square, p, r) \cdot \Pi(\square, q, s) = \Pi(\square, p, q) \cdot \Pi(\square, r, s) + \Pi(\square, p, s) \cdot \Pi(\square, q, r),$$

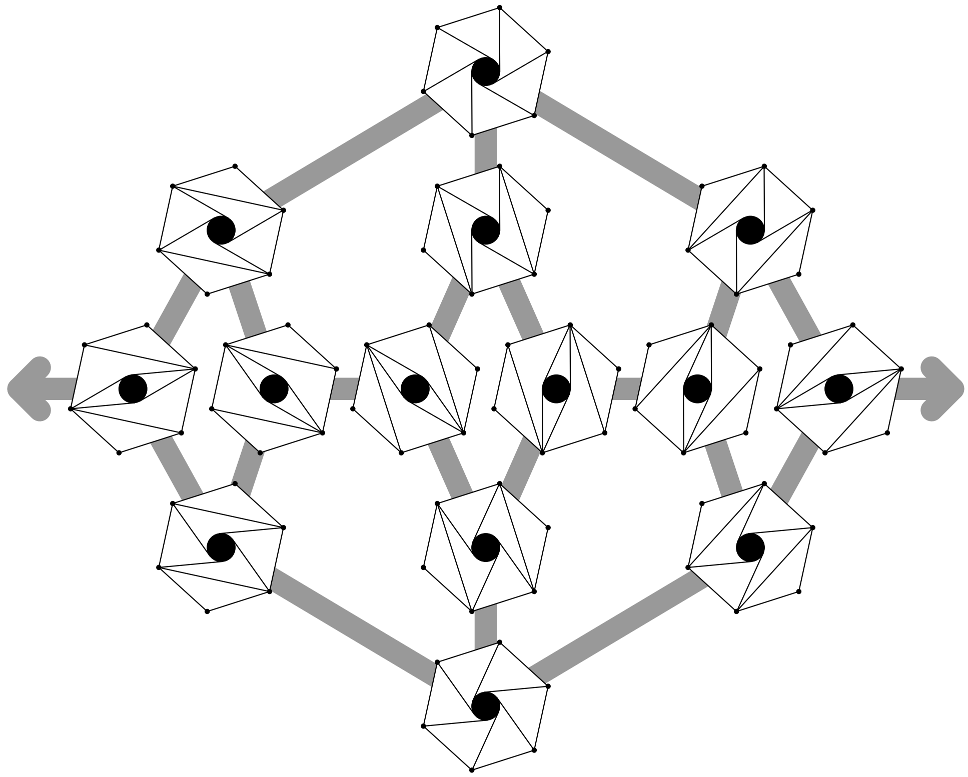


FIGURE 7. The type D_3 mutation graph interpreted geometrically with centrally symmetric pseudotriangulations of $\mathcal{D}_3$. The leftmost and rightmost triangulations are related by a (centrally symmetric) flip. Note that this graph is the 1-skeleton of the 3-dimensional associahedron since $D_3 = A_3$.

where

- $\Pi(\square, p, r)$ denotes the product of the cluster variables $\chi(\delta)$ corresponding to all chords δ which appear along the geodesic from p to r in $\square$ — and similarly for $\Pi(\square, q, s)$ — and
- $\Pi(\square, p, q)$ denotes the product of the cluster variables $\chi(\delta)$ corresponding to all chords δ which appear on the concave chain from p to q in $\square$ — and similarly for $\Pi(\square, q, r)$, $\Pi(\square, r, s)$, and $\Pi(\square, p, s)$.

For example, the four flips in Figure 8 result in the following relations:

$$\begin{aligned}
\chi([p,r]) \cdot \chi([q,s]) &= \chi([p,q]) \cdot \chi([r,s]) + \chi([p,s]) \cdot \chi([q,r]), \\
\chi([p,r]) \cdot \chi(q^{\mathrm{R}}) &= \chi([p,q]) \cdot \chi(r^{\mathrm{R}}) + \chi(p^{\mathrm{R}}) \cdot \chi([q,r]), \\
\chi([p,r]) \cdot \chi([q,\bar{p}]) &= \chi([p,q]) \cdot \chi([r,\bar{p}]) + \chi(\bar{p}^{\mathrm{L}}) \cdot \chi(p^{\mathrm{R}}) \cdot \chi([q,r]), \\
\chi(\bar{p}^{\mathrm{L}}) \cdot \chi(p^{\mathrm{R}}) \cdot \chi(q^{\mathrm{R}}) &= \chi([p,q]) \cdot \chi(\bar{p}^{\mathrm{R}}) + \chi([q,\bar{p}]) \cdot \chi(p^{\mathrm{R}}).
\end{aligned}$$

Note that the last relation will always simplify by $\chi(p^{\mathrm{R}}) = \chi(\bar{p}^{\mathrm{R}})$. For a concrete example, in the flip presented in Figure 6, we obtain the relation

$$\chi(\bar{0}^{\mathrm{L}}) \cdot \chi(0^{\mathrm{R}}) \cdot \chi(2^{\mathrm{R}}) = \chi([0,2]) \cdot \chi(\bar{0}^{\mathrm{R}}) + \chi([2,\bar{0}]) \cdot \chi(0^{\mathrm{R}}).$$

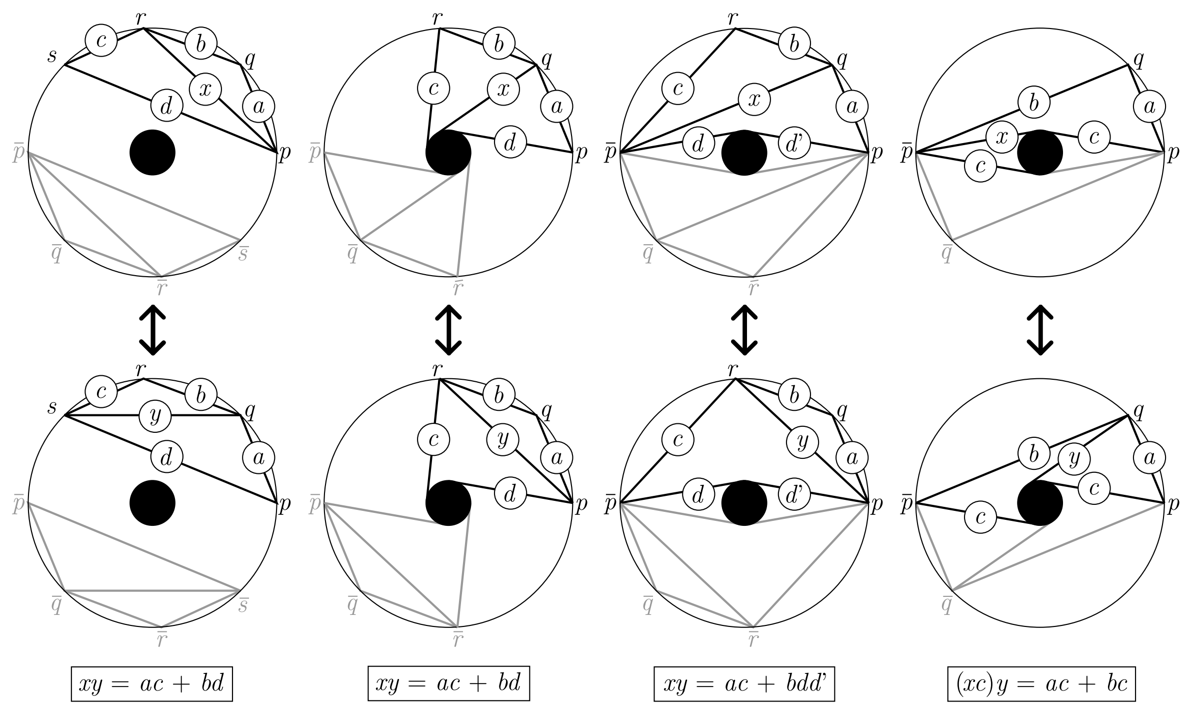


FIGURE 8. Different types of flips in type D.

which simplifies to

$$\chi(\bar{0}^{\text{L}}) \cdot \chi(2^{\text{R}}) = \chi([0,2]) + \chi([2,\bar{0}]).$$

(iv) Given any two centrally symmetric pairs of chords θ, δ, the compatibility degree $d(\chi(\theta), \chi(\delta))$ between the corresponding cluster variables $\chi(\theta)$ and $\chi(\delta)$ is given by the *crossing number* $[\,\theta \,\|\, \delta\,]$ of the pairs of chords θ and δ. By definition, $[\,\theta \,\|\, \delta\,]$ is equal to -1 if $\theta = \delta$, and to the number of times that a representative diagonal of the pair δ crosses the chords of θ if $\theta \neq \delta$.

(v) Given any initial centrally symmetric seed pseudotriangulation $T := \{\theta_1, \dots, \theta_n\}$ and any centrally symmetric pair of chords δ, the d-vector of the cluster variable $\chi(\delta)$ with respect to the initial cluster seed $\chi(T)$ is the *crossing vector* $\mathbf{d}(T, \delta) := ([\,\theta_1 \,\|\, \delta\,], \dots, [\,\theta_n \,\|\, \delta\,])$ of δ with respect to T.

(vi) The rotation map τ on the cluster variables corresponds to the rotation of angle $\frac{\pi}{n}$ in the geometric picture, except that the chords p^{L} and p^{R} are exchanged after rotation.

See Figure 11 and Tables 7–8 in Section 7.4 for illustrations.

Remark 7.1. Our geometric interpretation of type D cluster algebras slightly differs from that of S. Fomin and A. Zelevinsky in [FZ03b, Section 3.5][FZ03a, Section 12.4]. Namely, to obtain their interpretation, we can just remove the disk in the configuration $\mathcal{D}_n$ and replace the centrally symmetric pairs of chords $\{p^{\text{L}}, \bar{p}^{\text{L}}\}$ and $\{p^{\text{R}}, \bar{p}^{\text{R}}\}$ by long diagonals $[p, \bar{p}]$ colored in red and blue respectively. Long diagonals of the same color are then allowed to cross, while long diagonals of different colors cannot. Flips and exchange relations can then be worked out, with special

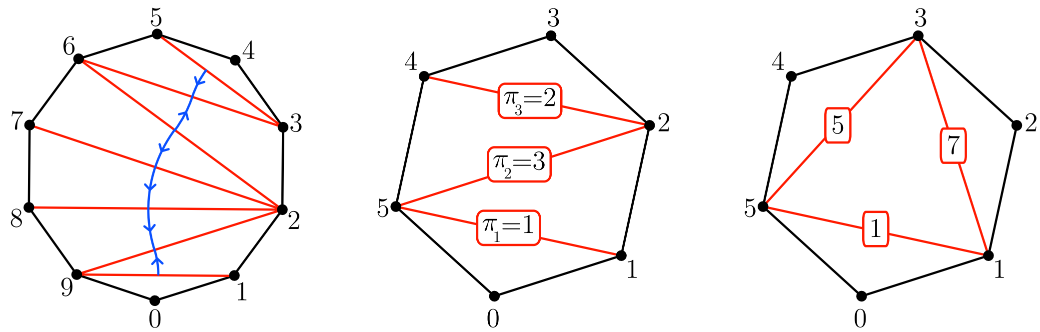


FIGURE 9. TYPE A. The accordion Z_c for $c = \tau_5\tau_7\tau_6\tau_4\tau_3\tau_1\tau_2$ (left) and for $c = \tau_1\tau_3\tau_2$ (middle) and the initial seed triangulation $T = \zeta_{\tau_1\tau_3\tau_2}(\{1,5,7\}) = \{[1,5],[3,5],[1,3]\}$ (right). See Tables 1 – 2.

rules for colored long diagonals, see [FZ03b, Section 3.5][FZ03a, Section 12.4]. Although our presentation is only slightly different from the classical presentation, we believe that it has certain advantages:

(i) It simplifies certain combinatorial and algebraic aspects. Compared to the classical interpretation, we have no color code. Therefore, the notion of crossing, the d-vector, the cluster mutations, and the exchange relations are simpler to express (compare [FZ03a, Section 12.4]). In a forthcoming paper, we will moreover interpret quivers on the pseudotriangulations and cluster variables in terms of perfect matchings of weighted graphs.

(ii) It makes an additional link between cluster algebras and pseudotriangulations (compare [PP12]). In particular, it ensures that types A, B, C and D generalized associahedra can all be realized as pseudotriangulation polytopes by the construction of G. Rote, F. Santos, and I. Streinu [RSS03] (more precisely, by the centrally symmetric version of it in types B, C and D).

7.2. **Positions in the word $\mathrm{Q_c}$.** Points (iii) and (iv) in the previous section already give two ways to compute d-vectors in the classical types A, B, C and D. According to Corollary 3.4, the d-vectors can also be interpreted in terms of root functions associated to the word $\mathrm{Q_c}$. In order to illustrate this, we present explicit bijections between positions in $\mathrm{Q_c}$ and (c.s. pairs of) diagonals in the geometric picture. These bijections and the computation of d-vectors are illustrated in Tables 1 to 8, for the Coxeter elements and the initial cluster seeds from Figures 9, 10 and 11.

7.2.1. *Type A_n.* Let c be a Coxeter element of A_n, let c be a reduced expression of c, and let $\mathrm{Q_c} := \mathrm{c}\,\mathrm{w_\circ}(\mathrm{c})$ be the corresponding word. For $i \in [n]$, we denote by π_i the position of the transposition τ_i in c.

We first define an *accordion* Z_c in the $(n+3)$-gon (*i.e.* a triangulation whose dual tree is a path) as follows. The (internal) diagonals of Z_c are labeled by $\tau_1, \dots, \tau_n$ in the order they appear along its dual path. The angle between two consecutive diagonals labeled by τ_i and τ_{i+1} is in clockwise direction if $\pi_i > \pi_{i+1}$ and counterclockwise direction if $\pi_i < \pi_{i+1}$. See Figure 9. The map ζ_{c} sends position π_i to the diagonal of Z_c labeled by τ_i. More explicitly, if we label the vertices of the regular

$(n+3)$-gon counter-clockwise from 0 to $n+2$, then we have $\zeta_{\mathrm{c}}(\pi_i) = [p_i, q_i]$, where

$$p_i := \quad 1 + |\{j \in [n] \mid j < i \text{ and } \pi_j < \pi_{j+1}\}| \pmod{n+3},$$
$$q_i := -1 - |\{j \in [n] \mid j < i \text{ and } \pi_j > \pi_{j+1}\}| \pmod{n+3}.$$

This only defines ζ_{c} on the first n positions in the word $\mathrm{Q_c}$. The other values of ζ_{c} are then determined using the rotation map τ_{c} from Definition 2.3. Namely, $\zeta_{\mathrm{c}}(\tau_{\mathrm{c}}(i))$ is defined as the rotation of angle $\frac{2\pi}{n+3}$ of the diagonal $\zeta_{\mathrm{c}}(i)$. Therefore, the map ζ_{c} sends the position of the kth appearance of the generator τ_i in $\mathrm{Q_c}$ to the rotation of angle $\frac{(k-1)2\pi}{n+3}$ of the diagonal $[p_i, q_i]$. See Table 1 and Figure 9.

Under this bijection, a set $I \subset [m]$ is the set of positions of the complement of a reduced expression of $w_\circ$ in $\mathrm{Q_c}$ (*i.e.* a c-cluster), if and only if the corresponding diagonals $\zeta_{\mathrm{c}}(I)$ form a triangulation of the $(n+3)$-gon.

7.2.2. *Types B_n and C_n.* Let c be a Coxeter element of B_n, let c be a reduced expression of c, and let $\mathrm{Q_c} := \mathrm{c}w_\circ(\mathrm{c})$ be the corresponding word. We still denote by π_i the position of τ_i in c. Moreover, we still label the vertices of the regular $(2n+2)$-gon counter-clockwise from 0 to $2n+1$, and denote by $\bar{p} := p+n+1 \pmod{2n+2}$, for $p \in \{0, \dots, 2n+1\}$.

The construction of the bijection ζ_{c} is similar to the construction of ζ_{c} in type A, except that we start from the long diagonal $[0, \bar{0}]$. We first construct a centrally symmetric accordion Z_c associated to c as illustrated in Figure 10. The bijection ζ_{c} is defined on the first n positions by $\zeta_{\mathrm{c}}(\pi_i) = [p_i, q_i] \cup [\overline{p}_i, \overline{q}_i]$, where

$$p_i := \qquad |\{j \in [0, n-1] \mid j < i \text{ and } \pi_j < \pi_{j+1}\}|,$$
$$q_i := n+1 - |\{j \in [0, n-1] \mid j < i \text{ and } \pi_j > \pi_{j+1}\}|.$$

The images of the remaining positions in $\mathrm{Q_c}$ are again determined by rotation. Namely, $\zeta_{\mathrm{c}}(\tau_{\mathrm{c}}(i))$ is defined as the rotation of angle $\frac{\pi}{n+1}$ of the pair of diagonals $\zeta_{\mathrm{c}}(i)$. Therefore, the map ζ_{c} sends the position of the kth appearance of the generator τ_i in $\mathrm{Q_c}$ to the rotation of angle $\frac{(k-1)\pi}{n+1}$ of the pair of diagonals $[p_i, q_i] \cup [\overline{p}_i, \overline{q}_i]$. See Table 3 and Figure 10.

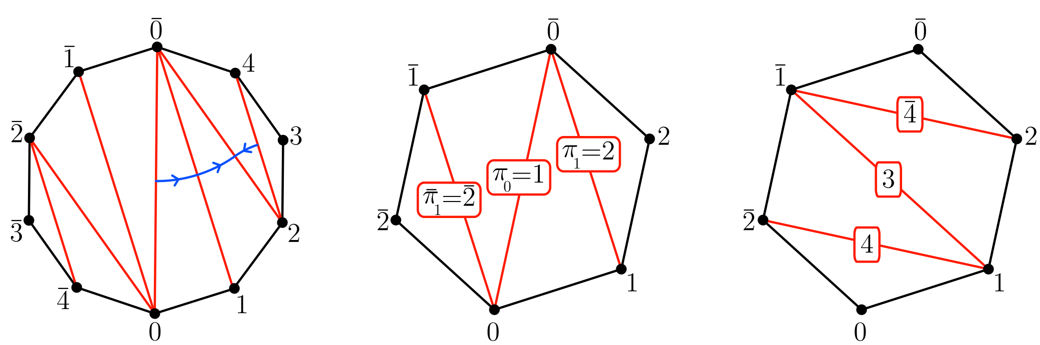


FIGURE 10. TYPE B AND C. The centrally symmetric accordion Z_c for $c = \tau_0\tau_1\tau_3\tau_2$ (left) and for $c = \tau_0\tau_1$ (middle) and the initial seed triangulation $T = \zeta_{\tau_0\tau_1}(\{3,4\}) = \{[1,\bar{1}], [1,\bar{2}], [\bar{1},2]\}$ (right). See also Tables 3–4–5–6.

Under this bijection, a set $I \subset [m]$ is the set of positions of the complement of a reduced expression of $w_\circ$ in $\mathrm{Q_c}$ (*i.e.* a c-cluster), if and only if the corresponding centrally symmetric pairs of diagonals $\zeta_{\mathrm{c}}(I)$ form a centrally symmetric triangulation of the $(2n+2)$-gon.

7.2.3. *Type D_n.* Let c be a Coxeter element of D_n, let c be a reduced expression of c, and let $\mathrm{Q_c} := \mathrm{c}w_\circ(\mathrm{c})$ be the corresponding word. As before, we denote by π_i the position of τ_i in c. Moreover, we still label the vertices of the $2n$-gon counterclockwise from 0 to $2n-1$, and define $\bar{p} := p + n \pmod{2n}$ for $p \in \{0, \dots, 2n-1\}$.

The bijection ζ_{c} is defined as follows. The positions π_0 and π_1 are sent to

$$\zeta_{\mathrm{c}}(\pi_0) = \begin{cases} 0^{\mathrm{L}} \cup \bar{0}^{\mathrm{L}} & \text{if } \pi_0 > \pi_2, \\ (n-1)^{\mathrm{R}} \cup \overline{(n-1)}^{\mathrm{R}} & \text{if } \pi_0 < \pi_2, \end{cases}$$

$$\zeta_{\mathrm{c}}(\pi_1) = \begin{cases} 0^{\mathrm{R}} \cup \bar{0}^{\mathrm{R}} & \text{if } \pi_1 > \pi_2, \\ (n-1)^{\mathrm{L}} \cup \overline{(n-1)}^{\mathrm{L}} & \text{if } \pi_1 < \pi_2, \end{cases}$$

and the positions $\pi_2, \cdots, \pi_{n-1}$ are sent to $\zeta_{\mathrm{c}}(\pi_i) := [p_i, q_i] \cup [\bar{p}_i, \bar{q}_i]$, where

$$p_i := \quad |\{j \in [2, n-1] \mid j < i \text{ and } \pi_j < \pi_{j+1}\}|,$$
$$q_i := n - 1 - |\{j \in [2, n-1] \mid j < i \text{ and } \pi_j > \pi_{j+1}\}|.$$

In other words, the pairs of diagonals $\zeta_{\mathrm{c}}(\pi_2), \dots, \zeta_{\mathrm{c}}(\pi_{n-1})$ form a centrally symmetric pair of accordions based on the diagonals $\zeta_{\mathrm{c}}(\pi_2) = [0, n-1] \cup [\bar{0}, \overline{(n-1)}]$. We denote by Z_c the centrally symmetric pseudotriangulation formed by the diagonals $\zeta_{\mathrm{c}}(\pi_0), \dots, \zeta_{\mathrm{c}}(\pi_{n-1})$.

Finally, the other values of ζ_{c} are determined using the rotation map τ_{c} from Definition 2.3. Namely, $\zeta_{\mathrm{c}}(\tau_{\mathrm{c}}(i))$ is obtained by rotating by $\frac{\pi}{n}$ the pair of chords $\zeta_{\mathrm{c}}(i)$, and exchanging p^{L} with p^{R}. See Table 7 and Figure 11.

Under this bijection, a set $I \subset [m]$ is the set of positions of the complement of a reduced expression of $w_\circ$ in $\mathrm{Q_c}$ (*i.e.* a c-cluster), if and only if the corresponding centrally symmetric pairs of chords $\zeta_{\mathrm{c}}(I)$ form a centrally symmetric pseudotriangulation of the $(2n+2)$-gon.

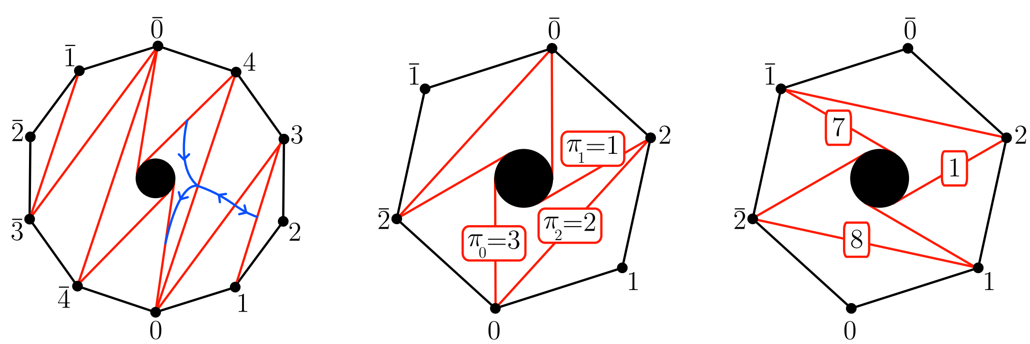


FIGURE 11. TYPE D. The centrally symmetric "accordion" pseudotriangulation Z_c for $c = \tau_3\tau_4\tau_0\tau_2\tau_1$ (left) and for $c = \tau_1\tau_2\tau_0$ (middle) and the initial centrally symmetric seed pseudotriangulation $T = \zeta_{\tau_1\tau_2\tau_0}(\{1, 7, 8\}) = \{2^{\mathrm{L}}, \bar{2}^{\mathrm{L}}, 1^{\mathrm{L}}, \bar{1}^{\mathrm{L}}, [1, \bar{2}], [\bar{1}, 2]\}$ (right). See also Tables 7–8.

7.3. **Almost positive roots and c-cluster complexes.** Composing the bijection of the previous section with the bijection between positions of the word Q_{c} and almost positive roots of Section 2.1 determines a bijection between almost positive roots and (c.s. pairs of) diagonals in the geometric picture for the classical types A, B, C, and D. Using this bijection, which is described in detailed in this section, we obtain a simple geometric description of c-cluster complexes of [Rea07, Section 7] for any Coxeter element c.

7.3.1. *Type A_n.* The almost positive roots of type A_n are in bijection with the diagonals of an $(n+3)$-gon. The almost positive root associated to a diagonal δ is the crossing vector $\mathbf{d}(Z_c, \delta)$ of δ with respect to the accordion Z_c constructed in Section 7.2.1. Under this bijection, a set of almost positive roots forms a c-cluster if and only if the corresponding diagonals form a triangulation of the $(n+3)$-gon.

7.3.2. *Types B_n and C_n.* The almost positive roots of type B_n (or C_n) are in bijection with the centrally symmetric pairs of diagonals of a regular $(2n+2)$-gon. The almost positive root associated to a pair of diagonals δ is the crossing vector $\mathbf{d}(Z_c, \delta)$ of δ with respect to the centrally symmetric accordion Z_c constructed in Section 7.2.2. Under this bijection, a set of almost positive roots forms a c-cluster if and only if the corresponding centrally symmetric pairs of diagonals form a centrally symmetric triangulation of the $(2n+2)$-gon.

7.3.3. *Type D_n.* The almost positive roots of type D_n are in bijection with the centrally symmetric pairs of chords of the geometric configuration $\mathcal{D}_n$. The almost positive root associated to a pair of chords δ is the crossing vector $\mathbf{d}(Z_c, \delta)$ of δ with respect to the centrally symmetric "accordion" pseudotriangulation Z_c constructed in Section 7.2.3. Under this bijection, a set of almost positive roots forms a c-cluster if and only if the corresponding centrally symmetric pairs of chords form a centrally symmetric pseudotriangulation of $\mathcal{D}_n$.

7.4. **Examples.** Our goal in all examples below is to illustrate how to explicitly compute all d-vectors with respect to a non-acyclic cluster seed. For this, we use four different methods:

(i) either using direct computations of cluster variables;
(ii) or using compatibility degrees and applying Corollary 3.3
(iii) or using root functions on the word Q_{c} and applying Corollary 3.4;
(iv) or using the interpretation of the d-vector in terms of crossings of diagonals on the corresponding geometric picture.

These different methods are illustrated in types A_3, B_2, C_2 and D_3 in Tables 1 to 8. To illustrate further our approach to type D cluster algebras, which differs from the classical one of [FZ03b, Section 3.5][FZ03a, Section 12], we worked out in Appendix A an example in type D_4. See Figures 12 – 13 – 14 and Table 9. The geometric interpretation of denominator vectors is also illustrated for specific examples of higher rank in Figure 2.

TYPE A_3, Coxeter element $\mathrm{c} = \tau_1\tau_3\tau_2$, Cluster seed $I = \{1,5,7\}$										
Word $\mathrm{Q_c}$	Position j in $[9]$	1	2	3	4	5	6	7	8	9
	letter q_j of $\mathrm{Q_c}$	τ_1	τ_3	τ_2	τ_1	τ_3	τ_2	τ_1	τ_3	τ_2
	root $\mathsf{r}(I,j)$	α_1	α_3	$\alpha_2+\alpha_3$	$\alpha_1+\alpha_2+\alpha_3$	α_2	α_1	$-\alpha_2-\alpha_3$	$\alpha_1+\alpha_2$	α_2
	root $\mathsf{r}(I,j)$ in the basis $\mathsf{R}(I)$	β_1	$-\beta_2-\beta_3$	$-\beta_3$	$\beta_1-\beta_3$	β_2	β_1	β_3	$\beta_1+\beta_2$	β_2
diagonal $\zeta_\mathrm{c}(j)$ of the hexagon		$[1,5]$	$[2,4]$	$[2,5]$	$[0,2]$	$[3,5]$	$[0,3]$	$[1,3]$	$[0,4]$	$[1,4]$
Almost positive root $\vartheta_\mathrm{c}(j)$		$-\alpha_1$	$-\alpha_3$	$-\alpha_2$	α_1	α_3	$\alpha_1+\alpha_2+\alpha_3$	$\alpha_2+\alpha_3$	$\alpha_1+\alpha_2$	α_2
cluster variable $\psi_\mathrm{c}(j)$		x_1	$\frac{x_1+x_2+x_3}{x_2x_3}$	$\frac{x_1+x_2}{x_3}$	$\frac{x_1+x_2+x_3}{x_1x_3}$	x_2	$\frac{x_2+x_3}{x_1}$	x_3	$\frac{x_1+x_2+x_3}{x_1x_2}$	$\frac{x_1+x_3}{x_2}$
d-vector		$\begin{pmatrix}-1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\-1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\-1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$

Table 1. Correspondence between positions of letters in $\mathrm{Q}_{\tau_1\tau_3\tau_2}$, diagonals of the hexagon, almost positive roots, and cluster variables in $\mathcal{A}(A_3)$. The d-vectors of the cluster variables correspond to the crossing vectors of the corresponding diagonals with respect to the seed triangulation $T = \zeta_{\tau_1\tau_3\tau_2}(\{1,5,7\}) = \{[1,5],[3,5],[1,3]\}$. The column corresponding to the diagonals of this triangulation are shaded. See also Figure 9.

TYPE A_3, Coxeter element $\mathrm{c} = \tau_1\tau_3\tau_2$, Compatibility table											
position j in the word $\mathrm{Q_c}$			1	2	3	4	5	6	7	8	9
	almost positive root $\vartheta_\mathrm{c}(j)$		$-\alpha_1$	$-\alpha_3$	$-\alpha_2$	α_1	α_3	$\alpha_1+\alpha_2+\alpha_3$	$\alpha_2+\alpha_3$	$\alpha_1+\alpha_2$	α_2
		diagonal $\zeta_\mathrm{c}(j)$	$[1,5]$	$[2,4]$	$[2,5]$	$[0,2]$	$[3,5]$	$[0,3]$	$[1,3]$	$[0,4]$	$[1,4]$
1	$-\alpha_1$	$[1,5]$	-1	0	0	1	0	1	0	1	0
2	$-\alpha_3$	$[2,4]$	0	-1	0	0	1	1	1	0	0
3	$-\alpha_2$	$[2,5]$	0	0	-1	0	0	1	1	1	1
4	α_1	$[0,2]$	1	0	0	-1	0	0	1	0	1
5	α_3	$[3,5]$	0	1	0	0	-1	0	0	1	1
6	$\alpha_1+\alpha_2+\alpha_3$	$[0,3]$	1	1	1	0	0	-1	0	0	1
7	$\alpha_2+\alpha_3$	$[1,3]$	0	1	1	1	0	0	-1	0	0
8	$\alpha_1+\alpha_2$	$[0,4]$	1	0	1	0	1	0	0	-1	0
9	α_2	$[1,4]$	0	0	1	1	1	1	0	0	-1
d-vector $\mathbf{d}(X,y) = \mathbf{d}_\mathrm{c}(I,j) = \mathbf{d}_c(B,\beta) = \mathbf{d}(T,\delta)$			$\begin{pmatrix}-1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\-1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\-1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$

Table 2. c-compatibility table for the Coxeter element $\mathrm{c} = \tau_1\tau_3\tau_2$ of type A_3. The last line shows the d-vectors with respect to the c-cluster $I = \{1,5,7\}$ (corresponding to the cluster $X = \psi_\mathrm{c}(I)$, to the c-cluster $B = \vartheta_\mathrm{c}(I) = \{-\alpha_1, \alpha_3, \alpha_2+\alpha_3\}$, and to the triangulation $T = \zeta_\mathrm{c}(I) = \{[1,5],[3,5],[1,3]\}$). In general, the d-vectors are the column vectors of the submatrix of the matrix of c-compatibility degrees, with rows corresponding to I (or X, B, or T).

TYPE B_2, Coxeter element $c = \tau_0\tau_1$, Cluster seed $I = \{3, 4\}$							
Word Q_c	Position j in $[6]$	1	2	3	4	5	6
	letter q_j of Q_c	τ_0	τ_1	τ_0	τ_1	τ_0	τ_1
	root $\mathsf{r}(I, j)$	α_0	$2\alpha_0 + \alpha_1$	$\alpha_0 + \alpha_1$	$-2\alpha_0 - \alpha_1$	$\alpha_0 + \alpha_1$	α_1
	root $\mathsf{r}(I, j)$ in the basis $\mathsf{R}(I)$	$-\beta_1 - \beta_2$	$-\beta_2$	β_1	β_2	β_1	$2\beta_1 + \beta_2$
c.s. pairs $\zeta_c(j)$ of diagonals		$[0, \bar{0}]$	$[0, \bar{1}] \cup [\bar{0}, 1]$	$[1, \bar{1}]$	$[1, \bar{2}] \cup [\bar{1}, 2]$	$[2, \bar{2}]$	$[0, 2] \cup [\bar{0}, \bar{2}]$
Almost positive root $\vartheta_c(j)$		$-\alpha_0$	$-\alpha_1$	α_0	$2\alpha_0 + \alpha_1$	$\alpha_0 + \alpha_1$	α_1
cluster variable $\psi_c(j)$		$\frac{1 + x_1^2 + x_2}{x_1 x_2}$	$\frac{1 + x_1^2}{x_2}$	x_1	x_2	$\frac{1 + x_2}{x_1}$	$\frac{(1 + x_2)^2 + x_1^2}{x_1^2 x_2}$
d-vector		$\binom{1}{1}$	$\binom{0}{1}$	$\binom{-1}{0}$	$\binom{0}{-1}$	$\binom{1}{0}$	$\binom{2}{1}$

TABLE 3. Correspondence between positions of letters in $Q_{\tau_0\tau_1}$, centrally symmetric pairs of diagonals of the hexagon, almost positive roots, and cluster variables in $\mathcal{A}(B_2)$. The d-vectors of the cluster variables correspond to the crossing vectors of the corresponding diagonals with respect to the seed triangulation $T = \zeta_{\tau_0\tau_1}(\{3, 4\}) = \{[1, \bar{1}], [1, \bar{2}], [\bar{1}, 2]\}$. The column corresponding to the diagonals of this triangulation are shaded. See also Figure 10.

TYPE B_2, Coxeter element $c = \tau_0\tau_1$, Compatibility table								
position j in the word Q_c			1	2	3	4	5	6
	almost positive root $\vartheta_c(j)$		$-\alpha_0$	$-\alpha_1$	α_0	$2\alpha_0 + \alpha_1$	$\alpha_0 + \alpha_1$	α_1
		diagonal $\zeta_c(j)$	$[0, \bar{0}]$	$[0, \bar{1}] \cup [\bar{0}, 1]$	$[1, \bar{1}]$	$[1, \bar{2}] \cup [\bar{1}, 2]$	$[2, \bar{2}]$	$[0, 2] \cup [\bar{0}, \bar{2}]$
1	$-\alpha_0$	$[0, \bar{0}]$	-1	0	1	2	1	0
2	$-\alpha_1$	$[0, \bar{1}] \cup [\bar{0}, 1]$	0	-1	0	1	1	1
3	α_0	$[1, \bar{1}]$	1	0	-1	0	1	2
4	$2\alpha_0 + \alpha_1$	$[1, \bar{2}] \cup [\bar{1}, 2]$	1	1	0	-1	0	1
5	$\alpha_0 + \alpha_1$	$[2, \bar{2}]$	1	2	1	0	-1	0
6	α_1	$[0, 2] \cup [\bar{0}, \bar{2}]$	0	1	1	1	0	-1
d-vector $\mathbf{d}(X, y) = \mathbf{d}_c(I, j) = \mathbf{d}_c(B, \beta) = \mathbf{d}(T, \delta)$			$\binom{1}{1}$	$\binom{0}{1}$	$\binom{-1}{0}$	$\binom{0}{-1}$	$\binom{1}{0}$	$\binom{2}{1}$

TABLE 4. c-compatibility table for the Coxeter element $c = \tau_0\tau_1$ of type B_2. The last line shows the d-vectors with respect to the c-cluster $I = \{3, 4\}$ (corresponding to the cluster $X = \psi_c(I)$, to the c-cluster $B = \vartheta_c(I) = \{\alpha_0, 2\alpha_0 + \alpha_1\}$, and to the centrally symmetric triangulation $T = \zeta_c(I) = \{[1, \bar{1}], [1, \bar{2}], [\bar{1}, 2]\}$). In general, the d-vectors are the column vectors of the submatrix of the matrix of c-compatibility degrees, with rows corresponding to I (or X, B, or T).

TYPE C_2, Coxeter element $c = \tau_0\tau_1$, Cluster seed $I = \{3,4\}$							
Word Q_c	Position j in [6]	1	2	3	4	5	6
	letter q_j of Q_c	τ_0	τ_1	τ_0	τ_1	τ_0	τ_1
	root $\mathsf{r}(I,j)$	α_0	$\alpha_0+\alpha_1$	$\alpha_0+2\alpha_1$	$-\alpha_0-\alpha_1$	$\alpha_0+2\alpha_1$	α_1
	root $\mathsf{r}(I,j)$ in the basis $\mathsf{R}(I)$	$-\beta_1-2\beta_2$	$-\beta_2$	β_1	β_2	β_1	$\beta_1+\beta_2$
c.s. pairs $\zeta_c(j)$ of diagonals		$[0,\bar{0}]$	$[0,\bar{1}]\cup[\bar{0},1]$	$[1,\bar{1}]$	$[1,\bar{2}]\cup[\bar{1},2]$	$[2,\bar{2}]$	$[0,2]\cup[\bar{0},\bar{2}]$
Almost positive root $\vartheta_c(j)$		$-\alpha_0$	$-\alpha_1$	α_0	$\alpha_0+\alpha_1$	$\alpha_0+2\alpha_1$	α_1
cluster variable $\psi_c(j)$		$\frac{(1+x_1)^2+x_2^2}{x_1x_2^2}$	$\frac{1+x_1}{x_2}$	x_1	x_2	$\frac{1+x_2^2}{x_1}$	$\frac{1+x_1+x_2^2}{x_1x_2}$
d-vector		$\binom{1}{2}$	$\binom{0}{1}$	$\binom{-1}{0}$	$\binom{0}{-1}$	$\binom{1}{0}$	$\binom{1}{1}$

TABLE 5. Correspondence between positions of letters in $Q_{\tau_0\tau_1}$, centrally symmetric pairs of diagonals of the hexagon, almost positive roots, and cluster variables in $\mathcal{A}(C_2)$. The d-vectors of the cluster variables correspond to the crossing vectors of the corresponding diagonals with respect to the seed triangulation $T = \zeta_{\tau_0\tau_1}(\{3,4\}) = \{[1,\bar{1}],[1,\bar{2}],[\bar{1},2]\}$. The column corresponding to the diagonals of this triangulation are shaded. See also Figure 10.

TYPE C_2, Coxeter element $c = \tau_0\tau_1$, Compatibility table								
position j in the word Q_c			1	2	3	4	5	6
	almost positive root $\vartheta_c(j)$		$-\alpha_0$	$-\alpha_1$	α_0	$\alpha_0+\alpha_1$	$\alpha_0+2\alpha_1$	α_1
		diagonals $\zeta_c(j)$	$[0,\bar{0}]$	$[0,\bar{1}]\cup[\bar{0},1]$	$[1,\bar{1}]$	$[1,\bar{2}]\cup[\bar{1},2]$	$[2,\bar{2}]$	$[0,2]\cup[\bar{0},\bar{2}]$
1	$-\alpha_0$	$[0,\bar{0}]$	-1	0	1	1	1	0
2	$-\alpha_1$	$[0,\bar{1}]\cup[\bar{0},1]$	0	-1	0	1	2	1
3	α_0	$[1,\bar{1}]$	1	0	-1	0	1	1
4	$\alpha_0+\alpha_1$	$[1,\bar{2}]\cup[\bar{1},2]$	2	1	0	-1	0	1
5	$\alpha_0+2\alpha_1$	$[2,\bar{2}]$	1	1	1	0	-1	0
6	α_1	$[0,2]\cup[\bar{0},\bar{2}]$	0	1	2	1	0	-1
d-vector $\mathbf{d}(X,y)=\mathbf{d}_c(I,j)=$ $\mathbf{d}_c(B,\beta)=\mathbf{d}(T,\delta)$			$\binom{1}{2}$	$\binom{0}{1}$	$\binom{-1}{0}$	$\binom{0}{-1}$	$\binom{1}{0}$	$\binom{1}{1}$

TABLE 6. c-compatibility table for the Coxeter element $c = \tau_0\tau_1$ of type C_2. The last line shows the d-vectors with respect to the c-cluster $I = \{3,4\}$ (corresponding to the cluster $X = \psi_c(I)$, to the c-cluster $B = \vartheta_c(I) = \{\alpha_0, \alpha_0+\alpha_1\}$, and to the centrally symmetric triangulation $T = \zeta_c(I) = \{[1,\bar{1}],[1,\bar{2}],[\bar{1},2]\}$). In general, the d-vectors are the column vectors of the submatrix of the matrix of c-compatibility degrees, with rows corresponding to I (or X, B, or T).

TYPE D_3, Coxeter element $c = \tau_1\tau_2\tau_0$, Cluster seed $I = \{1,7,8\}$									
Word Q_c: Position j in $[9]$	1	2	3	4	5	6	7	8	9
letter q_j of Q_c	τ_1	τ_2	τ_0	τ_1	τ_2	τ_0	τ_1	τ_2	τ_1
root $\mathsf{r}(I,j)$	α_1	α_2	$\alpha_0+\alpha_2$	$\alpha_1+\alpha_2$	$\alpha_0+\alpha_1+\alpha_2$	α_1	α_0	$-\alpha_0-\alpha_2$	α_0
root $\mathsf{r}(I,j)$ in the basis $\mathsf{R}(I)$	β_1	$-\beta_2-\beta_3$	$-\beta_3$	$\beta_1-\beta_2-\beta_3$	$\beta_1-\beta_3$	β_1	β_2	β_3	β_2
c.s. pairs $\zeta_c(j)$ of chords of $\mathcal{D}_3$	$2^{\mathrm{L}} \cup \bar{2}^{\mathrm{L}}$	$[0,2] \cup [\bar{0},\bar{2}]$	$0^{\mathrm{L}} \cup \bar{0}^{\mathrm{L}}$	$0^{\mathrm{R}} \cup \bar{0}^{\mathrm{R}}$	$[0,\bar{1}] \cup [\bar{0},1]$	$1^{\mathrm{R}} \cup \bar{1}^{\mathrm{R}}$	$1^{\mathrm{L}} \cup \bar{1}^{\mathrm{L}}$	$[1,\bar{2}] \cup [\bar{1},2]$	$2^{\mathrm{R}} \cup \bar{2}^{\mathrm{R}}$
Almost pos. root $\vartheta_c(j)$	$-\alpha_1$	$-\alpha_2$	$-\alpha_0$	α_1	$\alpha_1+\alpha_2$	$\alpha_0+\alpha_1+\alpha_2$	α_2	$\alpha_0+\alpha_2$	α_0
cluster variable $\psi_c(j)$	x_1	$\frac{x_1+x_2+x_1x_3}{x_2x_3}$	$\frac{x_1+x_2}{x_3}$	$\frac{(x_1+x_2)(1+x_3)}{x_1x_2x_3}$	$\frac{x_1+x_2+x_2x_3}{x_1x_3}$	$\frac{1+x_3}{x_1}$	x_2	x_3	$\frac{1+x_3}{x_2}$
d-vector	$\begin{pmatrix}-1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\-1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\-1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$

TABLE 7. Correspondence between positions of letters in $Q_{\tau_1\tau_2\tau_0}$, centrally symmetric pairs of chords in $\mathcal{D}_3$, almost positive roots, and cluster variables in $\mathcal{A}(D_3)$. The d-vectors of the cluster variables correspond to the crossing vectors of the corresponding chords with respect to the seed pseudotriangulation $T = \zeta_{\tau_1\tau_2\tau_0}(\{1,7,8\}) = \{2^{\mathrm{L}}, \bar{2}^{\mathrm{L}}, 1^{\mathrm{L}}, \bar{1}^{\mathrm{L}}, [1,\bar{2}], [\bar{1},2]\}$. The column corresponding to the chords of this pseudotriangulation are shaded. See also Figure 11.

TYPE D_3, Coxeter element $c = \tau_1\tau_2\tau_0$, Compatibility table											
position j in the word Q_c			1	2	3	4	5	6	7	8	9
	almost positive root $\vartheta_c(j)$		$-\alpha_1$	$-\alpha_2$	$-\alpha_0$	α_1	$\alpha_1+\alpha_2$	$\alpha_0+\alpha_1+\alpha_2$	α_2	$\alpha_0+\alpha_2$	α_0
		chords $\zeta_c(j)$	$2^{\mathrm{L}} \cup \bar{2}^{\mathrm{L}}$	$[0,2] \cup [\bar{0},\bar{2}]$	$0^{\mathrm{L}} \cup \bar{0}^{\mathrm{L}}$	$0^{\mathrm{R}} \cup \bar{0}^{\mathrm{R}}$	$[0,\bar{1}] \cup [\bar{0},1]$	$1^{\mathrm{R}} \cup \bar{1}^{\mathrm{R}}$	$1^{\mathrm{L}} \cup \bar{1}^{\mathrm{L}}$	$[1,\bar{2}] \cup [\bar{1},2]$	$2^{\mathrm{R}} \cup \bar{2}^{\mathrm{R}}$
1	$-\alpha_1$	$2^{\mathrm{L}} \cup \bar{2}^{\mathrm{L}}$	-1	0	0	1	1	1	0	0	0
2	$-\alpha_2$	$[0,2] \cup [\bar{0},\bar{2}]$	0	-1	0	0	1	1	1	1	0
3	$-\alpha_0$	$0^{\mathrm{L}} \cup \bar{0}^{\mathrm{L}}$	0	0	-1	0	0	1	0	1	1
4	α_1	$0^{\mathrm{R}} \cup \bar{0}^{\mathrm{R}}$	1	0	0	-1	0	0	1	1	0
5	$\alpha_1+\alpha_2$	$[0,\bar{1}] \cup [\bar{0},1]$	1	1	0	0	-1	0	0	1	1
6	$\alpha_0+\alpha_1+\alpha_2$	$1^{\mathrm{R}} \cup \bar{1}^{\mathrm{R}}$	1	1	1	0	0	-1	0	0	0
7	α_2	$1^{\mathrm{L}} \cup \bar{1}^{\mathrm{L}}$	0	1	0	1	0	0	-1	0	1
8	$\alpha_0+\alpha_2$	$[1,\bar{2}] \cup [\bar{1},2]$	0	1	1	1	1	0	0	-1	0
9	α_0	$2^{\mathrm{R}} \cup \bar{2}^{\mathrm{R}}$	0	0	1	0	1	0	1	0	-1
d-vector $\mathbf{d}(X,y) = \mathbf{d}_c(I,j) = \mathbf{d}_c(B,\beta) = \mathbf{d}(T,\delta)$			$\begin{pmatrix}-1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\-1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\-1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$

TABLE 8. c-compatibility table for the Coxeter element $c = \tau_1\tau_2\tau_0$ of type D_3. The last line shows the d-vectors with respect to the c-cluster $I = \{1,7,8\}$ (corresponding to the cluster $X = \psi_c(I)$, to the c-cluster $B = \vartheta_c(I) = \{-\alpha_1, \alpha_2, \alpha_0+\alpha_2\}$, and to the centrally symmetric triangulation $T = \zeta_c(I) = \{2^{\mathrm{L}}, \bar{2}^{\mathrm{L}}, 1^{\mathrm{L}}, \bar{1}^{\mathrm{L}}, [1,\bar{2}], [\bar{1},2]\}$). In general, the d-vectors are the column vectors of the submatrix of the matrix of c-compatibility degrees, with rows corresponding to I (or X, B, or T).

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Appendix A. Type D_4 cluster algebra

In this appendix, we illustrate further our interpretation of type D_n cluster algebras in terms of centrally symmetric pseudotriangulations, with an example in type D_4.

We consider the Coxeter element $\tau_0\tau_3\tau_2\tau_1$ of D_4. The corresponding centrally symmetric "accordion" pseudotriangulation $Z_{\tau_0\tau_3\tau_2\tau_1}$ is represented in Figure 12 (left), and the map $\zeta_{\tau_0\tau_3\tau_2\tau_1}$ is given in Table 9.

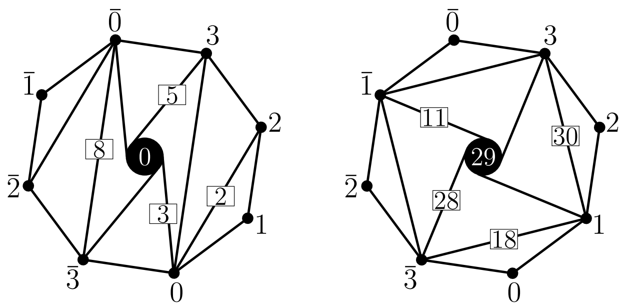

Figure 12. The centrally symmetric "accordion" pseudotriangulation $Z_{\tau_0\tau_3\tau_2\tau_1}$ (left) and the initial seed pseudotriangulation $T = \{[1,3],[\bar{1},\bar{3}],1^{\text{L}},\bar{1}^{\text{L}},[1,\bar{3}],[\bar{1},3],3^{\text{L}},\bar{3}^{\text{L}}\}$ (right). In each pseudotriangulation, the number at the center of the disk is its label in the mutation graph represented on Figure 14, and each pair of chords is labeled with the pseudotriangulation obtained when flipping it. We use the same conventions in Figures 13 and 14.

We also compute cluster variables with respect to the seed pseudotriangulation

$$T = \zeta_{\tau_0\tau_3\tau_2\tau_1}(\{6,8,14,16\}) = \{[1,3],[\bar{1},\bar{3}],1^{\text{L}},\bar{1}^{\text{L}},[1,\bar{3}],[\bar{1},3],3^{\text{L}},\bar{3}^{\text{L}}\},$$

represented in Figure 12 (right). The remaining 48 centrally symmetric pseudotriangulations of $\mathcal{D}_4$ are represented in Figure 13, and their flip graph in Figure 14.

Finally, we have computed in Table 9 the d-vectors of all cluster variables with respect to the initial seed T. For this, we can now use either the direct computation of the cluster variables, or the geometric description of d-vectors as crossing vectors, or the root function in the word Q_{c}.

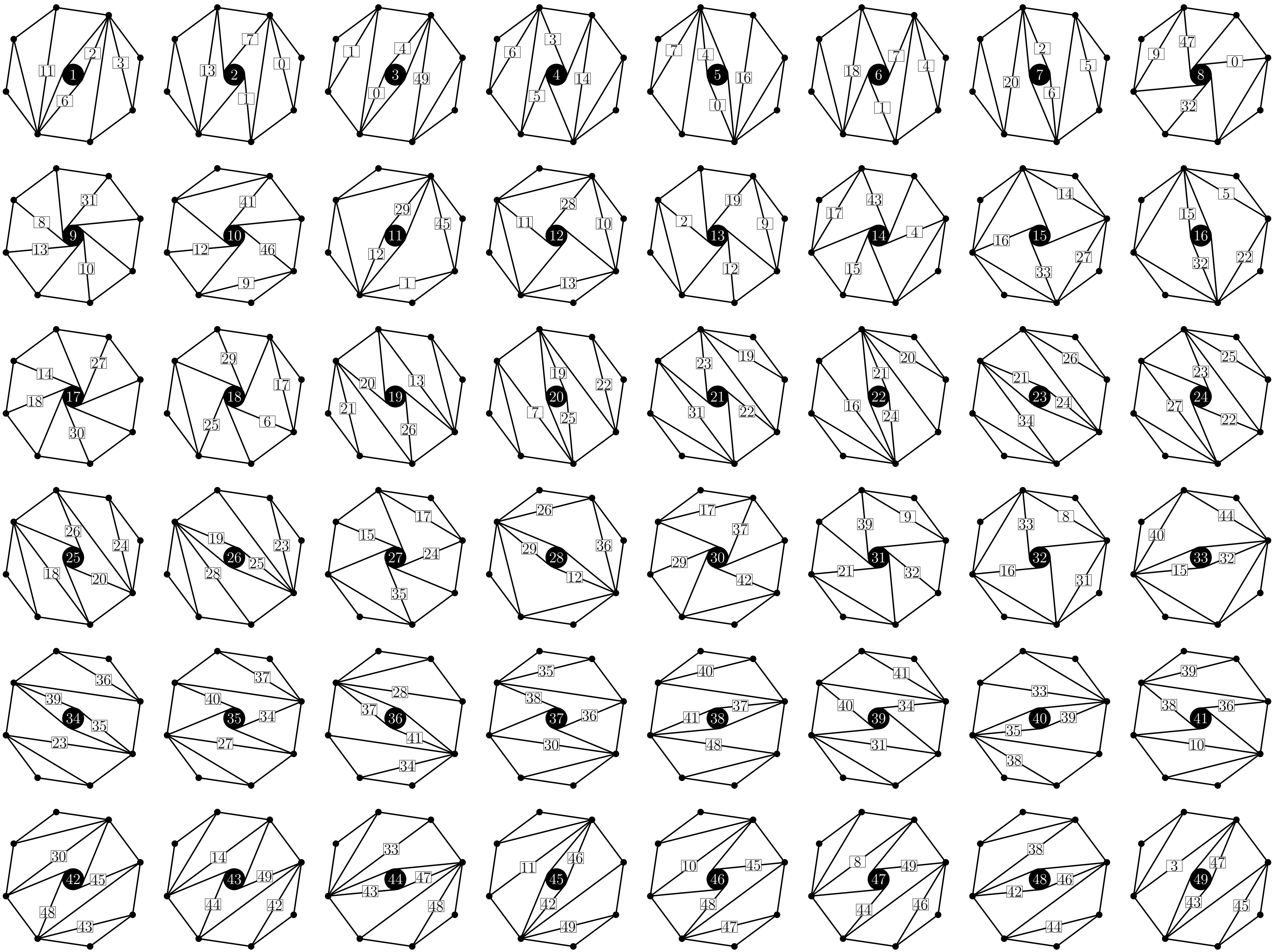

FIGURE 13. The remaining 48 centrally symmetric pseudotriangulations of the configuration $\mathcal{D}_4$. See Figure 12 for an explanation of the labeling conventions, and Figure 14 for the mutation graph.

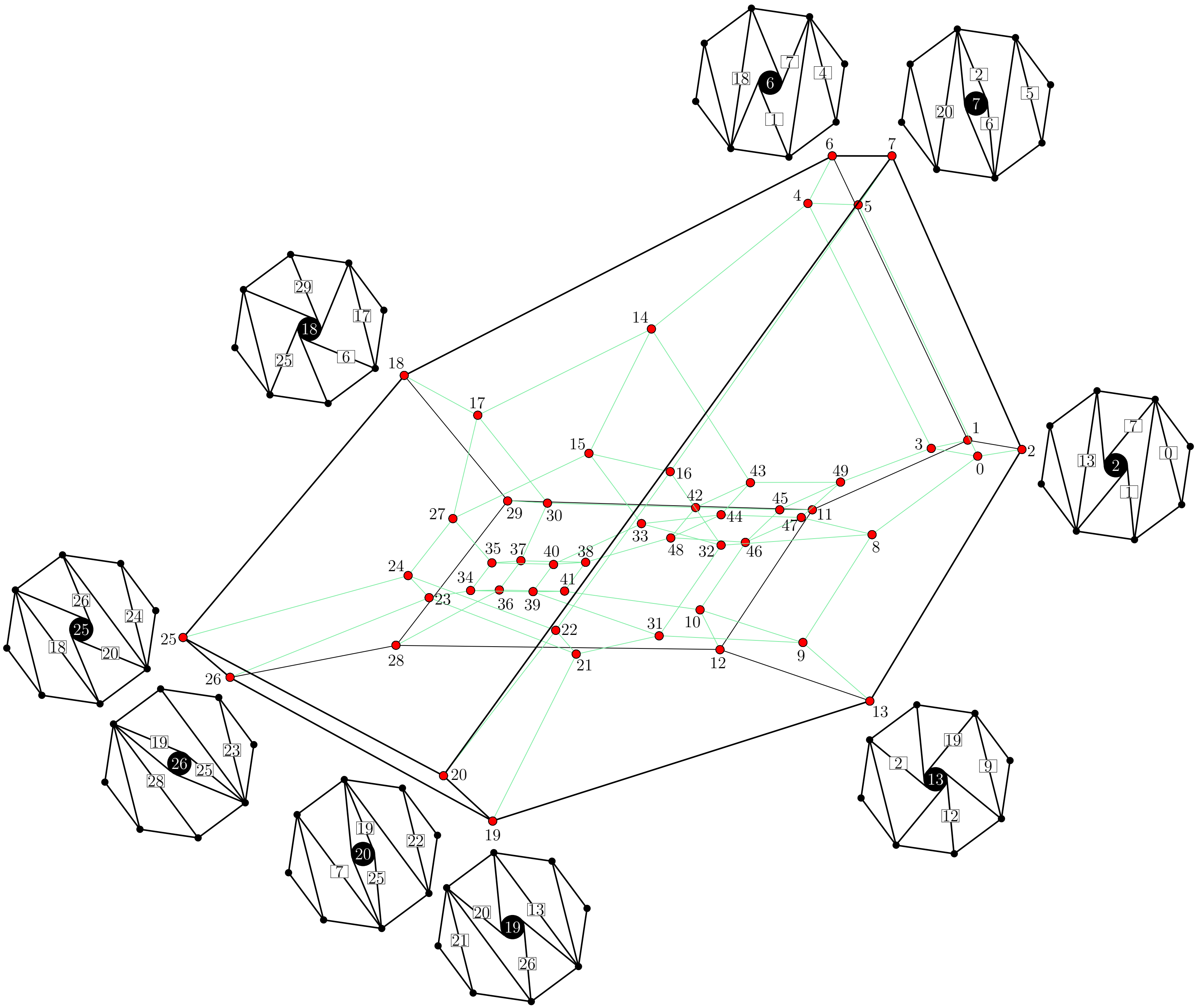

FIGURE 14. The type D_4 mutation graph. We have represented some of the corresponding centrally symmetric pseudotriangulations of $\mathcal{D}_4$ on this picture, while the others can be found on Figures 12 and 13. The underlying graph used for the representation is a Schlegel diagram of the generalized associahedron [CFZ02, HLT11, PS11].

TYPE D_4, Coxeter element $c = \tau_0\tau_3\tau_2\tau_1$, Cluster seed $I = \{6, 8, 14, 16\}$ — First half								
Word Q_c: Position j in $[16]$	1	2	3	4	5	6	7	8
Word Q_c: letter q_j of Q_c	τ_0	τ_3	τ_2	τ_1	τ_0	τ_3	τ_2	τ_1
Word Q_c: root $\mathsf{r}(I, j)$	α_0	α_3	$\alpha_0 + \alpha_2 + \alpha_3$	$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$	$\alpha_2 + \alpha_3$	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2 + \alpha_3$	$-\alpha_0$
Word Q_c: root $\mathsf{r}(I, j)$ in the basis $\mathsf{R}(I)$	$-\beta_2$	$-\beta_1 - \beta_2 - \beta_3$	$-\beta_2 - \beta_3$	$-\beta_2 - \beta_3 - \beta_4$	$-\beta_3$	β_1	$-\beta_3 - \beta_4$	β_2
c.s. pairs $\zeta_c(j)$ of chords of $\mathcal{D}_4$	$3^{\mathrm{R}} \cup \bar{3}^{\mathrm{R}}$	$[0, 2] \cup [\bar{0}, \bar{2}]$	$[0, 3] \cup [\bar{0}, \bar{3}]$	$0^{\mathrm{R}} \cup \bar{0}^{\mathrm{R}}$	$0^{\mathrm{L}} \cup \bar{0}^{\mathrm{L}}$	$[1, 3] \cup [\bar{1}, \bar{3}]$	$[0, \bar{1}] \cup [\bar{0}, 1]$	$1^{\mathrm{L}} \cup \bar{1}^{\mathrm{L}}$
Almost positive root $\vartheta_c(j)$	$-\alpha_0$	$-\alpha_3$	$-\alpha_2$	$-\alpha_1$	α_0	α_3	$\alpha_0 + \alpha_2 + \alpha_3$	$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$
cluster variable $\psi_c(j)$	$\frac{x_1 + x_3}{x_2}$	$\frac{x_1x_2 + x_1x_4 + x_2x_3 + x_3x_4}{x_1x_2x_3}$	$\frac{x_1x_2 + x_1x_4 + x_3x_4}{x_2x_3}$	$\frac{x_1x_2 + x_1x_4 + x_2x_3 + x_3x_4}{x_2x_3x_4}$	$\frac{x_2 + x_4}{x_3}$	x_1	$\frac{x_1x_2 + x_1x_4 + x_2x_3}{x_3x_4}$	x_2
d-vector	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}-1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0\\-1\\0\\0\end{pmatrix}$

TYPE D_4, Coxeter element $c = \tau_0\tau_3\tau_2\tau_1$, Cluster seed $I = \{6, 8, 14, 16\}$ — Second half								
Word Q_c: Position j in $[16]$	9	10	11	12	13	14	15	16
Word Q_c: letter q_j of Q_c	τ_0	τ_3	τ_2	τ_1	τ_0	τ_3	τ_2	τ_1
Word Q_c: root $\mathsf{r}(I, j)$	α_1	$\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3$	$\alpha_0 + \alpha_1 + \alpha_2$	$\alpha_1 + \alpha_2$	$\alpha_0 + \alpha_2$	$-\alpha_2 - \alpha_3$	α_2	$-\alpha_1$
Word Q_c: root $\mathsf{r}(I, j)$ in the basis $\mathsf{R}(I)$	$-\beta_4$	$\beta_1 - \beta_3 - \beta_4$	$\beta_1 - \beta_4$	$\beta_1 + \beta_2 - \beta_4$	β_1	β_3	$\beta_1 + \beta_2$	β_4
c.s. pairs $\zeta_c(j)$ of chords of $\mathcal{D}_4$	$1^{\mathrm{R}} \cup \bar{1}^{\mathrm{R}}$	$[0, \bar{2}] \cup [\bar{0}, 2]$	$[1, \bar{2}] \cup [\bar{1}, 2]$	$2^{\mathrm{R}} \cup \bar{2}^{\mathrm{R}}$	$2^{\mathrm{L}} \cup \bar{2}^{\mathrm{L}}$	$[1, \bar{3}] \cup [\bar{1}, 3]$	$[2, \bar{3}] \cup [\bar{2}, 3]$	$3^{\mathrm{L}} \cup \bar{3}^{\mathrm{L}}$
Almost positive root $\vartheta_c(j)$	$\alpha_2 + \alpha_3$	$\alpha_0 + \alpha_2$	$\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3$	α_2	$\alpha_0 + \alpha_1 + \alpha_2$	$\alpha_1 + \alpha_2 + \alpha_3$	$\alpha_1 + \alpha_2$	α_1
cluster variable $\psi_c(j)$	$\frac{x_1 + x_3}{x_4}$	$\frac{x_1x_2 + x_1x_4 + x_2x_3 + x_3x_4}{x_1x_3x_4}$	$\frac{x_1x_2 + x_2x_3 + x_3x_4}{x_1x_4}$	$\frac{x_1x_2 + x_1x_4 + x_2x_3 + x_3x_4}{x_1x_2x_4}$	$\frac{x_2 + x_4}{x_1}$	x_3	$\frac{x_1x_4 + x_2x_3 + x_3x_4}{x_1x_2}$	x_4
d-vector	$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\-1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}$

TABLE 9. Correspondence between positions of letters in $Q_{\tau_0\tau_3\tau_2\tau_1}$, centrally symmetric pairs of chords in $\mathcal{D}_4$, almost positive roots, and cluster variables in $\mathcal{A}(D_4)$. The d-vectors of the cluster variables correspond to the crossing vectors of the corresponding chords with respect to the seed pseudotriangulation $T = \zeta_{\tau_0\tau_3\tau_2\tau_1}(\{6, 8, 14, 16\}) = \{[1, 3], [\bar{1}, \bar{3}], 1^{\mathrm{L}}, \bar{1}^{\mathrm{L}}, [1, \bar{3}], [\bar{1}, 3], 3^{\mathrm{L}}, \bar{3}^{\mathrm{L}}\}$. The column corresponding to the chords of this pseudotriangulation are shaded. See also Figure 12.

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