MINIMUM MAXIMAL MATCHINGS IN PERMUTAHEDRA

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Abstract. We prove that the minimal size $\mathcal{M}(\Pi_n)$ of a maximal matching in the permutahedron Π_n is asymptotically $n!/3$. On the one hand, we obtain a lower bound $\mathcal{M}(\Pi_n) \geq n!(n-1)/(3n-2)$ by considering 4-cycles in the permutahedron. On the other hand, we obtain an asymptotical upper bound $\mathcal{M}(\Pi_n) \leq n!(1/3 + o(1))$ by multiple applications of Hall's theorem (similar to the approach of Forcade for the hypercube [?]) and an exact upper bound $\mathcal{M}(\Pi_n) \leq n!/3$ by an explicit construction. We also derive bounds on minimum maximal matchings in products of permutahedra.

Keywords. Maximal matching, Permutahedron, Cartesian product.

1. Introduction

Matchings are a fundamental concept in mathematics and computer science. Different variants of matchings can be used to model problems or as subroutines in algorithms. While perfect matchings and maximum cardinality matchings are well understood, the landscape for small maximal matchings is less clear. More specifically, given a graph G, we are interested in

$$\mathcal{M}(G) := \min \{|M| \mid M \text{ is a maximal matching in } G\}.$$

(The quantity $\mathcal{M}(G)$ is also known as the *edge domination number* of G [?].)

For example, while a maximum cardinality matching can be found in polynomial time [?], finding a minimum maximal matching is an NP-hard problem [?], even when the graph is regular and bipartite [?]. Assuming the Unique Games Conjecture, it is also NP-hard to approximate a minimum maximal matching with a constant better than two [?]. For general bounds, there are a couple of results. First, $\mathcal{M}(G) \geq \frac{m}{2\Delta-1}$ where m is the number of edges and Δ is the maximum degree [?]. Second, $\frac{m}{\Delta^L} \leq \mathcal{M}(G) \leq m - \Delta^L$ where m is the number of edges and Δ^L

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is the maximum degree of the line graph of G [?]. Thus, it is natural to ask for estimates and solutions for this problem for special graph classes.

Let Q_n be the graph with 2^n vertices corresponding to the binary strings of length n, and where two vertices are adjacent if their corresponding strings differ in exactly one position. It is the skeleton of the n-dimensional *hypercube*, and the cover graph of the *Boolean lattice*. Let Π_n be the graph with $n!$ vertices corresponding to the permutations of $[n]$, and where two vertices are adjacent if they differ by a transposition of two adjacent elements. It is the skeleton of the $(n-1)$-dimensional *permutahedron*, and the cover graph of the *weak order*.

Concerning the minimal size of a maximal matching in the hypercube Q_n, Forcade [?] showed that

$$\lim_{n\to\infty} \frac{1}{2^n}\mathcal{M}(Q_n) = \frac{1}{3}.$$

Our main contribution is an analogous result for the permutahedron Π_n. We obtain the following asymptotically tight bounds for $\mathcal{M}(\Pi_n)$.

Theorem 1. *The minimal size $\mathcal{M}(\Pi_n)$ of a maximal matching of the permutahedron Π_n is bounded by*

$$\frac{n-1}{3n-2}n! \le \mathcal{M}(\Pi_n) \le \frac{1}{3}n!.$$

The lower bound is obtained from a more general bound, where we argue in terms of the distribution of 4-cycles in any graph (Section 3). This generalizes Forcade's lower bound argument for the hypercube [?] and is better than the aforementioned lower bounds.

The upper bound is obtained by a simple and explicit construction, obtained by combining maximal matchings of Π_4 in subgraphs of Π_n (Section 5). For any vertex, it takes linear time in n to output its neighbor in the matching, if any.

We also discuss how Forcade's argument for the upper bound for the hypercube [?] can be packaged into a general framework for certain bipartite graphs (Section 4), based on Hall's theorem. Note that our general lower bound above works best on some regular graphs, while this framework for the upper bound only works on some bipartite graphs. Hence, as a next step, we focus on the bipartite regular graphs for which we can obtain asymptotically tight bounds.

For this, we turn to the Cartesian products of permutahedra. These objects also correspond to a rich class of polytopes, namely the bipartite regular quotientopes [?]. While the general bounds above are asymptotically tight in this case, we additionally give an explicit and tighter upper bound construction, based on our construction for the permutahedron (Section 6).

Theorem 2. *Let $n_1, \dots, n_k \ge 2$ be integers, and let $n := n_1 + \dots + n_k$. Let Π be the Cartesian product of $\Pi_{n_1}, \dots, \Pi_{n_k}$. Then*

$$\frac{n-k}{3n-3k+1}|V(\Pi)| \le \mathcal{M}(\Pi) \le \begin{cases} \left(\frac{1}{3} + O(n^{-1/2})\right)|V(\Pi)| & \text{if } \Pi \text{ is a hypercube,} \\ \frac{1}{3}|V(\Pi)| & \text{otherwise.} \end{cases}$$

2. Notation and preliminaries

In this section, we introduce the notation used throughout this paper. We denote the disjoint union by $\sqcup$.

Matchings. A *matching* M in a graph $G := (V, E)$ is a subset of E such that every vertex in V is incident to at most one edge in M. A vertex in V is *covered* (resp. *exposed*) if it is incident to one (resp. no) edge in M. The matching M *saturates* a subset X of V if all vertices in X are covered. A *maximal matching* is a matching that is maximal with respect to inclusion.

Permutations. Let S_n denote the *symmetric group*, that is, the group of permutations of $[n] := \{1, \dots, n\}$. We write a permutation σ in *one-line notation*, meaning as a word $\sigma_1 \dots \sigma_n$. For a permutation σ, let $\mathrm{inv}(\sigma)$ denote its *inversion set*, i.e., the set of pairs (i, j) such that $i < j$ and $\sigma_i > \sigma_j$. For $i \in [n-1]$, let $\tau_i := (i\ i+1)$ be the *simple transposition* that only exchanges i and $i+1$. We obtain $\sigma\tau_i$ from σ by swapping the entries at positions i and $i+1$.

Permutahedra. The *permutahedron* Π_n is the Cayley graph of S_n for the simple transpositions $\{\tau_1, \dots, \tau_{n-1}\}$. In other words, its vertices are the permutations of $[n]$ and its edges are the pairs of permutations $\{\sigma, \sigma'\}$ such that $\sigma = \sigma'\tau_i$ for some $i \in [n-1]$. In particular, Π_n is $(n-1)$-regular. Moreover, every edge in Π_n corresponds to a unique transposition τ_i for $i \in [n-1]$, and we call such an edge a *τ_i-edge*. For $i \neq j$ in $[n-1]$, we call *$\tau_i\tau_j$-cycle* any cycle obtained by alternating between τ_i-edges and τ_j-edges. The $\tau_i\tau_j$-cycles are 4-cycles if $|i-j| > 1$ and 6-cycles otherwise.

3. Lower bound for general graphs

In this section, we derive a lower bound for the size of a maximal matching in a graph G in terms of the number of its 4-cycles (Proposition 1). Our bound specializes to that of [?] for the hypercube (Example 1). Applied to the permutahedron, this yields an asymptotically tight lower bound (Corollary 1).

Definition 1. For $\alpha \in \mathbb{Z}_{>0}$, we say that a graph G is *α-heavy* if, for every edge e in G, there are at least α induced 4-cycles such that e is the only common edge of any two of these cycles.

Proposition 1. *If $G = (V, E)$ is α-heavy and has average degree d and maximum degree Δ, then any maximal matching of G has cardinality at least*

$$\frac{d}{4\Delta - \alpha - 2}|V|.$$

Proof. Let M be a maximal matching in G. An *M-edge* is an edge in M. A *1-edge* (resp. *2-edge*) is an edge in G that is incident to exactly one M-edge (resp. two M-edges). Note in particular that 1- and 2-edges are not in M since they are adjacent to some M-edges. We denote by m_0, m_1 and m_2 the number of M-edges, 1-edges, and 2-edges, respectively.

Since G has average degree d and all edges are either M-edges, 1-edges, or 2-edges, we obtain

$$d|V| = 2(m_0 + m_1 + m_2). \tag{1}$$

Double counting the number of adjacent pairs of edges with precisely one M-edge, we obtain

$$m_1 + 2m_2 \leq 2(\Delta - 1)m_0, \tag{2}$$

since each 1-edge (resp. 2-edge) is adjacent to one (resp. two) M-edges, while each M-edge is adjacent to at most $2(\Delta - 1)$ edges that are either 1- or 2-edges.

Finally, double counting the number of adjacent pairs of edges with one M-edge and one 2-edge, we obtain

$$\alpha m_0 \leq 2m_2. \tag{3}$$

Indeed, each 2-edge is adjacent to precisely two M-edges. Conversely, consider an M-edge e and a 4-cycle C that contains e. One of the two edges of C adjacent to e must be a 2-edge, since otherwise we could add the remaining edge of C to M to create a larger matching, contradicting the maximality of M. The inequality (3) thus follows from the fact that G is α-heavy.

The sum (1) + 2(2) + (3) gives $d|V| \leq (4\Delta - \alpha - 2)m_0$. □

Example 1. For the n-dimensional hypercube Q_n, we have $d = \Delta = n$ and $\alpha = n - 1$ so that $m \geq |V|n/(3n-1)$, recovering the bound in [?].

Inner nodes n	Vertices	Edges	Matching	Independent
2	2	1	1	1
3	5	5	2	2
4	14	21	5	6
5	42	84	14	16
6	132	330	44	50

Table 1. The sizes of minimum maximal matchings and maximum independent sets in the associahedron for $n \leq 6$.

In general, the lower bound of Proposition 1 works best when the graph is regular (i.e., $d = \Delta$) and $\alpha = \Delta - c$, for some constant c, ideally $c = 1$ (as in the case of the hypercube). Applying Proposition 1 to the permutahedron yields the following bound.

Corollary 1. *Every maximal matching of Π_n has at least $n!(n-1)/(3n-2)$ edges.*

Proof. This is a direct application of Proposition 1 since Π_n is $(n-1)$-regular and $(n-4)$-heavy. Indeed, any edge e of Π_n is a τ_i-edge for some $i \in [n-1]$. For any $j \in [n-1]$ with $|i-j| > 1$, let C_j be the 4-cycle obtained by alternating τ_i- and τ_j-edges, starting with e. Then e is the only common edge of any two C_j and C_k with $j \neq k$, and there are at least $n-4$ such cycles given by the different $j \in [n-1] \smallsetminus \{i-1, i, i+1\}$. $\square$

Remark 1. Besides the hypercubes and the permutahedra, Proposition 1 can also be applied to the graphs of other classical polytopes:

- **Associahedra.:** The *associahedron* is the graph whose vertices are the binary trees with n internal nodes and whose edges are tree rotations. It is regular with degree $n-1$ and $(n-5)$-heavy, so that any maximal matching in the associahedron has at least $\frac{(n-1)}{(3n-1)(n+1)}\binom{2n}{n}$ edges. Using integer linear programming we calculated the minimum size of maximal matchings in the associahedron for $n \leq 6$; see Table 1. Since exposed vertices form an independent set, we also included the maximal size of an independent set.
- **Coxeter permutahedra.:** A *Coxeter permutahedron* is the convex hull of a generic point under the action of a finite Coxeter group W [?, ?]. It is regular of degree n and $(n-\delta)$-heavy, where n is the rank of W and δ is the maximal degree of the Dynkin diagram of W, hence any maximal matching in the Coxeter permutahedron has at least $\frac{n}{3n+\delta-2}|W|$ edges.

In contrast, it does not apply as such to all graphical zonotopes, since they can fail to be α-heavy (some edges might appear in no 4-cycle). For graphical zonotopes, we would need an improved version of Proposition 1 averaging the number of disjoint 4-cycles containing an edge.

4. Upper bound for bipartite graphs

In this section, we obtain an upper bound on the minimal size of a maximal matching of certain bipartite graphs (Proposition 2). Our approach is similar to that of [?] for the hypercube (Example 2). Applied to the permutahedron, this yields an asymptotically tight upper bound (Corollary 2). The proof of Proposition 2 uses Hall's classical matching theorem.

Theorem 3 (Hall's theorem [?])**.** *Let G be a bipartite graph with two parts X and Y. Then G has a matching that saturates X if and only if for every $X' \subseteq X$, the number of neighbors of X' in Y is at least $|X'|$.*

Proposition 2. *Consider a graph $G := (V, E)$ such that*

(i) $V = V_0 \dot{\bigcup} \dots \dot{\bigcup} V_\ell$ *and* $E \cap (V_i \times V_j) \neq \varnothing$ *implies* $|i - j| = 1$,
(ii) *for any* $0 \le i < \lfloor \ell/2 \rfloor$ *and* $X \subseteq V_i$*, the number of neighbors of* X *in* V_{i+1} *is at least* $|X|$,
(iii) *for any* $\lceil \ell/2 \rceil < i \le \ell$ *and* $X \subseteq V_i$*, the number of neighbors of* X *in* V_{i-1} *is as least* $|X|$.

Then G *admits a maximal matching of cardinality at most*

$$|V|/3 + 6 \max\{|V_{\lfloor \ell/2 \rfloor}|, |V_{\lceil \ell/2 \rceil}|\}.$$

Proof. For $k \in [\ell - 1]$, we write G_k^- (resp. G_k^+, resp. $G_k^\pm$) for the subgraph of G induced by $V_{k-1} \cup V_k$ (resp. $V_k \cup V_{k+1}$, resp. $V_{k-1} \cup V_k \cup V_{k+1}$). For $0 < k < \lfloor \ell/2 \rfloor$, we obtain by two applications of Theorem 3 (Hall's theorem) using (ii) that

- there exists a matching M_k^- of G_k^- which saturates V_{k-1},
- there exists a matching M_k^+ of G_k^+ so that a vertex of V_k is covered in M_k^+ if and only if it is exposed in M_k^-.

Therefore, the union $M_k^\pm := M_k^- \cup M_k^+$ is a matching of $G_k^\pm$ which covers both V_{k-1} and V_k. Moreover, $|M_k^\pm| = |V_k|$ since all edges of $M_k^\pm$ are incident to V_k. Similarly, for $\lceil \ell/2 \rceil < k < \ell$, there is a matching $M_k^\pm$ of $G_k^\pm$ which covers both V_k and V_{k+1}, and with $|M_k^\pm| = |V_k|$.

Let $p \in [3]$ be such that $\sum_{k \in [\ell],\, k \equiv p \pmod 3} |V_k| \le |V|/3$. Let M be the union of the matchings $M_k^\pm$ for $k \in [\ell - 1] \smallsetminus \{\lfloor \ell/2 \rfloor, \lceil \ell/2 \rceil\}$ with $k \equiv p \pmod 3$. Note that M is a matching (since the graphs $G_k^\pm$ for $k \equiv p \pmod 3$ are vertex disjoint) and that $|M| \le |V|/3$ (since $|M_k^\pm| = |V_k|$). Let M' be a maximal matching containing M. Since $M_k^\pm$ covers V_{k-1} and V_k when $0 < k < \lfloor \ell/2 \rfloor$ (resp. V_k and V_{k+1} when $\lceil \ell/2 \rceil < k < \ell$), all edges of $M' \smallsetminus M$ are incident to $V_0 \cup V_{\lfloor \ell/2 \rfloor} \cup V_{\lfloor \ell/2 \rfloor - 1} \cup V_{\lceil \ell/2 \rceil} \cup V_{\lceil \ell/2 \rceil + 1} \cup V_\ell$. As (ii) and (iii) imply that $|V_k| \le \max\{|V_{\lfloor \ell/2 \rfloor}|, |V_{\lceil \ell/2 \rceil}|\}$ for all k, we indeed obtained a maximal matching M' with $|M'| \le |V|/3 + 6 \max\{|V_{\lfloor \ell/2 \rfloor}|, |V_{\lceil \ell/2 \rceil}|\}$. ☐

Example 2. The n-dimensional hypercube Q_n satisfies the assumptions of Proposition 2. For $0 \le k \le n$, denote by V_k the set of binary strings of length n with precisely k occurrences of 1. If $X \subseteq V_k$ and Y denotes its neighborhood in V_{k+1}, we have $(n-k)|X| = (k+1)|Y|$, which proves (ii) and (iii) in Proposition 2. It follows that Q_n admits a maximal matching of cardinality at most $2^n/3 + 6\binom{n}{\lfloor n/2 \rfloor} = 2^n/3\big(1 + O(n^{-1/2})\big)$. This was precisely the approach of [?].

Finally, we observe that Proposition 2 yields an asymptotically tight bound for the minimal size of a maximal matching in the permutahedron.

Corollary 2. *The permutahedron* Π_n *satisfies the assumptions of Proposition 2. Hence, it admits a maximal matching of cardinality at most*

$$n!\Big(\frac{1}{3} + \frac{6}{\sqrt{2\pi}} n^{-3/2} + o(n^{-3/2})\Big).$$

Proof. Recall that we denote by $\operatorname{inv}(\sigma) := \big\{(i,j) \in [n]^2 \;\big|\; i < j \text{ and } \sigma_i > \sigma_j\big\}$ the inversion set and by $\operatorname{asc}(\sigma) := \{i \in [n-1] \mid \sigma_i < \sigma_{i+1}\}$ the ascent set of a permutation σ of $[n]$. For $0 \le k \le \binom{n}{2} =: \ell$, we denote by $V_i := \{\sigma \in S_n \mid |\operatorname{inv}(\sigma)| = k\}$ the set of permutations with precisely k inversions. Let $\mathbb{R}S_n$ be the real vector space with basis indexed by S_n. Consider the linear map $U : \mathbb{R}S_n \to \mathbb{R}S_n$ defined by $U(\sigma) = \sum_{i \in \operatorname{asc}(\sigma)} i \cdot \sigma\tau_i$. Gaetz and Gao [?] showed that $U^{\ell - 2k}$ is an isomorphism between $\mathbb{R}V_k$ and $\mathbb{R}V_{\ell-k}$ for $0 \le k \le \lfloor k/2 \rfloor$. This immediately implies that U is injective.

Consider now a subset X of V_k and its neighborhood Y in V_{k+1}. Since U is injective, the image by U of the subspace of $\mathbb{R}V_i$ generated by X has dimension $|X|$. But by definition of U, this image is contained in the subspace of $\mathbb{R}V_{k+1}$ generated by Y, which has dimension $|Y|$. Hence, we obtain that $|X| \le |Y|$. This shows (ii), and the proof of (iii) is symmetric.

Finally, $M_n := \max\{|V_{\lfloor \ell/2 \rfloor}|, |V_{\lceil \ell/2 \rceil}|\}$ is known as the *Kendall-Mann number* (`A000140`), the row maximum of table of Mahonian numbers (`A008302`). It is known (it follows for instance from [?]) that $M_n \sim 6n^{n-1}/e^n \sim n! \cdot n^{-3/2} \cdot 6/\sqrt{2\pi}$, which yields our asymptotic bound. □

Remark 2. By using Hall's theorem, the proof of Proposition 2 is not constructive. For the application to the hypercube in Example 2, we can make it constructive by using a symmetric chain decomposition in the hypercube. (Note that this decomposition is a well known concept in partial order theory; we define it here in graph theoretic terms.) A *symmetric chain* in the n-dimensional hypercube is a path $(x_t, x_{t+1}, \dots, x_{n-t})$ for some $t \in \lfloor n/2 \rfloor$ such that x_i has exactly i occurrences of 1 for $i \in \{t, \dots, n-t\}$. A *symmetric chain decomposition* in the hypercube is a partition of its vertices into symmetric chains. Greene and Kleitman [?] described a simple construction of such a decomposition as follows. For a binary word x, we interpret the 0s in x as opening brackets and the 1s as closing brackets. For example, if $x = 1000110$, we interpret x as)((())(. By matching the opening and closing brackets in the natural way, we can obtain the symmetric chain containing x by flipping the rightmost unmatched 1 or the leftmost unmatched 0, until no more unmatched bits can be flipped. In the example above, the chain that contains x is $(0000110, 1000110, 1100110, 1100111)$. The chains in this decomposition can be used to explicitly describe the matchings M_k^+ and M_k^- in the proof of Proposition 2 for the hypercube. For the permutahedron, however, it is unknown whether an analogous symmetric chain decomposition exists, and as far as we are aware, there is no constructive proof of (ii) and (iii).

5. Explicit construction for permutahedra

In this section, we construct an explicit maximal matching of size $n!/3$ in the permutahedron Π_n. This bound is constructive and tighter than the upper bound in Corollary 2, and thus gives an alternative proof that the lower bound of Corollary 1 is asymptotically tight. Our construction involves three steps:

(1) We first define two maximal matchings M_+ and M_- of Π_4 with 8 edges.
(2) We then construct a matching M of Π_n with $n!/3$ edges by transporting M_+ and M_- to certain maximal matchings in each subgraph of Π_n induced by permutations with a fixed suffix of length $n-4$.
(3) We finally prove that M is maximal.

5.1. Two maximal matchings of Π_4.

We consider the two matchings M^+ and M^- in Π_4 of Fig. 1. Note that

- both M^+ and M^- are maximal matchings of Π_4 with 8 edges,
- the sets E^+ and E^- of exposed vertices of M^+ and M^- are disjoint.

5.2. Combining maximal matchings.

We now combine copies of the matchings M^+ and M^- to create a matching M of Π_n with $n!/3$ edges. Let S be the set of duplicate-free strings in $[n]$ of length $n-4$. For $s \in S$, denote by Π^s the subgraph of Π_n induced by permutations with suffix s. Note that $\Pi_n = \bigsqcup_{s \in S} \Pi^s$. Denote by $\bar{s} = \{\bar{s}_1 < \bar{s}_2 < \bar{s}_3 < \bar{s}_4\}$ the set of elements in $[n]$ that do not occur in s. For a permutation π of $[4]$, let $\bar{s}_\pi := \bar{s}_{\pi_1}\bar{s}_{\pi_2}\bar{s}_{\pi_3}\bar{s}_{\pi_4}$ and define $\phi_s(\pi) := \bar{s}_\pi s \in \Pi^s$. Observe that ϕ_s defines a graph isomorphism from Π_4 to Π^s. Define $\varepsilon(s) := (-1)^{|\operatorname{inv}(s)| + \Sigma(\bar{s})}$, where $\Sigma(\bar{s}) := \bar{s}_1 + \bar{s}_2 + \bar{s}_3 + \bar{s}_4$ and $\operatorname{inv}(s) := \{(i,j) \in [n-4]^2 \mid i < j \text{ and } s_i > s_j\}$ is the inversion set of s. Finally, define

$$M := \bigsqcup_{s \in S} \phi_s(M^{\varepsilon(s)}).$$

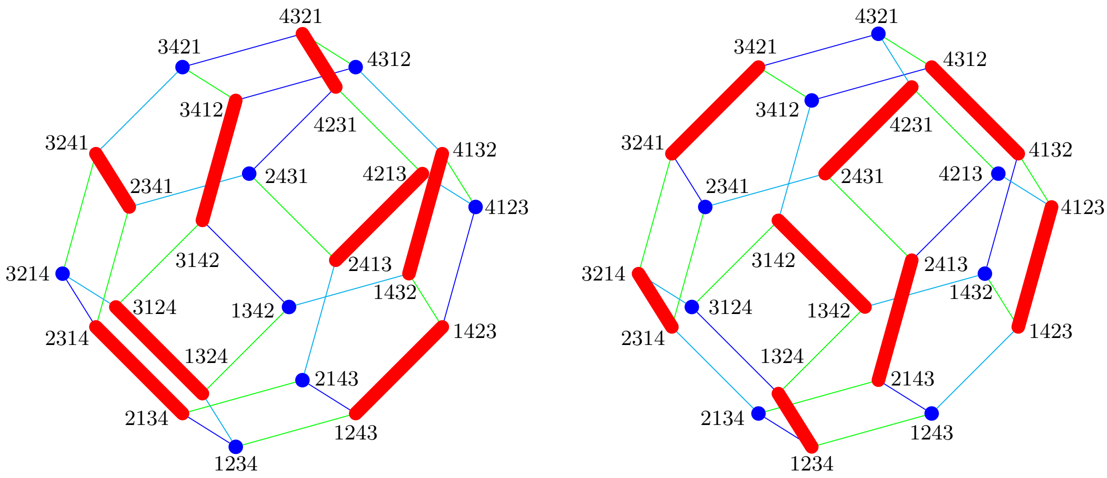

FIGURE 1. Matching edges (red) and exposed vertices (blue) for M^+ (left) and M^- (right). All other edges are colored according to the exchange position.

Example 3. For instance, Fig. 2 illustrates the matching M when $n = 5$. Its edges are

(23451, 24351)	(13542, 15342)	(12453, 14253)	(12534, 15234)	(12345, 13245)
(24531, 42531)	(15432, 51432)	(14523, 41523)	(15324, 51324)	(13425, 31425)
(25341, 52341)	(14352, 41352)	(15243, 51243)	(13254, 31254)	(14235, 41235)
(32541, 35241)	(31452, 34152)	(21543, 25143)	(21354, 23154)	(21435, 24135)
(34251, 43251)	(34512, 43512)	(24153, 42153)	(23514, 32514)	(23145, 32145)
(35421, 53421)	(35142, 53142)	(25413, 52413)	(25134, 52134)	(24315, 42315)
(43521, 45321)	(41532, 45132)	(42513, 45213)	(31524, 35124)	(32415, 34215)
(52431, 54231)	(53412, 54312)	(51423, 54123)	(52314, 53214)	(41325, 43125)

and its exposed vertices are

12354	13524	21345	23541	31245	34125	41253	43215	51234	53241
12435	14325	21453	24513	31542	34521	42135	45123	51342	54132
12543	14532	21534	25314	32154	35214	42351	45231	52143	54213
13452	15423	23415	25431	32451	35412	43152	45312	53124	54321

Theorem 4. *The set M is a maximal matching in Π_n of size $n!/3$.*

Proof. We have $|M| = \sum_{s\in S} |M^{\varepsilon(s)}| = 8|S| = n!/3$ since $M = \bigsqcup_{s\in S} \phi_s(M^{\varepsilon(s)})$. Moreover, M is a matching since $\Pi_n = \bigsqcup_{s\in S} \Pi^s$ and any edge of $\phi_s(M^{\varepsilon(s)})$ lies in Π^s. We thus just need to prove that M is maximal, and this is the purpose of the next section. $\square$

Remark 3. For later use, we point out that we can in fact define two maximal matchings $M^\bullet := \bigsqcup_{s\in S} \phi_s(M^{\varepsilon(s)})$ and $M^\circ := \bigsqcup_{s\in S} \phi_s(M^{-\varepsilon(s)})$ of the permutahedron Π_n of size $n!/3$. The sets of exposed vertices $E^\bullet$ and E° of these matchings $M^\bullet$ and M° are given by

$$E^\bullet = \bigsqcup_{s\in S} \phi_s(E^{\varepsilon(s)}) \qquad \text{and} \qquad E^\circ = \bigsqcup_{s\in S} \phi_s(E^{-\varepsilon(s)}).$$

Hence, $E^\bullet \cap E^\circ = \varnothing$ since $E^+ \cap E^- = \varnothing$.

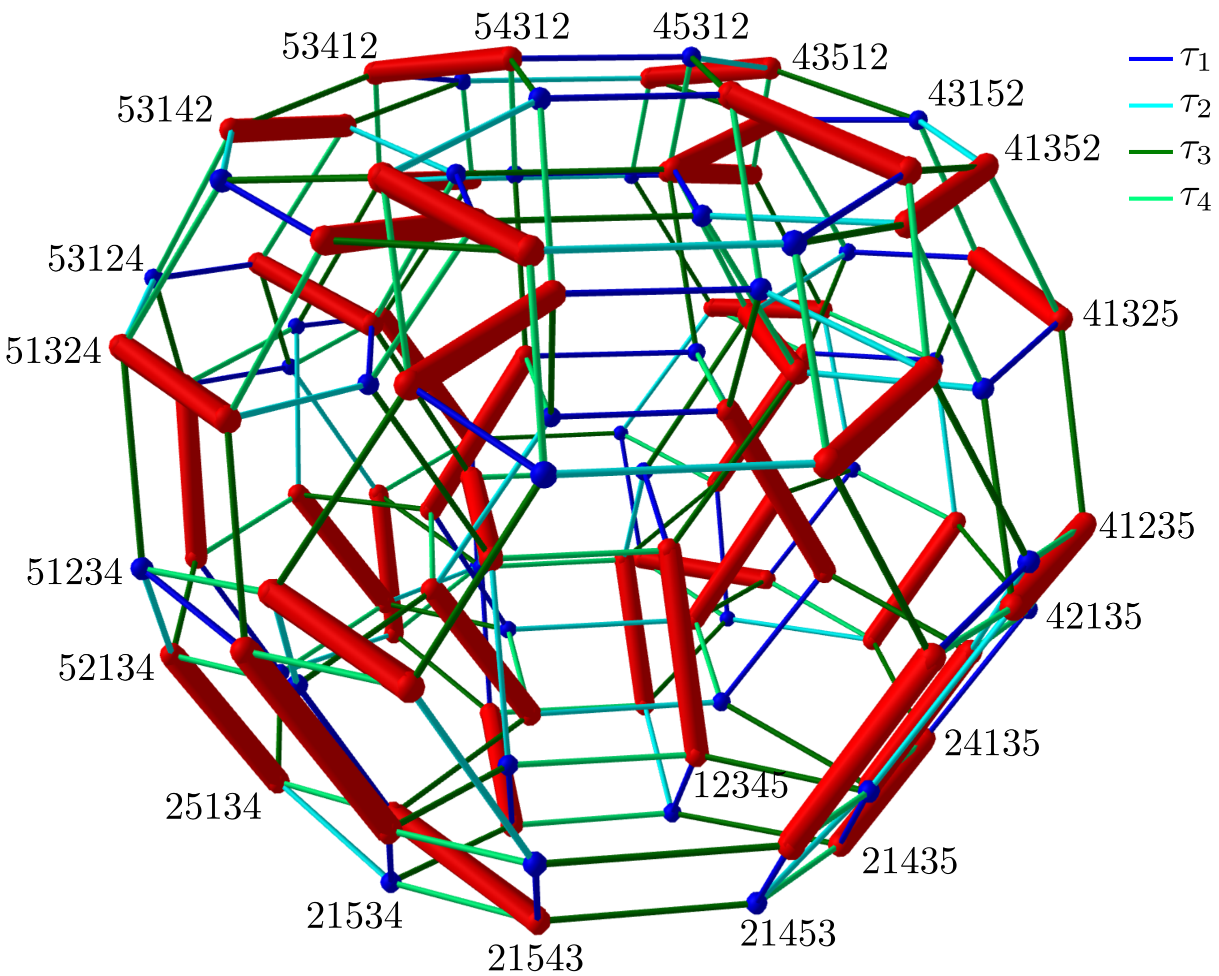

FIGURE 2. Matching edges (red) and exposed vertices (blue) for the matching M when $n = 5$. All other edges are colored according to the exchange position. The coordinates of the embedding were communicated to us by Nathan Carter. An animated 3d version can be found at `https://tinyurl.com/maximalMatchingPermutahedron`.

5.3. **Proof of maximality.** In this section, we prove that M is maximal. Assume by means of contradiction that M admits two exposed vertices $\sigma, \sigma' \in S_n$ with $\sigma = \sigma\tau_i$ for some $i \in [n-1]$. Let s be the suffix formed by the last $n-4$ letters of σ, and π be the permutation of $[4]$ such that $\sigma = \phi_s(\pi)$. Similarly, let s' and π' be such that $\sigma' = \phi_{s'}(\pi')$.

If $i < 4$, then $s = s'$ so that σ and σ' belong to the same $\Pi^s = \Pi^{s'}$. This contradicts the maximality of $M^{\varepsilon(s)}$. If $i > 4$, then $\bar{s} = \bar{s}'$ and $|\operatorname{inv}(s)| = |\operatorname{inv}(s')| \pm 1$, so that $\varepsilon(s) \neq \varepsilon(s')$. As $\bar{s} \in E^{\varepsilon(s)}$ and $\bar{s}' \in E^{\varepsilon(s')}$, this contradicts that $E^+ \cap E^- = \varnothing$.

We can thus assume from now on that $i = 4$. Assume moreover without loss of generality that $\sigma_4 < \sigma_5$ and set $t := |\{j \in [3] \mid \sigma_4 < \sigma_j < \sigma_5\}|$. For $j \in [3]$, we have $\bar{s}_{\pi_j} = \sigma_j = \sigma'_j = \bar{s}'_{\pi'_j}$. Hence, we have

$$\pi_j = \begin{cases} \pi'_j + 1 & \text{if } \sigma_4 < \sigma_j < \sigma_5, \\ \pi'_j & \text{otherwise.} \end{cases} \tag{4}$$

Observing E^+ and E^- in Fig. 1, we therefore obtain that (π, π') belongs to

- $\{(1\bar{3}\bar{4}\bar{2}, 1\bar{2}\bar{3}\bar{4}), (\bar{4}\bar{3}1\bar{2}, \bar{3}\bar{2}1\bar{4})\}$ if $\varepsilon(s) = \varepsilon(s') = +$,
- $\{(\bar{2}\bar{3}4\bar{1}, \bar{1}\bar{2}4\bar{3}), (4\bar{3}\bar{2}\bar{1}, 4\bar{2}\bar{1}\bar{3})\}$ if $\varepsilon(s) = \varepsilon(s') = -$,
- $\{(21\bar{4}\bar{3}, 21\bar{3}\bar{4}), (1\bar{3}4\bar{2}, 1\bar{2}4\bar{3}), (21\bar{4}\bar{3}, 21\bar{3}\bar{4}), (\bar{4}12\bar{3}, \bar{3}12\bar{4}), (4\bar{3}1\bar{2}, 4\bar{2}1\bar{3}), (34\bar{2}\bar{1}, 34\bar{1}\bar{2})\}$ if $\varepsilon(s) = +$ and $\varepsilon(s') = -$,
- $\{(12\bar{4}\bar{3}, 12\bar{3}\bar{4}), (\bar{2}\bar{3}\bar{4}\bar{1}, \bar{1}\bar{2}\bar{3}\bar{4}), (\bar{2}34\bar{1}, \bar{1}34\bar{2}), (\bar{4}21\bar{3}, \bar{3}21\bar{4}), (\bar{4}\bar{3}\bar{2}\bar{1}, \bar{3}\bar{2}\bar{1}\bar{4}), (43\bar{2}\bar{1}, 43\bar{1}\bar{2})\}$ if $\varepsilon(s) = -$ and $\varepsilon(s') = +$.

In each such pair (π, π'), we have overlined the positions $j \in [4]$ where $\pi_j \neq \pi'_j$. Hence, as $t = |\left\{ j \in [3] \;\middle|\; \pi_j \neq \pi'_j \right\}$— by Eq. (4), we obtain from this case analysis that

$$\varepsilon(s) = \varepsilon(s') \iff t \text{ is even.} \tag{5}$$

Observe now that since $\sigma = \sigma' \tau_4$ and $\sigma_4 < \sigma_5$, we have

$$|\operatorname{inv}(s)| = |\operatorname{inv}(s')| + \sigma_5 - \sigma_4 - t - 1 \qquad \text{and} \qquad \Sigma(\bar{s}) = \Sigma(\bar{s}') + \sigma_4 - \sigma_5.$$

Hence, as $\varepsilon(s)$ just records the parity of $|\operatorname{inv}(s)| + \Sigma(\bar{s})$, we obtain that

$$\varepsilon(s) = \varepsilon(s') \iff t \text{ is odd.} \tag{6}$$

This concludes the proof since (5) and (6) contradict each other.

6. Cartesian products of permutahedra

In this section, we prove Theorem 2 concerning minimum maximal matchings in Cartesian products of permutahedra. Let us first recall the definition.

Definition 2. The *Cartesian product* $G \,\square\, H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and edge set $\{(u,v)(u',v) \mid uu' \in E(G), v \in V(H)\} \cup \{(u,v)(u,v') \mid u \in V(G), vv' \in E(H)\}$.

We start with the lower bound.

Proposition 3. *Let $n_1, \dots, n_k$ be integers with $n_1 \geq \cdots \geq n_k \geq 2$ and $n_1 \geq 3$. Let $n \coloneqq n_1 + \cdots + n_k$ and $\Pi \coloneqq \Pi_{n_1} \square \dots \square \Pi_{n_k}$. Then any maximal matching of Π has cardinality at least*

$$\frac{n-k}{3n-3k+1} V(\Pi).$$

Proof. First, Π is $(n-k)$-regular since each Π_{n_i} is $(n_i - 1)$-regular. Next, we prove by induction on $i \in [k]$ that $G_i \coloneqq \Pi_{n_1} \square \dots \square \Pi_{n_i}$ is $(N_i - i - 3)$-heavy, for $N_i \coloneqq n_1 + \cdots + n_i$. For the base case $i = 1$, this follows from the proof of Corollary 1. For the inductive step $i \geq 2$, note that $G_i = G_{i-1} \square \Pi_{n_i}$. Hence, we can write each vertex of G_i as (x, y), where x and y are vertices in G_{i-1} and Π_{n_i}, respectively. Let e be an edge of G_i between two vertices (x, y) and (x', y'). By the definition of a Cartesian product, either $x = x'$ or $y = y'$.

Firstly consider the case that $x = x'$. By the proof of Corollary 1, there are $n_i - 4$ cycles of Π_{n_i} such that $x'y'$ is the only common edge of any two such cycles. Adding x as a prefix to all vertices to these cycles, we obtain a collection $\mathcal{C}_1$ of $n_i - 4$ cycles in G_i. Further, for each neighbor z of x in G_{i-1}, $(x, x'), (z, x'), (z, y'), (x', y')$ forms a cycle in G_i. Applying this argument for all $N_{i-1} - (i-1)$ neighbors of x in G_{i-1}, we obtain another collection $\mathcal{C}_2$ of $N_{i-1} - (i-1)$ cycles in G_i. Together, both collections form a set of $n_i - 4 + N_{i-1} - (i-1) = N_i - i - 3$ cycles in G_i such that e is the only common edge of any two such cycles.

Secondly, we use a similar argument for the case $y = y'$. By the inductive hypothesis, there are $N_{i-1} - (i-1) - 3$ cycles of G_{i-1} such that $\{x', y'\}$ is the only common edge of any two such cycles. Further, y has $n_i - 1$ neighbors in Π_{n_i}. Together, these induce $N_i - i - 3$ cycles in G_i that pairwise share e as the only common edge.

The preceding two paragraphs complete the inductive proof. This implies that Π is $(n-k-3)$-heavy. Together with Proposition 1 and the fact that Π is $(n-k)$-regular, we then obtain the statement of the proposition. $\square$

The main tool for the upper bound is the following proposition.

Proposition 4. *Let $G := H \,\Box\, B$ be the Cartesian product of a graph H and a bipartite graph B. Suppose that H has maximal matchings $N^\bullet$ and N° such that $|N^\bullet| = |N^\circ|$ and every vertex of H is covered by $N^\bullet$ or N°. Then G has maximal matchings $M^\bullet$ and M° such that $|M^\bullet| = |M^\circ| = |N^\bullet| \cdot |V(B)|$ and every vertex of G is covered by $M^\bullet$ or M°.*

Proof. For $b \in V(B)$, let H_b be the subgraph of G induced by $V(H) \times \{b\} \subseteq V(G)$. Let $N_b^\bullet$ and N_b° be maximal matchings in H_b corresponding to $N^\bullet$ and N°, respectively. Let $B^\bullet$ and B° be the parts of B. Set

$$M^\bullet := \bigcup_{b \in B^\bullet} N_b^\bullet \cup \bigcup_{b \in B^\circ} N_b^\circ, \qquad \text{and} \qquad M^\circ := \bigcup_{b \in B^\bullet} N_b^\circ \cup \bigcup_{b \in B^\circ} N_b^\bullet.$$

Clearly, $M^\bullet$ and M° are matchings in G, as they only use matching edges inside H_b. For the cardinalities, we have

$$|M^\bullet| = |N^\bullet| \cdot |B^\bullet| + |N^\circ| \cdot |B^\circ| = |N^\bullet| \cdot |V(B)| = |N^\circ| \cdot |B^\bullet| + |N^\bullet| \cdot |B^\circ| = |M^\circ|.$$

Moreover, $M^\bullet$ is maximal since the exposed vertices of $N_b^\bullet$ and $N_{b'}^\circ$ are disjoint if $bb' \in E(B)$ (and similarly, M° is maximal). Finally, $M^\bullet \cup M^\circ$ covers $V(G)$ since $N_b^\bullet \cup N_b^\circ$ covers $V(H)$ for each $b \in B$. $\square$

Remark 4. Note that Proposition 4 extends straightforward to a Cartesian product $G := H \,\Box\, K$ of two graphs H and K such that there are maximal matchings $M_1, \dots, M_k$ of G and a coloring $f : V(K) \to [k]$ such that $M_{f(u)} \cup M_{f(v)}$ covers $V(H)$ for any edge uv of K.

The upper bound in Theorem 2 then follows from Example 2 and the following proposition.

Proposition 5. *Let $n_1, \dots, n_k \geq 2$ be integers with $n_1 \geq 3$. Then the Cartesian product $\Pi := \Pi_{n_1} \Box \dots \Box \Pi_{n_k}$ has a maximal matching of size $|V(\Pi)|/3$.*

Proof. We show by induction on $i \in [k]$ that $G_i := \Pi_{n_1} \Box \dots \Box \Pi_{n_i}$ has maximal matchings $M_i^\bullet$ and M_i° of size $|V(G_i)|/3$ such that every vertex in G is covered by $M_i^\bullet$ or M_i°. For $i = 1$, this follows from Theorem 4 and Remark 3. For the induction step, assume that the above statement holds for G_{i-1}. By Proposition 4, using that Π_{n_i} is bipartite, we obtain maximal matchings $M_i^\bullet$ and M_i° of size $|V(G_i)|/3$ in G_i such that every vertex in G_i is covered by $M_i^\bullet$ or M_i°. In particular, $\Pi = G_k$ has a maximal matching of size $|V(\Pi)|/3$, which proves the claim. $\square$

7. Open questions

We conclude with a few open questions.

- Is there a simple or constructive proof for conditions (ii) and (iii) of Proposition 2 for the permutahedron? Is there a symmetric chain decomposition in the permutahedron? See Remark 2.
- Can we get upper and lower bounds on the size of a minimum maximal matching for larger classes of polytopes generalizing the permutahedron and associahedron, in particular quotientopes (even only bipartite quotientopes or regular quotientopes), graphical zonotopes, graph associahedra, or Coxeter permutahedra?
- What can be said about minimal maximal matchings in Cartesian products in general?

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