

INTERVAL HYPERGRAPHIC LATTICES

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ABSTRACT. For a hypergraph \mathbb{H} on $[n]$, the hypergraphic poset $P_{\mathbb{H}}$ is the transitive closure of the oriented skeleton of the hypergraphic polytope $\Delta_{\mathbb{H}}$ (the Minkowski sum of the standard simplices Δ_H for all $H \in \mathbb{H}$). Hypergraphic posets include the weak order for the permutahedron (when \mathbb{H} is the complete graph on $[n]$) and the Tamari lattice for the associahedron (when \mathbb{H} is the set of all intervals of $[n]$), which motivates the study of lattice properties of hypergraphic posets. In this paper, we focus on interval hypergraphs, where all hyperedges are intervals of $[n]$. We characterize the interval hypergraphs \mathbb{I} for which $P_{\mathbb{I}}$ is a lattice, a distributive lattice, a semidistributive lattice, and a lattice quotient of the weak order.

CONTENTS

1. Introduction	2
2. Hypergraphic posets	3
2.1. Hypergraphic polytopes	3
2.2. Acyclic orientations, increasing flips, and hypergraphic posets	3
2.3. Surjection map	5
3. Interval hypergraphic posets	6
3.1. Acyclic orientations for interval hypergraphs	7
3.2. Fibers of \mathcal{O} for interval hypergraphs	8
3.3. Source characterization for interval hypergraphs	8
3.4. Flips and cover relations for interval hypergraphs	9
4. Interval hypergraphic lattices	10
4.1. If $P_{\mathbb{I}}$ is a lattice, then \mathbb{I} is closed under intersection	11
4.2. Properties of $P_{\mathbb{I}}$ when \mathbb{I} is closed under intersection	13
4.3. If \mathbb{I} is closed under intersection, then $P_{\mathbb{I}}$ is a lattice	14
5. Distributive interval hypergraphic lattices	15
5.1. Some join irreducible acyclic orientations	15
5.2. If \mathbb{I} is distributive then $P_{\mathbb{I}}$ is distributive	19
5.3. If $P_{\mathbb{I}}$ is distributive then \mathbb{I} is distributive	20
5.4. Schröder hypergraphs	20
6. Semidistributive interval hypergraphic lattices	21
6.1. All join irreducible acyclic orientations	23
6.2. If \mathbb{I} is semidistributive then $P_{\mathbb{I}}$ is semidistributive	26
6.3. If $P_{\mathbb{I}}$ is semidistributive then \mathbb{I} is semidistributive	27
7. Quotient interval hypergraphic lattices	29
References	30

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1. INTRODUCTION

Fix an integer $n \geq 1$ and denote by $(e_i)_{i \in [n]}$ the standard basis of \mathbb{R}^n . The **hypergraphic polytope** of a hypergraph \mathbb{H} on $[n]$ is the Minkowski sum $\Delta_{\mathbb{H}} := \sum_{H \in \mathbb{H}} \Delta_H$, where Δ_H is the simplex given by the convex hull of the points e_h for $h \in H$. The face lattice of $\Delta_{\mathbb{H}}$ was described combinatorially in terms of acyclic orientations of \mathbb{H} in [BBM19]. Note that the singletons of \mathbb{H} are irrelevant for the combinatorics of $\Delta_{\mathbb{H}}$ as they just contribute to translations. It is convenient for us to assume that $\{i\} \in \mathbb{H}$ for all $i \in [n]$.

The **hypergraphic poset** $P_{\mathbb{H}}$ is the transitive closure of the skeleton of $\Delta_{\mathbb{H}}$ oriented in the direction $\omega := (n, n-1, \dots, 2, 1) - (1, 2, \dots, n-1, n) = (n-1, n-3, \dots, 3-n, 1-n)$. For instance,

- if $\mathbb{H} = \binom{[n]}{2}$ is the complete graph (or any hypergraph containing it), then $\Delta_{\mathbb{H}}$ is the **permutahedron** and $P_{\mathbb{H}}$ is the **weak order on permutations**,
- if $\mathbb{H} = \{[i, j] \mid 1 \leq i \leq j \leq n\}$ is the complete interval hypergraph, then $\Delta_{\mathbb{H}}$ is J.-L. Loday's **associahedron** [SS93, Lod04] and $P_{\mathbb{H}}$ is the **Tamari lattice on binary trees** [Tam51].

In view of these two examples, we would like to characterize the hypergraphs \mathbb{H} for which $P_{\mathbb{H}}$ is a lattice, a distributive lattice, a semidistributive lattice, a congruence-uniform lattice, a (semi-)lattice quotient of the weak order on permutations, etc. These questions were settled in [Pil24] for graphical zonotopes (*i.e.* when $\mathbb{H} \subseteq \binom{[n]}{2}$), and also partially studied in [BM21] for graph associahedra [CD06] (*i.e.* when \mathbb{H} is the set of all subsets of vertices that induce a connected subgraph of a fixed graph on $[n]$).

In this paper, we study the case of **interval hypergraphs** \mathbb{I} , *i.e.* when all hyperedges of \mathbb{I} are intervals of $[n]$. Note that the family of interval hypergraphic polytopes does not contain the permutahedron, but contains

- the classical associahedron of [SS93, Lod04] when \mathbb{I} contains all intervals of $[n]$,
- the Pitman–Stanley polytope [SP02] when \mathbb{I} is the set of all singletons $\{i\}$ and all initial intervals $[i]$ for $i \in [n]$,
- the freehedron of [San09] when \mathbb{I} is the set of all singletons $\{i\}$, all initial intervals $[i]$ for $i \in [n]$, and all final intervals $[n] \setminus [i]$ for $i \in [n-1]$,
- the fertilotopes of [Def23] when any two intervals of \mathbb{I} are either nested or disjoint.

In fact, it follows from [BMCLD⁺23, PPPP23] that the interval hypergraphic polytopes are precisely the weak Minkowski summands of the classical associahedron (recall that a polytope $P \subset \mathbb{R}^n$ is a **weak Minkowski summand** of a polytope $Q \subset \mathbb{R}^n$ if there exists a real $\lambda \geq 0$ and a polytope $R \subset \mathbb{R}^d$ such that $\lambda Q = P + R$).

We obtain the following characterizations, where we assume that $\{i\} \in \mathbb{I}$ for all $i \in [n]$ as mentioned earlier. See Sections 4 to 7 and Figures 4 to 8 for illustrations.

Theorem A. *For an interval hypergraph \mathbb{I} , the poset $P_{\mathbb{I}}$ is a lattice if and only if \mathbb{I} is closed under intersection (*i.e.* $I, J \in \mathbb{I}$ and $I \cap J \neq \emptyset$ implies $I \cap J \in \mathbb{I}$).*

Theorem B. *For an interval hypergraph \mathbb{I} , the poset $P_{\mathbb{I}}$ is a distributive lattice if and only if for all $I, J \in \mathbb{I}$ such that $I \not\subseteq J$, $I \not\supseteq J$ and $I \cap J \neq \emptyset$, the intersection $I \cap J$ is in \mathbb{I} and is initial or final in any $K \in \mathbb{I}$ with $I \cap J \subseteq K$.*

Theorem C. *For an interval hypergraph \mathbb{I} , the poset $P_{\mathbb{I}}$ is a join semidistributive lattice if and only if \mathbb{I} is closed under intersection and for all $[r, r'], [s, s'], [t, t'], [u, u'] \in \mathbb{I}$ such that $r < s \leq r' < s'$, $r < t \leq s' < t'$, $u < \min(s, t)$ and $s' < u'$, there is $[v, v'] \in \mathbb{I}$ such that $v < s$ and $s' < v' < t'$. A symmetric characterization holds for meet semidistributivity.*

Theorem D. *For an interval hypergraph \mathbb{I} , the poset morphism from the weak order to the poset $P_{\mathbb{I}}$ is a meet (resp. join) semilattice morphism if and only if \mathbb{I} is closed under initial (resp. final) subintervals (*i.e.* $[i, k] \in \mathbb{I}$ implies $[i, j] \in \mathbb{I}$ (resp. $[j, k] \in \mathbb{I}$) for any $1 \leq i < j < k \leq n$).*

For instance, among the four above-mentioned families of interval hypergraphic polytopes, we recover that the Pitman–Stanley polytope and all fertilotopes yield distributive lattices, the associahedron yields a semidistributive (but not distributive) lattice which is a quotient of the

weak order, while the freehedron is not even a lattice (this was actually the motivation for [PP23] to construct alternative realizations of the skeleton of the freehedron).

Once Theorem A is established, an important step for Theorems B and C is to understand join irreducible elements of $P_{\mathbb{I}}$. Section 6.1 provides a combinatorial description of the join irreducible elements of the lattice $P_{\mathbb{I}}$ for an arbitrary interval hypergraph \mathbb{I} closed under intersections. To prepare this slightly technical description, we already describe some join irreducible elements of $P_{\mathbb{I}}$ in Section 5.1, which happen to be all join irreducible elements of $P_{\mathbb{I}}$ under the condition of Theorem B.

The paper is organized as follows. In Section 2, we recall basic properties of hypergraphic polytopes, we define hypergraphic posets, and we recall the natural poset morphism from the weak order on permutations to the hypergraphic poset $P_{\mathbb{I}}$. In Section 3, we develop specific properties of interval hypergraphic polytopes, in particular a simple characterization of their vertices and a global description of the relations in their hypergraphic posets. In Section 4, we characterize the interval hypergraphs \mathbb{I} for which the interval hypergraphic poset $P_{\mathbb{I}}$ is a lattice, proving Theorem A. In Section 5, we describe a family of join irreducible elements of $P_{\mathbb{I}}$ and we characterize the interval hypergraphs \mathbb{I} for which $P_{\mathbb{I}}$ is a distributive lattice, proving Theorem B. In Section 6, we describe all join irreducible elements of $P_{\mathbb{I}}$ and we characterize the interval hypergraphs \mathbb{I} for which $P_{\mathbb{I}}$ is a join (or meet) semidistributive lattice, proving Theorem C. In Section 7, we characterize the interval hypergraphs \mathbb{I} for which the poset morphism from the weak order on permutations to $P_{\mathbb{I}}$ is a join (or meet) semilattice morphism, proving Theorem D.

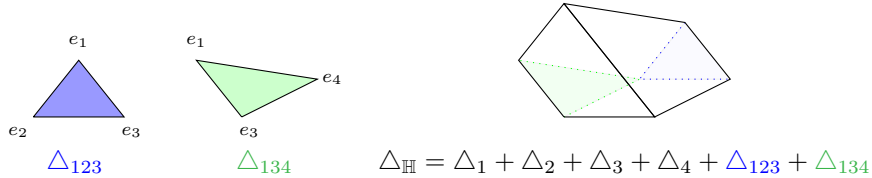
2. HYPERGRAPHIC POSETS

2.1. Hypergraphic polytopes. A **hypergraph** \mathbb{H} on $[n] := \{1, \dots, n\}$ is a collection of subsets of $[n]$. By convention, we always assume that \mathbb{H} contains all singletons $\{i\}$ for $i \in [n]$. The **hypergraphic polytope** $\Delta_{\mathbb{H}}$ is the Minkowski sum

$$\Delta_{\mathbb{H}} := \sum_{H \in \mathbb{H}} \Delta_H,$$

where Δ_H is the simplex given by the convex hull of the points $e_h \in \mathbb{R}^n$ for $h \in H$.

Example 2.1. For the hypergraph $\mathbb{H} = \{1, 2, 3, 4, 123, 134\}$, we have



which is a 3-dimensional polytope sitting in \mathbb{R}^4 . Note that as $n \leq 9$ in all our examples, we simplify notations and write 123 for the set $\{1, 2, 3\}$.

Remark 2.2. Note that the singletons hyperedges are irrelevant for our purposes. Namely, adding to \mathbb{H} the hyperedge $\{i\}$ for some $i \in [n]$ just translates the polytope $\Delta_{\mathbb{H}}$ in the direction e_i , which does not affect the face structure of the polytope. For our conditions on hypergraphs of Theorems A and B, it is convenient for us to assume that $\{i\} \in \mathbb{H}$ for all $i \in [n]$. When quoting results from [BBM19], the reader will have to be mindful that they took the opposite convention.

2.2. Acyclic orientations, increasing flips, and hypergraphic posets. We now recall from [BBM19, Thm. 2.18] a combinatorial model for the graph $(V_{\mathbb{H}}, E_{\mathbb{H}})$ of $\Delta_{\mathbb{H}}$. As we are only interested in the 1-skeleton of $\Delta_{\mathbb{H}}$, we simplify some of the general definitions of [BBM19] which were designed to deal with all faces of $\Delta_{\mathbb{H}}$.

Definition 2.3. An **orientation** of \mathbb{H} is a map O from \mathbb{H} to $[n]$ such that $O(H) \in H$ for all $H \in \mathbb{H}$. Equivalently, we often represent the orientation O as the set of pairs $\{(O(H), H) \mid H \in \mathbb{H}\}$. The orientation O is **acyclic** if there is no H_1, \dots, H_k with $k \geq 2$ such that $O(H_{i+1}) \in H_i \setminus \{O(H_i)\}$ for $i \in [k - 1]$ and $O(H_1) \in H_k \setminus \{O(H_k)\}$.

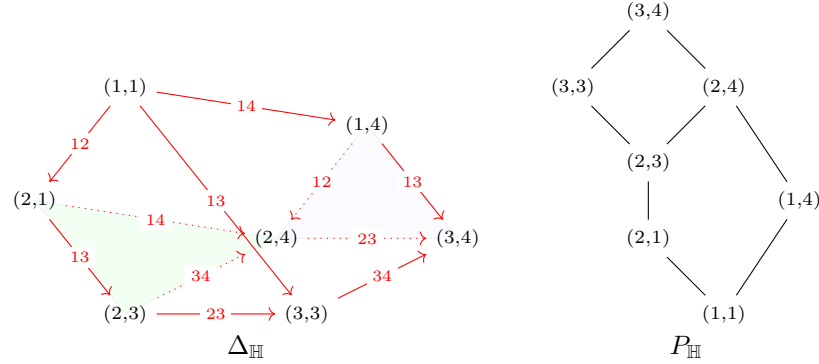


FIGURE 1. The polytope $\Delta_{\mathbb{H}}$ for $\mathbb{H} = \{1, 2, 3, 4, 123, 134\}$ has seven vertices corresponding to the acyclic orientations of \mathbb{H} and eleven (oriented) edges corresponding to the increasing flips between these orientations. The poset $P_{\mathbb{H}}$ is the transitive closure of the increasing flip graph.

Remark 2.4. The singleton $\{i\}$ of \mathbb{H} is always oriented $O(\{i\}) = i$, and plays no role in determining the acyclicity of O since $\{i\} \setminus \{O(\{i\})\} = \emptyset$. We will therefore omit the singletons when describing or drawing orientations (see also Example 3.3). If we list the non-singleton elements (H_1, H_2, \dots, H_k) of \mathbb{H} in some fixed order, it is then convenient to describe an orientation O as the tuple $O = (O(H_1), O(H_2), \dots, O(H_k))$.

Example 2.5. Using $\mathbb{H} = \{1, 2, 3, 4, 123, 134\}$ as in Example 2.1, we order the two non-singleton $(123, 134)$. There are 9 orientations of \mathbb{H} , 7 of which are acyclic as displayed in Figure 1. For instance, the orientation $(O(123), O(134)) = (1, 3)$ is cyclic since $O(123) = 1 \in 134$ and $O(134) = 3 \in 123$ is a cycle with $k = 2$.

Definition 2.6. Two orientations $O \neq O'$ of \mathbb{H} are related by an **increasing flip** if there exist $1 \leq i < j \leq n$ such that for all $H \in \mathbb{H}$,

- if $O(H) \neq O'(H)$, then $O(H) = i$ and $O'(H) = j$, and
- if $\{i, j\} \subseteq H$, then $O(H) = i \iff O'(H) = j$.

We denote such a flip by $O -_{ij} \rightarrow O'$.

Example 2.7. Figure 1 shows all the increasing flips between the acyclic orientations of the hypergraph $\mathbb{H} = \{1, 2, 3, 4, 123, 134\}$ of Examples 2.1 and 2.5. For instance, $(2, 3) -_{23} \rightarrow (3, 3)$ indicates that there is an increasing flip from $O = (2, 3)$ to $O' = (3, 3)$ with $i = 2 < 3 = j$.

The following correspondance was already observed in [BBM19, Thm. 2.18] (it even extends to all faces of $\Delta_{\mathbb{H}}$, but we do not need this level of generality in this paper). We provide an alternative short proof for convenience.

Proposition 2.8 ([BBM19, Thm. 2.18]). *The graph of the hypergraphic polytope $\Delta_{\mathbb{H}}$ oriented in the direction $\omega := (n, n-1, \dots, 2, 1) - (1, 2, \dots, n-1, n) = (n-3, n-1, \dots, 1-n, 3-n)$ is isomorphic to the increasing flip graph on acyclic orientations of \mathbb{H} .*

Proof. Recall that the face of a Minkowski sum $\sum_i P_i$ minimizing a direction \mathbf{v} is the Minkowski sum of the faces of the summands P_i minimizing \mathbf{v} .

The vertex of Δ_H minimizing a generic direction \mathbf{v} is \mathbf{e}_i for $i \in H$ such that $\mathbf{v}_i = \min \{\mathbf{v}_h \mid h \in H\}$. An acyclic orientation of \mathbb{H} corresponds to the choice of one vertex in each Δ_H , and the orientation is acyclic if and only if this choice corresponds to a generic orientation \mathbf{v} , hence to a vertex of $\Delta_{\mathbb{H}}$.

The edges of Δ_H are oriented in the directions $\mathbf{e}_i - \mathbf{e}_j$ for $i, j \in H$. The edges of $\Delta_{\mathbb{H}}$ are thus also oriented by $\mathbf{e}_i - \mathbf{e}_j$, and thus correspond to pairs of acyclic orientations which differ by a flip. \square

Finally, the main objects of this paper are the following posets.

Definition 2.9. The **hypergraphic poset** $P_{\mathbb{H}}$ is the transitive closure of the increasing flip graph on acyclic orientations of \mathbb{H} .

Example 2.10. For the hypergraph $\mathbb{H} = \{1, 2, 3, 4, 123, 134\}$ of Examples 2.1, 2.5 and 2.7, the poset $P_{\mathbb{H}}$ is represented in Figure 1 (right).

Remark 2.11. As seen in Figure 1 the edges in $E_{\mathbb{H}}$ are not necessarily cover relations in $P_{\mathbb{H}}$. The simplest case is given by $\mathbb{H} = \{1, 2, 3, 123\}$, whose hypergraphic polytope is

$$\Delta_{\mathbb{H}} = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_{123} = \begin{array}{c} (3) \\ \uparrow \\ 13 \\ \downarrow \\ (1) \end{array} \begin{array}{c} \nwarrow 23 \\ \nearrow 12 \end{array} \quad (2)$$

and the left edge $(1) - 13 \rightarrow (3)$ of $\Delta_{\mathbb{H}}$ is not a cover relation of $P_{\mathbb{H}}$.

We conclude with an elementary yet relevant symmetry.

Proposition 2.12. For $x \in [n]$, define $x^{\leftrightarrow} := n - x + 1$. For $H \subseteq [n]$, define $H^{\leftrightarrow} := \{h^{\leftrightarrow} \mid h \in H\}$. For an hypergraph \mathbb{H} , define $\mathbb{H}^{\leftrightarrow} := \{H^{\leftrightarrow} \mid H \in \mathbb{H}\}$. For an orientation O of \mathbb{H} , define the orientation O^{\leftrightarrow} of $\mathbb{H}^{\leftrightarrow}$ by $O^{\leftrightarrow}(H^{\leftrightarrow}) := O(H)^{\leftrightarrow}$. Then the map $A \mapsto A^{\leftrightarrow}$ is an anti-isomorphism from $P_{\mathbb{H}}$ to $P_{\mathbb{H}^{\leftrightarrow}}$.

Proof. Straightforward. □

2.3. Surjection map. The hypergraphic polytope $\Delta_{\mathbb{H}}$ is a deformed permutahedron (*aka.* generalized permutahedron [Pos09, PRW08], *aka.* polymatroid [Edm70]). This means that the normal fan of Δ coarsens the normal fan of the permutahedron. Hence, there is a natural surjection from the faces of the permutahedron to the faces of $\Delta_{\mathbb{H}}$, which was described in details in [BBM19, Lem. 2.9]. Here, we focus on the surjection \mathcal{O} from the permutations of $[n]$ to the acyclic orientations of \mathbb{H} .

Definition 2.13. For a permutation π of $[n]$, the orientation \mathcal{O}_{π} of \mathbb{H} is defined for all $H \in \mathbb{H}$ by

$$\mathcal{O}_{\pi}(H) := \pi(\min \{j \mid \pi(j) \in H\}).$$

Proposition 2.14 ([BBM19, Lem. 2.9]). For any hypergraph \mathbb{H} on $[n]$,

- the map \mathcal{O} is a surjection from the permutations of $[n]$ to the acyclic orientations of \mathbb{H} ,
- two acyclic orientations A, B of \mathbb{H} are related by a flip if and only if there are permutations π_A, π_B of $[n]$ which differ by a simple transposition such that $\mathcal{O}_{\pi_A} = A$ and $\mathcal{O}_{\pi_B} = B$.

In other words, the graph $(V_{\mathbb{H}}, E_{\mathbb{H}})$ of $\Delta_{\mathbb{H}}$ is isomorphic to the graph obtained by contracting the fibers of \mathcal{O} in the graph of the permutahedron.

Corollary 2.15. The map \mathcal{O} defines a poset morphism from the weak order on permutations to the hypergraphic poset $P_{\mathbb{H}}$.

Finally, we describe the fibers of the surjection $\mathcal{O} : \mathfrak{S}_n \rightarrow V_{\mathbb{H}}$. Given an acyclic orientation A of \mathbb{H} , define \triangleleft_A as the order on $[n]$ obtained by the transitive closure of the union of the orders $\{A(H) < h \mid h \in H \setminus \{A(H)\}\}$ for each $H \in \mathbb{H}$. That is,

$$\triangleleft_A = \text{Trans. cl.} \bigcup_{H \in \mathbb{H}} \begin{array}{c} h \in H \setminus \{A(H)\} \\ \swarrow \quad \downarrow \quad \searrow \\ A(H) \end{array}.$$

This is a well defined partial order since A is acyclic. The following lemma is straightforward.

Lemma 2.16. For any acyclic orientation A of \mathbb{H} , the preimage $\mathcal{O}^{-1}(A) := \{\pi \mid \mathcal{O}_{\pi} = A\}$ is the set of linear extensions of \triangleleft_A .

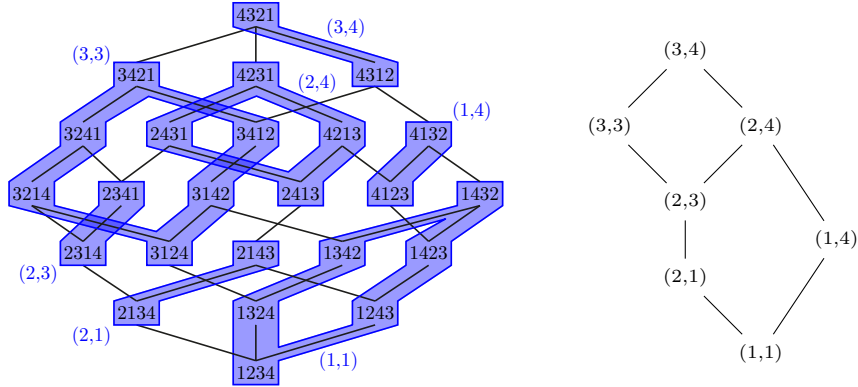


FIGURE 2. The poset morphism \mathcal{O} from the weak order (left) to the hypergraphical poset $P_{\mathbb{H}}$ (right), for $\mathbb{H} = \{1, 2, 3, 4, 123, 134\}$. The fibers of \mathcal{O} appear as blue bubbles on the weak order, labeled by their acyclic orientation of \mathbb{H} (left).

Example 2.17. Continuing with the hypergraph $\mathbb{H} = \{1, 2, 3, 4, 123, 134\}$ of Examples 2.1, 2.5, 2.7 and 2.10, let A be the acyclic orientation $(2, 4)$. The order \triangleleft_A is the following

$$\triangleleft_A = \text{Transitive closure} \left(\begin{array}{c} 1 \quad 3 \\ \backslash \quad / \\ 2 \end{array} \cup \begin{array}{c} 1 \quad 3 \\ \backslash \quad / \\ 4 \end{array} \right) = \begin{array}{c} 1 \\ | \\ 2 \end{array} \times \begin{array}{c} 3 \\ | \\ 4 \end{array}$$

The linear extensions of this order are the permutations 2413, 2431, 4213 and 4231. All other fibers are represented in Figure 2.

3. INTERVAL HYPERGRAPHIC POSETS

In this paper, we focus on the following family of hypergraphs on $[n]$.

Definition 3.1. An **interval hypergraph** \mathbb{I} is an hypergraph on $[n]$ where each $I \in \mathbb{I}$ is an interval of the form $I = [i, j] := \{i, i + 1, i + 2, \dots, j - 1, j\}$.

Example 3.2. Our running example $\mathbb{H} = \{1, 2, 3, 4, 123, 134\}$ of Examples 2.1, 2.5, 2.7, 2.10 and 2.17, is not an interval hypergraph, as 134 is not an interval.

Example 3.3. The hypergraph $\mathbb{I} = \{1, 2, 3, 4, 123, 23, 234, 1234\}$ is an interval hypergraph and $\mathcal{O}_{4132} = (1, 4, 3, 4)$ is an acyclic orientation of \mathbb{I} (for the lexicographic order $(123, 1234, 23, 234)$ of the non-singletons). We will represent interval hypergraphs and their orientations graphically as follows:

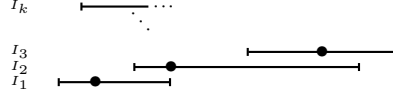


As before, we omit to draw the singletons $\{i\}$ for $i \in [n]$. See also Figure 3.

Example 3.4. We have represented further interval hypergraphic posets in Figures 4 to 8. These figures illustrate in particular the following relevant families of interval hypergraphic posets mentioned in the introduction:

- if $\mathbb{I} = \{[i, j] \mid 1 \leq i \leq j \leq n\}$ is the set of all intervals of $[n]$, then $P_{\mathbb{I}}$ is the Tamari lattice corresponding to the classical associahedron of [SS93, Lod04]; see Figure 5;
- if $\mathbb{I} = \{[1, i] \mid i \in [n]\}$ is the set of initial intervals of $[n]$, then $P_{\mathbb{I}}$ is the boolean lattice corresponding to the Pitman–Stanley polytope [SP02]; see Figure 6 (bottom right);
- if $\mathbb{I} = \{[1, i] \mid i \in [n]\} \cup \{[i, n] \mid i \in [n]\}$ is the set of all initial or final intervals of $[n]$, then $P_{\mathbb{I}}$ is a poset given by the freehedron of [San09], which is not even a lattice; see Figure 4 (right);
- if any two intervals of \mathbb{I} are either nested or disjoint, then $P_{\mathbb{I}}$ is a distributive lattice given by the fertitope of [Def23]; see Figure 6 (top right and bottom) and Section 5.4.

and $O(I_1) \in I_k \setminus \{O(I_k)\}$. Graphically



Assume that $k > 2$ and is minimal for this property. Note that $O(I_i) \neq O(I_{i+1})$ for all i . By symmetry, suppose $O(I_1) < O(I_2)$. If there is $i \in [k-1]$ such that $O(I_{i+1}) < O(I_i)$, then for the smallest such i , we have

- either $O(I_{i-1}) \in [O(I_{i+1}), O(I_i)] \subseteq I_i$, so that $O(I_{i-1}) \in I_i$ and $O(I_i) \in I_{i-1}$,
- or $O(I_{i+1}) \in [O(I_{i-1}), O(I_i)] \subseteq I_{i-1}$, so that we can drop I_i from our sequence, contradicting the minimality of k .

We thus obtain $O(I_1) < \dots < O(I_k)$. As $O(I_1) \in I_k$, we get $O(I_{k-1}) \in [O(I_1), O(I_k)] \subseteq I_k$, so that $O(I_{k-1}) \in I_k$ and $O(I_k) \in I_{k-1}$. \square

Remark 3.7. Proposition 3.6 fails when \mathbb{H} is not an interval hypergraph. For instance, the orientation O of the hypergraph $\mathbb{H} := \{1, 2, 3, 12, 23, 13\}$ defined by $O(12) = 1$, $O(23) = 2$ and $O(13) = 3$ has a 3-cycle but no 2-cycle.

3.2. Fibers of \mathcal{O} for interval hypergraphs. One striking property for interval hypergraphs is that the fibers of the surjection \mathcal{O} are intervals in the weak order. To describe this we first need to recall the following classical result of A. Björner and M. Wachs [BW91, Thm. 6.8]

Proposition 3.8 ([BW91, Thm. 6.8]). *The set of linear extensions of a poset \triangleleft on $[n]$ forms an interval $[\sigma, \tau]$ of the weak order if and only if $a \triangleleft c \implies a \triangleleft b$ or $b \triangleleft c$, and $a \triangleright c \implies a \triangleright b$ or $b \triangleright c$, for every $1 \leq a < b < c \leq n$. Moreover, the inversions of σ are the pairs (b, a) with $a < b$ and $a \triangleright b$, and the non-inversions of τ are the pairs (a, b) with $a < b$ and $a \triangleleft b$.*

This statement was refined in [CPP19] to describe Tamari interval posets.

Proposition 3.9 ([CPP19, Coro. 2.24]). *The set of linear extensions of a poset \triangleleft on $[n]$ forms an interval $[\sigma, \tau]$ of the weak order such that σ avoids the pattern 231 and τ avoids the pattern 213 if and only if $a \triangleleft c \implies a \triangleleft b$ and $a \triangleright c \implies b \triangleright c$ for every $1 \leq a < b < c \leq n$.*

Proposition 3.10. *The fiber $\mathcal{O}^{-1}(A)$ of any acyclic orientation A of an interval hypergraph \mathbb{I} is an interval of the weak order with minimum avoiding the pattern 231 and maximum avoiding the pattern 213 (in other words, a Tamari interval).*

Proof. From Lemma 2.16, the fiber $\mathcal{O}^{-1}(A)$ is the set of linear extensions of \triangleleft_A , so that we use the characterization of Proposition 3.9 to prove Proposition 3.10. Let $1 \leq a < b < c \leq [n]$. If we have $a \triangleleft_A c$, then, by definition of \triangleleft_A , there must be a sequence $I_1, \dots, I_k \in \mathbb{I}$ such that $a = A(I_1)$, $A(I_{i+1}) \in I_i$ for all $i \in [k-1]$, and $c \in I_k$. As $\bigcup_{i \in [k]} I_i$ is an interval containing a and c and $a < b < c$, it contains also b . Hence, there is $i \in [k]$ such that $b \in I_i$, and the sequence I_1, \dots, I_i proves that $a \triangleleft_A b$. The case $a \triangleright_A c$ is similar and implies that $b \triangleright_A c$. \square

Example 3.11. For the hypergraph $\mathbb{I} = \{1, 2, 3, 4, 123, 23, 234, 1234\}$ of Example 3.3, the fibers of the surjection map \mathcal{O} are represented in Figure 3 (bottom left).

Remark 3.12. Proposition 3.10 fails when \mathbb{H} is not an interval hypergraph. For $\mathbb{H} := \{1, 2, 3, 13\}$, there are two fibers $\{123, 213, 132\}$ and $\{231, 312, 321\}$ which are not intervals of the weak order.

3.3. Source characterization for interval hypergraphs. We now characterize the comparisons in the poset $P_{\mathbb{I}}$ in terms of the comparisons of the sources for each $I \in \mathbb{I}$.

Proposition 3.13. *For any acyclic orientations A and B of an interval hypergraph \mathbb{I} ,*

$$A \leq B \text{ in } P_{\mathbb{I}} \iff A(I) \leq B(I) \text{ for all } I \in \mathbb{I}.$$

Proof. The forward direction is immediate as it holds for increasing flip by Definition 2.6 and any cover of $P_{\mathbb{I}}$ is an increasing flip. For the backward direction, assume that $A(I) \leq B(I)$ for all $I \in \mathbb{I}$. The proof works by induction on $|\{I \in \mathbb{I} \mid A(I) < B(I)\}|$.

Choose $J \in \mathbb{I}$ such that $A(J) < B(J)$ and for any $I \in \mathbb{I}$ with $A(I) < B(I)$, we have $B(I) < B(J)$, or $B(I) = B(J)$ and $A(I) \leq A(J)$.

Let O be the orientation of \mathbb{I} obtained from A by flipping $A(J)$ to $B(J)$ as described in Definition 2.6. We claim that O is acyclic and that $O(I) \leq B(I)$ for all $I \in \mathbb{I}$. We conclude by induction that $O \leq B$, and thus $A \leq O \leq B$ as desired.

We first prove that O is acyclic. Otherwise, we would have $I, I' \in \mathbb{I}$ such that $O(I) \in I' \setminus \{O(I')\}$ and $O(I') \in I \setminus \{O(I)\}$. As A is acyclic, we have $A(I) \neq O(I)$ or $A(I') \neq O(I')$, but not both since $O(I) \neq O(I')$. Hence, we can assume by symmetry that $O(I) = A(I)$ while $A(I') = A(J)$ and $O(I') = B(J)$. Up to updating J to I' , we can thus also assume that $I' = J$. As B is acyclic, we have $B(I) \neq O(I)$. Since $A(I) = O(I) \neq B(I)$, our choice of J ensures that

- either $B(I) < B(J)$. We then obtain that $A(I) \leq B(I) < B(J)$. As $A(I) = O(I) \in I' = J$ and $B(J) \in J$, we thus get that $B(I) \in J$. Moreover, $B(J) = O(I') \in I$. As $B(I) \neq B(J)$, we obtain a contradiction with the acyclicity of B .
- or $B(I) = B(J)$ and $A(I) \leq A(J)$. We then have $A(I) \leq A(J) \leq B(J) = B(I)$ so that $A(J) \in I$. As $A(I) = O(I) \in I' = J$ and $A(I) \neq A(J)$, we obtain a contradiction with the acyclicity of A .

We now prove that $O(I) \leq B(I)$ for all $I \in \mathbb{I}$. We thus consider $I \in \mathbb{I}$ and distinguish two cases:

- Assume first that $A(I) = A(J)$ and $B(J) \in I$. By Definition 2.6, we then have $O(I) = B(J)$. Moreover, as I is an interval and contains $A(J)$ and $B(J)$, it contains $[A(J), B(J)] \subseteq J$. As B is acyclic, this implies that $B(I) \notin [A(J), B(J)[$. As $A(J) = A(I) \leq B(I)$, we thus obtain that $O(I) = B(J) \leq B(I)$.
- Otherwise, we have $O(I) = A(I) \leq B(I)$. □

Remark 3.14. Note that the forward direction holds for arbitrary hypergraphs. We are not sure whether the backward direction holds for arbitrary hypergraphs.

3.4. Flips and cover relations for interval hypergraphs. In this section, we exploit Proposition 3.6 to provide a simple description of the flips and cover relations of $P_{\mathbb{I}}$ for an interval hypergraph \mathbb{I} .

Proposition 3.15. *Consider an acyclic orientation A of an interval hypergraph \mathbb{I} , an increasing flip $A \rightarrow B$ (in the sense of Definition 2.6), and let $k := \max \{\max(I) \mid I \in \mathbb{I} \text{ and } A(I) = i\}$. Then B is acyclic if and only if there is no $J \in \mathbb{I}$ with $j \in J \setminus \{A(J)\}$ and $A(J) \in]i, k]$. A symmetric statement holds for decreasing flips.*

Proof. We prove the result for increasing flips, the result for decreasing flips follows by the symmetry of Proposition 2.12.

Assume first that there is $J \in \mathbb{I}$ with $j \in J \setminus \{A(J)\}$ and $A(J) \in]i, k]$. Let $I \in \mathbb{I}$ be such that $A(I) = i$ and $k = \max(I)$. Then $B(J) = A(J) \in]i, k] =]A(I), \max(I)] \subseteq I$ and $B(I) = j \in J$ and $B(I) = j \neq A(J) = B(J)$ implies that B is cyclic.

Conversely, assume that B is cyclic. By Proposition 3.6, there are $I, J \in \mathbb{I}$ with $B(I) \in J \setminus \{B(J)\}$ and $B(J) \in I \setminus \{B(I)\}$. As A is acyclic, we have $i = A(I) \neq B(I) = j$ or $i = A(J) \neq B(J) = j$, but not both since $B(I) \neq B(J)$. By symmetry, we can assume that $i = A(I) \neq B(I) = j$ and $A(J) = B(J)$. We obtain that $j = B(I) \in J \setminus \{B(J)\} = J \setminus \{A(J)\}$. Moreover, we have $i < A(J)$ as otherwise $A(J) = B(J) \in I$ and $A(I) = i \in [A(J), j] = [A(J), B(I)] \subseteq J$ and $A(I) \neq A(J)$ would contradict the acyclicity of A . As $A(J) = B(J) \in I \setminus \{B(I)\}$, we obtain that $i < A(J) < \max(I)$. We conclude that $A(J) \in]i, k]$ since $\max(I) \leq k$ as $A(I) = i$. □

Proposition 3.16. *Consider an acyclic orientation A of an interval hypergraph \mathbb{I} and $i \in [n]$ such that there is $I \in \mathbb{I}$ with $i = A(I) < \max(I)$. Then there exists $j > i$ such that the orientation of \mathbb{I} obtained by flipping i to j is acyclic. A symmetric statement holds with a decreasing flip if $i = A(I) > \min(I)$.*

Proof. We prove the result for increasing flips, the result for decreasing flips follows by the symmetry of Proposition 2.12.

Let $k := \max \{ \max(I) \mid I \in \mathbb{I} \text{ and } A(I) = i \}$ and consider

$$X := \bigcup_{\substack{J \in \mathbb{I} \\ A(J) \in]i, k]}} J \setminus \{A(J)\}.$$

Assume that $X =]i, k]$, and consider a minimal $j \in]i, k]$ for the poset \triangleleft_A . As $j \in X$, there is $J \in \mathbb{I}$ such that $A(J) \in]i, k]$ and $j \in J \setminus \{A(J)\}$. Hence, we have $A(J) \triangleleft_A j$ and $A(J) \in]i, k]$, a contradiction. We conclude that there is $j \in]i, k] \setminus X$. By Proposition 3.15, the orientation obtained from A by flipping of i to j is acyclic. \square

Remark 3.17. Proposition 3.16 fails when \mathbb{H} is not an interval hypergraph. For instance, for the hypergraph $\mathbb{H} = \{1, 2, 3, 4, 123, 134\}$ of Examples 2.1, 2.5, 2.7, 2.10 and 2.17, and the acyclic orientation A defined by $A(123) = 2$ and $A(134) = 1$, we have $2 = A(123) < \max(123)$, but no increasing flip from A to an acyclic orientation of \mathbb{H} flips 2.

Proposition 3.18. *An increasing flip $A \xrightarrow{-ij} B$ between two acyclic orientations of an interval hypergraph \mathbb{I} is a cover relation of $P_{\mathbb{I}}$ if and only if*

$$]i, j[\subseteq \bigcup_{\substack{J \in \mathbb{I} \\ A(J) \in]i, j]}} J \setminus \{A(J)\}.$$

Proof. Let $k := \max \{ \max(I) \mid I \in \mathbb{I} \text{ and } A(I) = i \}$ and $I \in \mathbb{I}$ be such that $A(I) = i$ and $k = \max(I)$. Note that $i < j \leq k$ since $A \xrightarrow{-ij} B$ is a flip.

Assume first that there is $\ell \in]i, j[\setminus \bigcup_{A(J) \in]i, j]} J \setminus \{A(J)\}$. We claim that there is no $J \in \mathbb{I}$ with $\ell \in J \setminus A(J)$ and $A(J) \in]i, k]$. Indeed, by definition of ℓ , we would have $A(J) > j$. Then $B(I) = j \in [\ell, A(J)] \subseteq J$ and $B(J) = A(J) \in]i, k] \subseteq I$ and $B(I) = j < A(J) = B(J)$ contradicts the acyclicity of A . We conclude from Proposition 3.15 that the orientation C of \mathbb{I} obtained by flipping i to ℓ is acyclic. This implies that $A \xrightarrow{-ij} B$ is not a cover relation as it is obtained transitively from $A \xrightarrow{-i\ell} C$ and $C \xrightarrow{-\ell j} B$.

Conversely, assume that $A \leq B$ is not a cover relation. Then there is a flip $A \xrightarrow{-i\ell} C$, where C is an acyclic orientation of \mathbb{I} such that $A < C < B$. Since $A \xrightarrow{-ij} B$ is a flip, we have $A(K) = C(K) = B(K)$ for any $K \in \mathbb{I}$, except if $A(K) = i$ and $B(K) = j$. Hence, there is $K \in \mathbb{I}$ such that $A(I) = i < C(I) = \ell < j = B(I)$. Since $A \xrightarrow{-i\ell} C$ is a flip, we obtain from Proposition 3.15 that there is no $J \in \mathbb{I}$ with $\ell \in J \setminus \{A(J)\}$ and $A(J) \in]i, k] \subseteq]i, j]$. Hence, $\ell \in]i, j[\setminus \bigcup_{A(J) \in]i, j]} J \setminus \{A(J)\}$. \square

4. INTERVAL HYPERGRAPHIC LATTICES

In this section, we prove Theorem A which we first introduce properly:

Definition 4.1. An interval hypergraph \mathbb{I} is **closed under intersection** if $I, J \in \mathbb{I}$ and $I \cap J \neq \emptyset$ implies $I \cap J \in \mathbb{I}$.

Definition 4.2. A poset P is a **lattice** if any subset of P admits a **join** (least upper bound) and a **meet** (greatest lower bound).

Theorem A. *For an interval hypergraph \mathbb{I} on $[n]$ (with our convention that $\{i\} \in \mathbb{I}$ for all $i \in [n]$), the poset $P_{\mathbb{I}}$ is a lattice if and only if \mathbb{I} is closed under intersection.*

Example 4.3. The interval hypergraphic posets of Figure 4 are not lattices, while those of Figures 5 to 8 are lattices.

Example 4.4. The hypergraphic poset of $\{1, 2, 3, 12, 23, 123\}$ (resp. of $\{1, 2, 3, 12, 123\}$) is a lattice. In general, the hypergraphic poset of all intervals (resp. all initial intervals) is the Tamari lattice (resp. the boolean lattice). See Figures 5 and 6 (bottom right).

Example 4.5. The hypergraphic posets of $\{1, 2, 3, 4, 123, 234\}$ and of $\{1, 2, 3, 4, 12, 123, 1234, 234, 34\}$ are not lattices. In general, the hypergraphic poset of all initial and final intervals is not a lattice. See Figure 4.

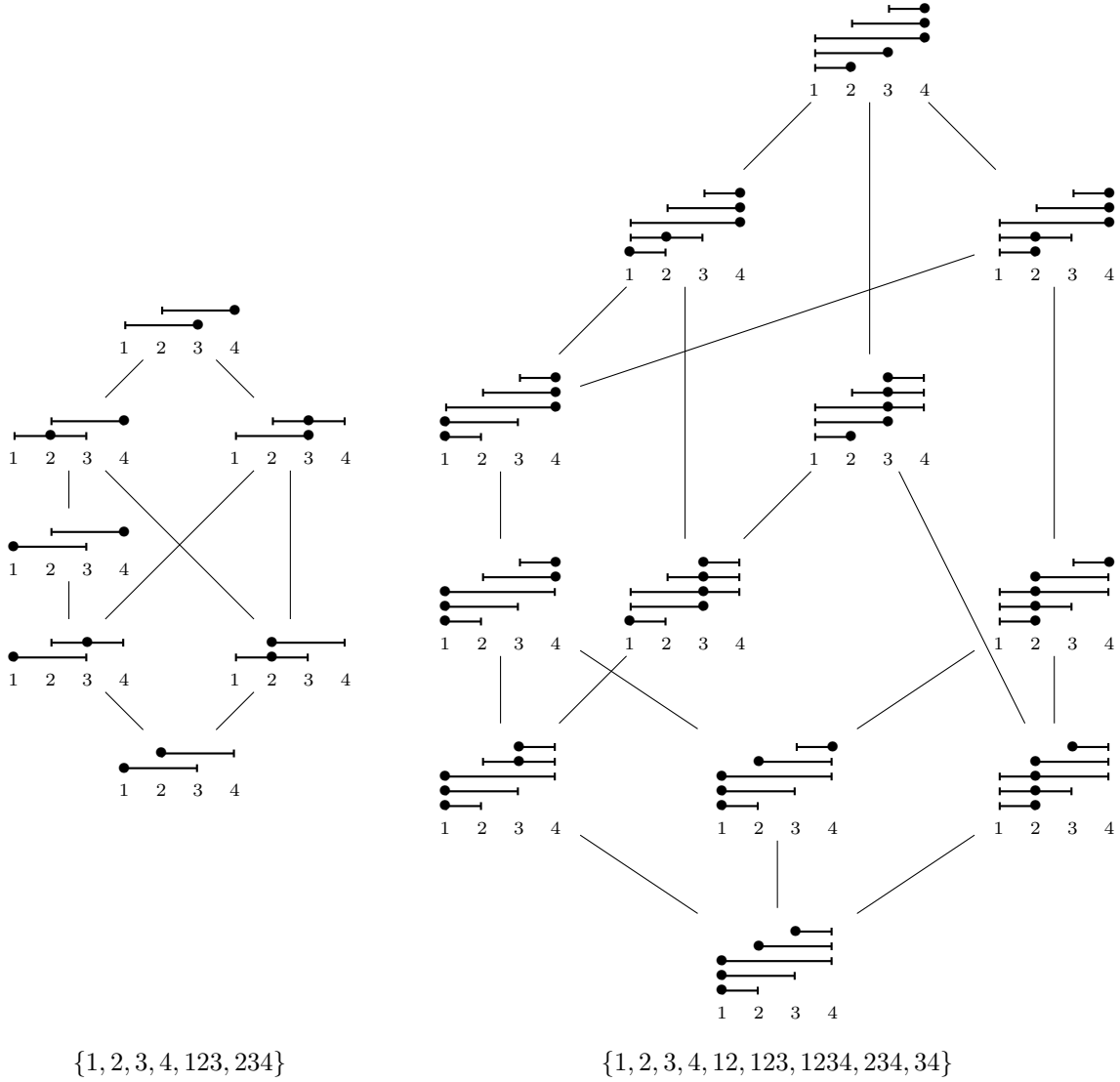


FIGURE 4. Two interval hypergraphic posets which are not lattices.

4.1. **If $P_{\mathbb{I}}$ is a lattice, then \mathbb{I} is closed under intersection.** We are now ready to show the forward implication of Theorem A. See Figure 4 for an illustration.

Proposition 4.6. *If \mathbb{I} is an interval hypergraph such that the poset $P_{\mathbb{I}}$ is a lattice, then \mathbb{I} is closed under intersection.*

Proof. See Example 4.7 for the smallest example of the proof. By contradiction, assume we have $I, J \in \mathbb{I}$ such that $\emptyset \neq I \cap J \notin \mathbb{I}$. Define $1 \leq a < b < c < d \leq n$ by

$$a := b - 1, \quad b := \min(I \cap J), \quad c := \max(I \cap J), \quad d := c + 1.$$

Note that $b \neq c$ since \mathbb{I} contains all singletons, hence $1 \leq a < b < c < d \leq n$. By symmetry, we assume that $a \in I \setminus J$ and $d \in J \setminus I$. Let X be the word formed by the complement of $\{a, b, c, d\}$ in $[n]$ written in increasing order. We now construct four permutations

$$\pi_A := bacdX, \quad \pi_B := acdbX, \quad \pi_C := dbacX, \quad \pi_D := cdbaX,$$

$K = [c, d] = I \cap J \notin \mathbb{I}$, a contradiction. This shows that $E = \mathcal{O}_{\pi_E} = \mathcal{O}_{\pi_F} = F$. Now we have $\pi_B < \pi_E$ and $\pi_F < \pi_C$ which gives $B < E = F < C$.

If the poset $P_{\mathbb{I}}$ is a lattice and $A \leq C$, $A \leq D$, $B \leq C$ and $B \leq D$, there is M such that $A \leq M$, $B \leq M$, $M \leq C$ and $M \leq D$ (anything between $A \vee B$ and $C \wedge D$ works). Let π_M be any permutation such that $M = \mathcal{O}_{\pi_M}$ and let

$$m = \pi(\min \{i \mid \pi_M(i) \in I \cup J\}).$$

If $m < b$, then $M(I) = m < b = A(I)$ and the easy forward implication of Proposition 3.13 implies that $A \not\leq M$. By similar arguments $b \leq m < c$ would imply $B \not\leq M$, $b < m \leq c$ would imply $M \not\leq C$, and $c < m$ would imply $M \not\leq D$. This is a contradiction to the existence of m . \square

Example 4.7. For the hypergraph $\{1, 2, 3, 4, 123, 234\}$ illustrated in Figure 4 (left), we have $a = 1$, $b = 2$, $c = 3$, $d = 4$ and

$$A = \begin{array}{c} \bullet \\ \hline 1 \quad 2 \quad 3 \quad 4 \end{array} \quad B = \begin{array}{c} \bullet \\ \hline 1 \quad 2 \quad 3 \quad 4 \end{array} \quad C = \begin{array}{c} \bullet \\ \hline 1 \quad 2 \quad 3 \quad 4 \end{array} \quad \text{and} \quad D = \begin{array}{c} \bullet \\ \hline 1 \quad 2 \quad 3 \quad 4 \end{array}.$$

We have $A, B \leq C, D$ and there is no M with $A, B \leq M \leq C, D$, so that A and B have no join, and C and D has no meet.

Remark 4.8. Proposition 4.6 fails when \mathbb{H} is not an interval hypergraph. For instance, the hypergraph $\mathbb{H} := \{1, 2, 3, 4, 5, 1234, 2345, 23, 24, 34\}$ is not closed under intersection, while its hypergraphic poset $P_{\mathbb{H}}$ is a lattice.

4.2. Properties of $P_{\mathbb{I}}$ when \mathbb{I} is closed under intersection. For the backward implication of Theorem A, we need to investigate the properties of interval hypergraphs that are closed under intersection. Recall from Proposition 3.10 that the fiber $\mathcal{O}^{-1}(A)$ of any acyclic orientation A of \mathbb{I} is an interval $[\pi_A^\downarrow, \pi_A^\uparrow]$ in the weak order. In the following we will keep this convention that π_A^\downarrow and π_A^\uparrow respectively denote the minimum and the maximum of the interval $\mathcal{O}^{-1}(A)$.

Theorem 4.9. *If \mathbb{I} is an interval hypergraph closed under intersection, then the following are equivalent for two acyclic orientations A and B of \mathbb{I} :*

- (a) $A \leq B$ in $P_{\mathbb{I}}$,
- (b) $A(I) \leq B(I)$ for all $I \in \mathbb{I}$,
- (c) $\pi_A^\downarrow \leq \pi_B^\uparrow$ in the weak order,
- (d) $i \triangleright_A j$ implies $i \not\triangleleft_B j$ for all $i < j$.

Proof. The equivalence (a) \iff (b) was established in Proposition 3.13.

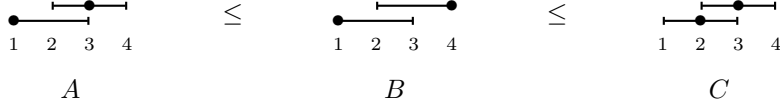
For the equivalence (c) \iff (d), Propositions 3.8 and 3.10 show that the inversion set of π_A^\downarrow is $\{(j, i) \mid i < j, i \triangleright_A j\}$ while the inversion set of π_B^\uparrow is $\{(j, i) \mid i < j, i \not\triangleleft_B j\}$. The equivalence of (c) \iff (d) thus follows from the characterization of the weak order in terms of inclusion of inversion sets.

The implication (c) \implies (a) follows from Corollary 2.15. Thus we only need to prove (b) \implies (d).

For a contradiction, assume that $A(I) \leq B(I)$ for all $I \in \mathbb{I}$ and that we have some $i < j$ such that $i \triangleright_A j$ and $i \triangleleft_B j$. Choose one such pair with $j - i$ minimal. Since $i \triangleright_A j$ there are $I_1, \dots, I_a \in \mathbb{I}$ such that $j = A(I_1)$, $A(I_{p+1}) \in I_p$ for all $p \in [a - 1]$, and $i \in I_a$. Since $i \triangleleft_B j$ there are $J_1, \dots, J_b \in \mathbb{I}$ such that $i = B(J_1)$, $B(J_{q+1}) \in J_q$ for all $q \in [b - 1]$, and $j \in J_b$. Note that $i \triangleright_A A(I_p)$ for all $p \in [a]$ and that $B(J_q) \triangleleft_B j$ for all $q \in [b]$. Moreover, as $\bigcup_{p \in [a]} I_p$ and $\bigcup_{q \in [b]} J_q$ are both intervals containing i and j , we have $i \triangleleft_B k \triangleright_A j$ for all $i \leq k \leq j$. By minimality of $j - i$, we thus obtain that $A(I_p) \notin]i, j[$ for $p \in [a]$ and $B(J_q) \notin]i, j[$ for $q \in [b]$. Hence, $]i, j[$ is contained in some I_p and some J_q . As A and B are acyclic, we obtain that $a = 1 = b$.

We can thus assume that we have $I, J \in \mathbb{I}$ such that $i = B(J) \in I$ and $j = A(I) \in J$. As \mathbb{I} is closed under intersection, we have $\{i, j\} \subseteq K := I \cap J \in \mathbb{I}$. We have $A(K) = j$, as otherwise $A(I) = j \in K$ and $A(K) \in K \subseteq I$ and $A(I) \neq A(K)$ would contradict the acyclicity of A . Similarly, we have $B(K) = i$. We conclude that $B(K) = i < j = A(K)$ and $K \in \mathbb{I}$ contradicting (b). \square

Remark 4.10. The implication (a) \implies (c) in Theorem 4.9 fails when \mathbb{I} is not closed under intersection. For instance, for $\mathbb{I} = \{1, 2, 3, 4, 123, 234\}$ and the acyclic orientations



we have $[\pi_A^\downarrow, \pi_A^\uparrow] = [1324, 1342]$, $[\pi_B^\downarrow, \pi_B^\uparrow] = [1423, 4132]$ and $[\pi_C^\downarrow, \pi_C^\uparrow] = [4213, 4231]$, so $\pi_A^\downarrow \not\leq \pi_C^\uparrow$.

4.3. If \mathbb{I} is closed under intersection, then $P_{\mathbb{I}}$ is a lattice. We now conclude the proof of Theorem A. This is a corollary of Theorem 4.9.

Proposition 4.11. *If \mathbb{I} is an interval hypergraph closed under intersection, then the poset $P_{\mathbb{I}}$ is a lattice where*

$$A \vee B = \mathcal{O}_{\pi_A^\downarrow \vee \pi_B^\downarrow} \quad \text{and} \quad A \wedge B = \mathcal{O}_{\pi_A^\uparrow \wedge \pi_B^\uparrow}.$$

Proof. Consider four acyclic orientations A, B, C, D of \mathbb{I} such that $A \leq C$, $A \leq D$, $B \leq C$ and $B \leq D$. Theorem 4.9 implies that $\pi_A^\downarrow \leq \pi_C^\uparrow$, $\pi_A^\downarrow \leq \pi_D^\uparrow$, $\pi_B^\downarrow \leq \pi_C^\uparrow$ and $\pi_B^\downarrow \leq \pi_D^\uparrow$. Hence

$$\pi_A^\downarrow \leq \pi_A^\downarrow \vee \pi_B^\downarrow, \quad \pi_B^\downarrow \leq \pi_A^\downarrow \vee \pi_B^\downarrow, \quad \pi_A^\downarrow \vee \pi_B^\downarrow \leq \pi_C^\uparrow, \quad \pi_A^\downarrow \vee \pi_B^\downarrow \leq \pi_D^\uparrow,$$

which implies by Corollary 2.15 that

$$A \leq \mathcal{O}_{\pi_A^\downarrow \vee \pi_B^\downarrow}, \quad B \leq \mathcal{O}_{\pi_A^\downarrow \vee \pi_B^\downarrow}, \quad \mathcal{O}_{\pi_A^\downarrow \vee \pi_B^\downarrow} \leq C, \quad \mathcal{O}_{\pi_A^\downarrow \vee \pi_B^\downarrow} \leq D.$$

This implies that A and B admit a join

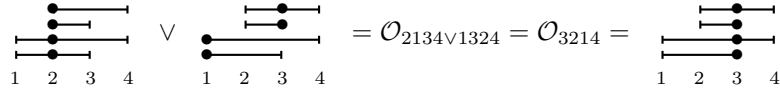
$$A \vee B \leq \mathcal{O}_{\pi_A^\downarrow \vee \pi_B^\downarrow}.$$

As $A \leq A \vee B$ and $B \leq A \vee B$, Theorem 4.9 ensures that $\pi_A^\downarrow \leq \pi_{A \vee B}^\uparrow$ and $\pi_B^\downarrow \leq \pi_{A \vee B}^\uparrow$, so that $\pi_A^\downarrow \vee \pi_B^\downarrow \leq \pi_{A \vee B}^\uparrow$. By Corollary 2.15, this implies

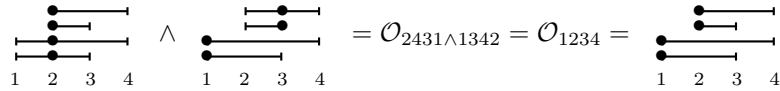
$$\mathcal{O}_{\pi_A^\downarrow \vee \pi_B^\downarrow} \leq A \vee B.$$

This shows the result for the join \vee . The proof for the meet \wedge is similar. \square

Example 4.12. For the hypergraph $\{1, 2, 3, 4, 123, 23, 234, 1234\}$ of Example 3.3 and Figure 3, we have



and



Remark 4.13. Proposition 4.11 fails when \mathbb{H} is not an interval hypergraph. For instance, the hypergraph $\mathbb{H} := \{1, 2, 3, 4, 12, 13, 24, 34\}$ is closed under intersection, while its hypergraphic poset $P_{\mathbb{H}}$ is not a lattice. More generally, all hypergraphs \mathbb{H} with $|H| \leq 2$ for all $H \in \mathbb{H}$ are closed under intersection, and the graphical zonotopes whose oriented skeleta are lattices were characterized in [Pil24].

Remark 4.14. In fact, our proof of Proposition 4.11 is a general statement about **quasi lattice maps**. Namely, if L is a lattice and P a poset, and $\Psi : L \rightarrow P$ is a poset morphism such that $\Psi^{-1}(A) = [\pi_A^\downarrow, \pi_A^\uparrow]$ is an interval for all $A \in P$, and $A \leq B$ implies $\pi_A^\downarrow \leq \pi_B^\uparrow$, then P is a lattice where $A \vee B = \Psi(\pi_A^\downarrow \vee \pi_B^\downarrow)$ and $A \wedge B = \Psi(\pi_A^\uparrow \wedge \pi_B^\uparrow)$. Note that lattice maps satisfy the stronger condition that $A \leq B$ implies $\pi_A^\downarrow \leq \pi_B^\downarrow$ and $\pi_A^\uparrow \leq \pi_B^\uparrow$. It would be interesting to characterize the quasi lattice maps of the weak order. Note that the lattice maps of the weak order were described by N. Reading in [Rea04, Rea15].

Proposition 4.15. *For an interval hypergraph \mathbb{I} closed under intersection, any acyclic orientations A_1, \dots, A_q of \mathbb{I} , and any $I \in \mathbb{I}$, we have*

$$\left(\bigvee_{p \in [q]} A_p \right)(I) = \min \left(I \setminus \left(\bigcup_{p \in [q]} \bigcup_{\substack{J \in \mathbb{I} \\ A_p(J) \in I}} [\min(J), A_p(J)[\right) \right).$$

Proof. Let O be the orientation of \mathbb{I} defined by the formula of the statement. We first prove that O is acyclic. Otherwise, Proposition 3.6 ensures the existence of $I, J \in \mathbb{I}$ such that $O(I) \in J \setminus \{O(J)\}$ and $O(J) \in I \setminus \{O(I)\}$. Assume by symmetry that $O(I) < O(J)$. As $O(I) \in J$ and $O(J) > O(I)$, there exists $p \in [q]$ and $K \in \mathbb{I}$ such that $A_p(K) \in J$ and $\min(K) \leq O(I) < A_p(K) \leq O(J)$. As I is an interval containing $O(I)$ and $O(J)$, and $O(I) < A_p(K) \leq O(J)$, it also contains $A_p(K)$. We thus obtain that $A_p(K) \in I$ and $\min(K) \leq O(I) < A_p(K)$ contradicting our definition of O .

We now prove that $O = \bigvee_{p \in [q]} A_p$. For $p \in [q]$ and $I \in \mathbb{I}$, we have $A_p(I) \in I$, hence $O(I) \geq A_p(I)$, so that $O \geq A_p$ by Proposition 3.13. Consider now an acyclic orientation O' of \mathbb{I} such that $A_p \leq O'$ for all $p \in [q]$. Assume that there are $I, J \in \mathbb{I}$ and $p \in [q]$ such that $A_p(J) \in I$. Then $A_p(J) \in I \cap J \in \mathbb{I}$, so that $A_p(I \cap J) = A_p(J)$ by acyclicity of A_p . By Proposition 3.13, we thus obtain $O'(I \cap J) = A_p(J)$. By acyclicity of O' , this implies that $O'(I) \notin [\min(I \cap J), A_p(J)[$, so that $O'(I) \notin [\min(J), A_p(J)[$. We conclude that $O(I) \leq O'(I)$. Hence, $O \leq O'$ by Proposition 3.13. \square

5. DISTRIBUTIVE INTERVAL HYPERGRAPHIC LATTICES

In this section, we prove Theorem B which we first introduce properly:

Definition 5.1. We say that an interval hypergraph \mathbb{I} is **distributive** if for all $I, J \in \mathbb{I}$ such that $I \not\subseteq J$, $I \not\supseteq J$ and $I \cap J \neq \emptyset$, the intersection $I \cap J$ is in \mathbb{I} and is initial or final in any $K \in \mathbb{I}$ with $I \cap J \subseteq K$.

Definition 5.2. A lattice (L, \leq, \vee, \wedge) is **distributive** if

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad \text{and} \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

for all $a, b, c \in L$ (the two conditions are in fact equivalent). Equivalently, L is distributive if and only if it is isomorphic to the inclusion lattice on lower sets of its poset of join irreducible elements.

Theorem B. *For an interval hypergraph \mathbb{I} on $[n]$ (with our convention that $\{i\} \in \mathbb{I}$ for all $i \in [n]$), the poset $P_{\mathbb{I}}$ is a distributive lattice if and only if \mathbb{I} is distributive.*

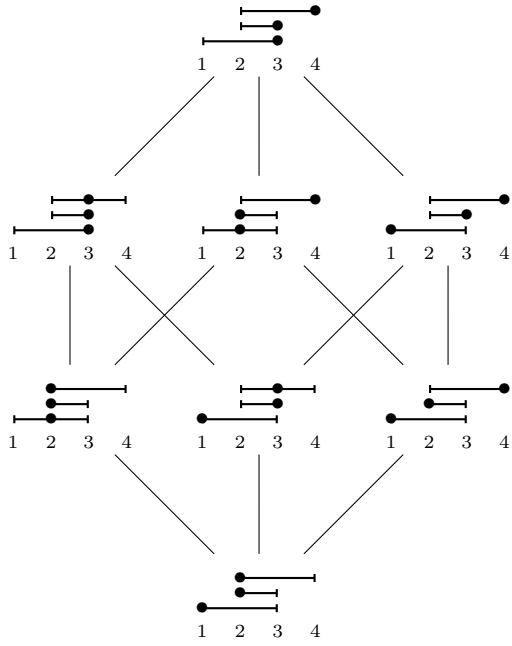
Example 5.3. The interval hypergraphic posets of Figure 6 are distributive lattices, while those of Figures 4, 5, 7 and 8 are not.

Example 5.4. The hypergraphic poset of $\{1, 2, 3, 12, 123\}$ is a distributive lattice (it is square). In general, the hypergraphic poset of all initial intervals is the boolean lattice, which is distributive. See Figure 6 and Section 5.4 for more examples of distributive hypergraphs.

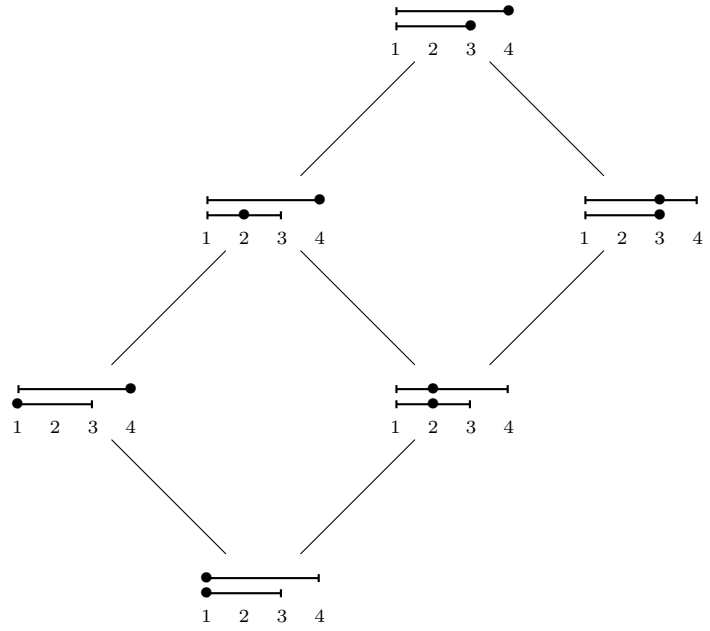
Example 5.5. The hypergraphic poset of $\{1, 2, 3, 12, 23, 123\}$ is a semidistributive but not distributive lattice (it is a pentagon). In general, the hypergraphic poset of all intervals is the Tamari lattice, which is semidistributive but not distributive. See Figure 5.

5.1. Some join irreducible acyclic orientations. In this section, we assume that \mathbb{I} is an interval hypergraph closed under intersection so that the hypergraph poset $P_{\mathbb{I}}$ is a lattice by Proposition 4.11. Our first task will be to identify some join irreducible acyclic orientations of \mathbb{I} (a complete but more technical description of all join irreducible acyclic orientations of \mathbb{I} will appear later in Section 6.1). The following notations are illustrated in Examples 5.10 to 5.14 below.

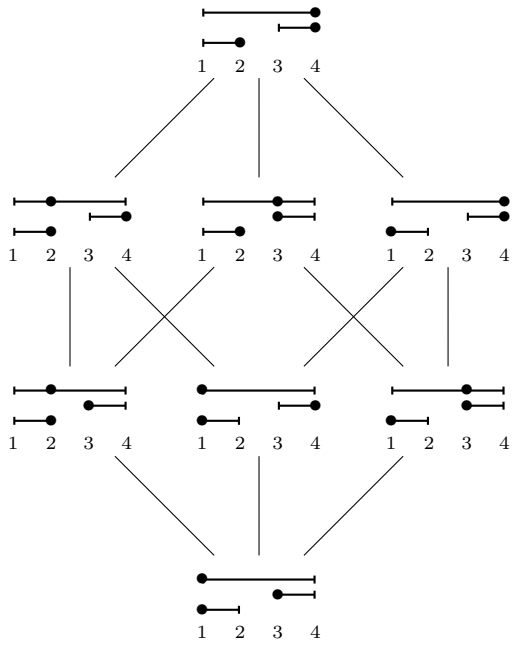
Notation 5.6. Define $\mathcal{J}_{\mathbb{I}} := \bigcup_{I \in \mathbb{I}} I \setminus \{\min(I)\}$.



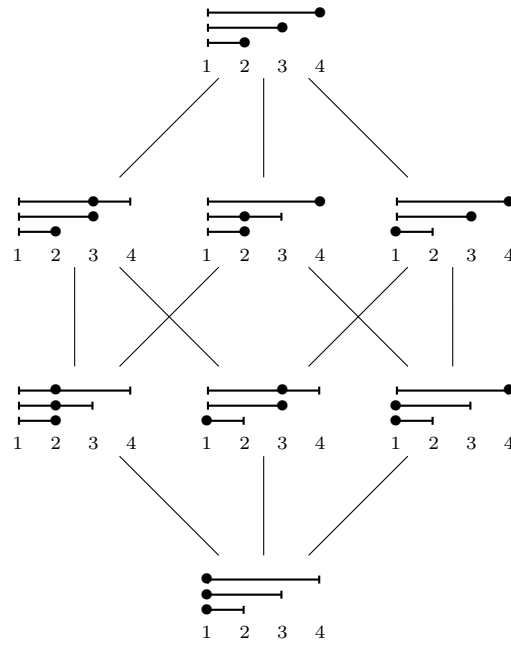
$\{1, 2, 3, 4, 123, 23, 234\}$
boolean lattice



$\{1, 2, 3, 4, 123, 1234\}$
distributive lattice



$\{1, 2, 3, 4, 12, 34, 1234\}$
boolean lattice



$\{1, 2, 3, 4, 12, 123, 1234\}$
boolean lattice

FIGURE 6. Four distributive interval hypergraphic lattices.

Notation 5.7. For $j \in \mathcal{J}_{\mathbb{I}}$, we let

$$J_j = \bigcap_{\substack{I \in \mathbb{I} \\ j \in I \setminus \{\min(I)\}}} I,$$

(note that $J_j \in \mathbb{I}$ since \mathbb{I} is closed under intersection), and we set $\mu_j := \min(J_j)$ and $\nu_j := \max(J_j)$. By definition $\min(I) \leq \mu_j < j \leq \nu_j \leq \max(I)$ for any $I \in \mathbb{I}$ such that $j \in I \setminus \{\min(I)\}$.

Notation 5.8. For $j \in \mathcal{J}_{\mathbb{I}}$, consider the acyclic orientation $A_j := \mathcal{O}_{(\mu_j, \mu_j+1, \dots, j)}$, obtained as the image by the surjection map \mathcal{O} of Definition 2.13 of the cycle permutation

$$(\mu_j, \mu_j + 1, \dots, j) = 12 \cdots (\mu_j - 1)j \mu_j \cdots (j - 1)(j + 1) \cdots n$$

obtained from the identity permutation by placing j just before μ_j . Note that that for all $J \in \mathbb{I}$

$$A_j(J) = \begin{cases} j & \text{if } j \in J \text{ and } \min(J) = \mu_j, \\ \min(J) & \text{otherwise.} \end{cases}$$

Notation 5.9. For $i, j \in \mathcal{J}_{\mathbb{I}}$, we write $i \preceq j$ if and only if $J_i = J_j$ and $i \leq j$.

Example 5.10. For the hypergraph $\mathbb{I} = \{1, 2, 3, 12, 123\}$, we have $\mathcal{J}_{\mathbb{I}} = \{2, 3\}$ and the corresponding acyclic orientations are

$$A_2 = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \end{array} \quad \text{and} \quad A_3 = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \end{array}$$

and are incomparable.

Example 5.11. For the hypergraph $\mathbb{I} = \{1, 2, 3, 12, 23, 123\}$ illustrated in Figure 5 (left), we have $\mathcal{J}_{\mathbb{I}} = \{2, 3\}$ and the corresponding acyclic orientations are

$$A_2 = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \end{array} \quad \text{and} \quad A_3 = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \end{array}$$

and are incomparable.

Example 5.12. For the hypergraph $\mathbb{I} := \{1, 2, 3, 12, 23, 34, 123, 234, 1234\}$ illustrated in Figure 5 (right), we have $\mathcal{J}_{\mathbb{I}} = \{2, 3, 4\}$, and the corresponding acyclic orientations are

$$A_2 = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \quad A_3 = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \quad \text{and} \quad A_4 = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array}$$

and are incomparable.

Example 5.13. For the hypergraph $\mathbb{I} := \{1, 2, 3, 4, 123, 23, 234\}$ illustrated in Figure 6 (top left), we have $\mathcal{J}_{\mathbb{I}} = \{2, 3, 4\}$, and the corresponding acyclic orientations are

$$A_2 = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \quad A_3 = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \quad \text{and} \quad A_4 = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array}$$

and are incomparable.

Example 5.14. For the hypergraph $\mathbb{I} := \{1, 2, 3, 4, 123, 1234\}$ illustrated in Figure 6 (top right), we have $\mathcal{J}_{\mathbb{I}} = \{2, 3, 4\}$, and the corresponding acyclic orientations are

$$A_2 = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \quad A_3 = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \quad \text{and} \quad A_4 = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array}$$

and we have $A_2 < A_3$.

Lemma 5.15. For $i \neq j \in \mathcal{J}_{\mathbb{I}}$, we have $A_i \neq A_j$.

Proof. If $i \neq j$, then we have $A_i(J_j) \in \{\mu_j, i\}$ while $A_j(J_j) = j \notin \{\mu_j, i\}$. \square

Lemma 5.16. For $i, j \in \mathcal{J}_{\mathbb{I}}$, we have $A_i \leq A_j \iff i \preceq j$.

Proof. By Proposition 3.13 we have $A_i \leq A_j$ if and only if $A_i(I) \leq A_j(I)$ for all $I \in \mathbb{I}$. The forward direction is thus a direct consequence of the following four observations:

- if $\mu_i \neq \mu_j$, then $A_i(J_i) = i > \mu_i = \min(J_i) = A_j(J_i)$,
- if $\nu_i < \nu_j$, then $j \notin J_i \setminus \{\mu_i\}$, so that $A_i(J_i) = i > \mu_i = \min(J_i) = A_j(J_i)$,
- if $i > j$, then $A_i(J_i) = i > j \geq A_j(J_i)$,
- $i \leq j$ implies that $\mu_i \leq \mu_j$ and $\nu_i \leq \nu_j$.

For the backward direction, assume $i \preccurlyeq j$, and consider $I \in \mathbb{I}$. As $i < j$, $A_i(I) \in \{\min(I), i\}$ and $A_j(I) \in \{\min(I), j\}$, we have $A_i(I) \leq A_j(I)$ except if $A_i(I) = i$ and $A_j(I) = \min(I)$. As $A_i(I) = i$, we would have $i \in I$ and $\min(I) = \mu_i$. As $A_j(I) = \min(I)$ and $\min(I) = \mu_i = \mu_j$, we would have $j \notin I$. This contradicts the fact that $\nu_i = \nu_j$. \square

Lemma 5.17. *For any $j \in \mathcal{J}_\mathbb{I}$ and any acyclic orientation A of \mathbb{I} , we have*

- $A \leq A_j \iff A = \min(P_\mathbb{I})$ or $A = A_i$ with $i \preccurlyeq j$,
- $A_j \leq A \iff j \leq A(J_j)$.

Proof. By Proposition 3.13, $A \leq A_j$ implies that $A(I) \leq A_j(I)$ for all $I \in \mathbb{I}$. For $I, I' \in \mathbb{I}$ such that $j \in I \cap I'$ and $\min(I) = \min(I') = \mu_j$, we have $A(I) = A(I')$ since A is acyclic. If $A < A_j$, we conclude that there is $i \in [\mu_j, j]$ such that $A(I) = i$ if $j \in I$ and $\min(I) = \mu_j$, and $A(I) = \min(I)$ otherwise. Hence, $A = \min(P_\mathbb{I})$ or $A = A_i$. The first point thus follows from Lemma 5.16.

For the second point, $A_j \leq A$ implies $j = A_j(J_j) \leq A(J_j)$ by Proposition 3.13. Conversely, if $j \leq A(J_j)$, then for any $I \in \mathbb{I}$,

- if $j \in I$ and $\min(I) = \mu_j$, we have $A_j(I) = j \leq A(I)$ (by acyclicity of A),
- otherwise, $A_j(I) = \min(I) \leq A(I)$.

Hence, $A_j \leq A$ by Proposition 3.13. \square

Proposition 5.18. *If \mathbb{I} is an interval hypergraph closed under intersections, then the acyclic orientation A_j is join irreducible for any $j \in \mathcal{J}_\mathbb{I}$.*

Proof. By Lemma 5.17, the lower set of A_j in $P_\mathbb{I}$ is the chain $\min(P_\mathbb{I}) < A_{i_1} < \dots < A_{i_p} < A_j$ where $\{i_1 < \dots < i_p\} = \{i \in \mathcal{J}_\mathbb{I} \mid J_i = J_j \text{ and } i < j\}$. Hence, A_j is join irreducible. \square

Corollary 5.19. *If \mathbb{I} is an interval hypergraph closed under intersections, then the map $j \mapsto A_j$ is an injective poset morphism from $(\mathcal{J}_\mathbb{I}, \preccurlyeq)$ to the subposet of join irreducibles of $P_\mathbb{I}$.*

Proof. Immediate from Lemmas 5.15 and 5.16 and Proposition 5.18. \square

In the next two propositions, we assume \mathbb{I} is closed under intersections.

Proposition 5.20. *For any two lower sets X and Y of $(\mathcal{J}_\mathbb{I}, \preccurlyeq)$, if $\bigvee_{x \in X} A_x = \bigvee_{y \in Y} A_y$ then $X = Y$.*

Proof. Assume $\bigvee_{x \in X} A_x = \bigvee_{y \in Y} A_y$ and let $x \in X$. Then $x = A_x(J_x) \leq (\bigvee_{x \in X} A_x)(J_x) = (\bigvee_{y \in Y} A_y)(J_x)$.

From the description of the join of Proposition 4.15, we thus obtain that there exists $y \in Y$ and $J \in \mathbb{I}$ such that $A_y(J) \in J_x$ and $\min(J) < x \leq A_y(J)$. As $\min(J) \neq A_y(J)$, we obtain that $A_y(J) = y$ and $\min(J) = \mu_y$. We get that $\mu_y < x \leq y \leq \nu_x$, hence that $x \in J_y \setminus \{\mu_y\}$ and $y \in J_x \setminus \{\mu_x\}$ so that $J_x = J_y$. As $x \leq y$, we conclude that $x \preccurlyeq y$, so that $x \in Y$. We thus obtained that $X \subseteq Y$, and thus $X = Y$ by symmetry. \square

Proposition 5.21. $\max(P_\mathbb{I}) = \bigvee_{j \in \mathcal{J}_\mathbb{I}} A_j$.

Proof. Assume by contradiction $\max(P_\mathbb{I}) \neq \bigvee_{j \in \mathcal{J}_\mathbb{I}} A_j =: A$. Then there is $I \in \mathbb{I}$ with $A(I) < \max(I)$. Let $j := A(I) + 1$. As $j \in I \setminus \{\min(I)\}$, we have $A(J_j) \in J_j \subseteq I$ and $A(I) = j - 1 \in J_j$. As $A_j \leq A$, we have $A(I) = j - 1 < j = A_j(J_j) \leq A(J_j)$ so that $A(I) \neq A(J_j)$. We thus obtain that A is cyclic, a contradiction. \square

Corollary 5.22. *For any interval hypergraph \mathbb{I} closed under intersection, $P_\mathbb{I}$ contains a distributive sublattice containing $\min(P_\mathbb{I})$ and $\max(P_\mathbb{I})$.*

5.2. If \mathbb{I} is distributive then $P_{\mathbb{I}}$ is distributive. We now prove the backward direction of Theorem B. The following strengthen Proposition 3.16.

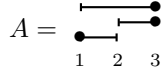
Proposition 5.23. *For a distributive interval hypergraph \mathbb{I} , an acyclic orientation A of \mathbb{I} , and $J \in \mathbb{I}$ such that $j := A(J) \neq \min(J)$, there exists i such that*

$$\min(J) \leq i \leq \max \{ \max(I) \mid I \in \mathbb{I}, \min(I) = \min(J), \max(I) < j \}$$

and the orientation obtained from A by flipping j to i is acyclic.

Proof. Let $I \in \mathbb{I}$ be maximal such that $\min(I) = \min(J)$ and $\max(I) < j$, and let $i := A(I)$. Let O be the orientation obtained from A by flipping j to i as in Definition 2.6. If O is cyclic, there exists $K, K' \in \mathbb{I}$ such that $O(K) \in K'$, $O(K') \in K$, and $O(K) \neq O(K')$. As A is acyclic, we have either $A(K) \neq O(K)$ or $A(K') \neq O(K')$, but not both since $O(K) \neq O(K')$. We can thus assume that $A(K) = O(K)$ while $A(K') = j$ and $O(K') = i$. If $j \in K$, then we have $A(K) \neq j = A(K')$ (otherwise, $O(K) = i = O(K')$), and $A(K) = O(K) \in K'$ and $A(K') = j \in K$, contradicting the acyclicity of A . As $i \in K$ and $j \notin K$, we obtain that $\max(K) < j$. As $A(I) = i \in K$ and $A(K) = O(K) \neq O(K') = i = A(I)$, we have $K \not\subseteq I$ by acyclicity of A . If $\min(I) < \min(K)$, then $I \not\subseteq K$ and $I \not\supseteq K$ and $I \cap K \ni i$, and K is neither initial nor final in J , contradicting the distributivity of \mathbb{I} . If $\min(K) < \min(J)$ and $\min(K') < \min(J)$, then we have $J \not\subseteq K$ and $J \not\supseteq K$ and $J \cap K \ni i$, and $J \cap K$ is neither initial nor final in K' , contradicting the distributivity of \mathbb{I} . As \mathbb{I} is closed under intersection and we have $A(K) = O(K) \in K'$, we can thus assume that $\min(I) = \min(K)$. We thus obtain that $K \subseteq I$. As $A(K) = O(K) \neq O(K') = i = A(I)$, this contradicts the acyclicity of A . We conclude that O is acyclic, which proves the statement. \square

Remark 5.24. Proposition 5.23 fails when \mathbb{I} is not distributive. For instance, for the interval hypergraph $\mathbb{I} = \{1, 2, 3, 12, 23, 123\}$ of Example 5.11 and Figure 5 (left), for the interval $J = 23$ and for the acyclic orientation



Proposition 5.25. *If \mathbb{I} is a distributive interval hypergraph, then the map $j \mapsto A_j$ is a poset isomorphism from $(\mathcal{J}_{\mathbb{I}}, \preceq)$ to the subposet of join irreducibles of $P_{\mathbb{I}}$.*

Proof. We have already seen in Corollary 5.19 that $j \mapsto A_j$ is an injective poset morphism. We thus just need to show that the distributivity of \mathbb{I} implies the surjectivity of this morphism. Consider an increasing flip $A \xrightarrow{-ij} B$. If there is $J \in \mathbb{I}$ such that $B(J) \notin \{\min(J), j\}$, then $A(J) = B(J)$ and Proposition 3.16 ensures that B admits a decreasing flip flipping $B(J)$ to some $k < B(J) = A(J)$. If there is $J \in \mathbb{I}$ such that $B(J) = j$ and $\min(J) < \mu_j$, then the distributivity of \mathbb{I} implies that $\max(I) < \mu_j$ for any $I \in \mathbb{I}$ such that $\min(I) = \min(J)$ and $\max(I) < j$ (otherwise, $I \not\subseteq J_j$ and $I \not\supseteq J_j$ and $\mu_j \in I \cap J_i \subseteq J$, and $I \cap J_j$ is neither initial nor final in J , contradicting the distributivity of \mathbb{I}). Hence, Proposition 5.23 ensures that B admits a decreasing flip flipping j to some $k < \mu_j$. As B is acyclic, there is no $J \in \mathbb{I}$ such that $B(J) = \min(J) = \mu_j$ and $j \in J$. We conclude that if B admits a single decreasing flip, then $B(J) = j$ if $j \in J$ and $\min(J) = \mu_j$, and $B(J) = \min(J)$ otherwise, so that $B = A_j$. \square

Remark 5.26. We will see in Lemma 5.29 that the surjectivity in Proposition 5.25 systematically fails when \mathbb{I} is not distributive. See Example 5.30 for an example.

Proposition 5.27. *If \mathbb{I} is a distributive interval hypergraph, then the maps*

$$\Phi : A \mapsto \{j \in \mathcal{J}_{\mathbb{I}} \mid A_j \leq A\} \quad \text{and} \quad \Psi : X \mapsto \bigvee_{x \in X} A_x$$

are inverse bijections between the acyclic orientations of \mathbb{I} and the lower sets of $(\mathcal{J}_{\mathbb{I}}, \preceq)$. Hence $P_{\mathbb{I}}$ is a distributive lattice.

Proof. In a finite lattice, any element can always be written as the join of the join irreducible elements below it. Proposition 5.25 thus implies that $\Psi(\Phi(A)) = A$. The statement follows since Ψ is injective by Proposition 5.20. \square

Corollary 5.28. *If \mathbb{I} is a distributive hypergraph, then $P_{\mathbb{I}}$ is a Cartesian product of chains.*

Proof. The poset $(\mathcal{J}_{\mathbb{I}}, \preceq)$ is a disjoint union of chains, hence its lattice of lower sets is a Cartesian product of chains. \square

5.3. If $P_{\mathbb{I}}$ is distributive then \mathbb{I} is distributive. We now prove the forward direction of Theorem B. We first show that the surjectivity in Proposition 5.25 systematically fails when \mathbb{I} is not distributive.

Lemma 5.29. *If an interval hypergraph \mathbb{I} is closed under intersection but not distributive, there is a join irreducible acyclic orientation A such that $A \not\preceq A_j$ for all $j \in \mathcal{J}_{\mathbb{I}}$.*

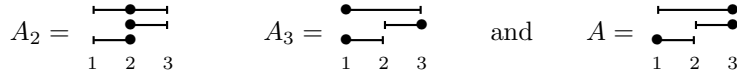
Proof. As \mathbb{I} is closed under intersection but not distributive, there are $I, J, K \in \mathbb{I}$ with $I \not\subseteq J, I \not\supseteq J, \emptyset \neq I \cap J \subseteq K$, and $I \cap J$ is neither initial nor final in K . By symmetry, we can assume that $\min(I) < \min(J) \leq \max(I) < \max(J)$. As $I \cap J$ is neither initial nor final in K , we have $\min(K) < \min(J) \leq \max(I) < \max(K)$. As \mathbb{I} is closed under intersection, we can even assume that $\min(I) = \min(K)$ and $\max(J) = \max(K)$. Let $i := \min(I) = \min(K)$ and $j := \min(J \setminus I)$. Let $A := \mathcal{O}_{(i, \dots, j)}$ be the acyclic orientation of \mathbb{I} obtained as the image by the surjection map \mathcal{O} of Definition 2.13 of the cycle permutation $(i, \dots, j) = 12 \cdots (i-1)j i \cdots (j-1)(j+1) \cdots n$. In other words,

$$A(K) = \begin{cases} j & \text{if } j \in J \text{ and } \min(J) \geq i \\ \min(J) & \text{otherwise.} \end{cases}$$

We claim that the flip of j to i is the only decreasing flip from A . Let B be an acyclic orientation of \mathbb{I} obtained by a decreasing flip from A . Let $H \in \mathbb{I}$ be such that $B(H) < A(H)$. Since $A(H) \in \{\min(H), j\}$, we have $A(H) = j$, so that $i \leq \min(H) \leq B(H) < A(H) = j$. If $B(H) \neq i$, then $B(K) = B(H) \in I$ (since $B(H) \in K$ and $A(K) = j$) and $B(I) \leq A(I) = \min(I) = i \in K$ (since $j \notin I$), contradicting the acyclicity of B . We conclude that $B(H) = i$, so that B is indeed obtained from A by flipping j to i .

Finally, as $A(K) = j$ while $A_j(K) = \min(K) < j$, we have $A \not\preceq A_j$ for all $j \in \mathcal{J}_{\mathbb{I}}$ by Proposition 3.13. \square

Example 5.30. Following on Remark 5.24, the join irreducible acyclic orientations of the interval hypergraph $\mathbb{I} = \{1, 2, 3, 12, 23, 123\}$ of Example 5.11 and Figure 5 (left) are



Proposition 5.31. *If \mathbb{I} is an interval hypergraph such that $P_{\mathbb{I}}$ is a distributive lattice, then \mathbb{I} is distributive.*

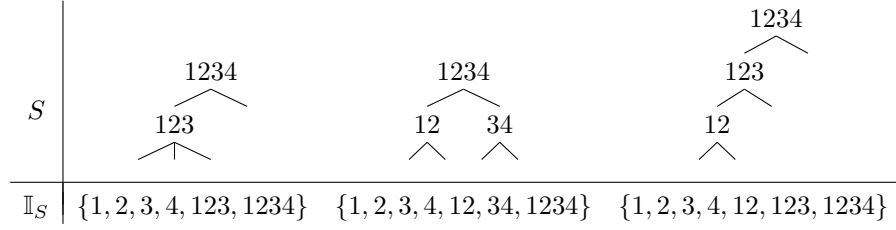
Proof. If \mathbb{I} is not closed under intersection, then $P_{\mathbb{I}}$ is not even a lattice by Proposition 4.6. If \mathbb{I} is closed under intersection but not distributive, consider the join irreducible acyclic orientation A of Lemma 5.29. As $A \not\preceq A_j$ for all $j \in \mathcal{J}_{\mathbb{I}}$, and $\max(P_{\mathbb{I}}) = \bigvee_{j \in \mathcal{J}_{\mathbb{I}}} A_j = A \vee \bigvee_{j \in \mathcal{J}_{\mathbb{I}}} A_j$, we obtain that $P_{\mathbb{I}}$ is not distributive. \square

Corollary 5.32. *For an interval hypergraph \mathbb{I} , the poset $P_{\mathbb{I}}$ is a distributive lattice if and only if it is a Cartesian product of chains.*

5.4. Schröder hypergraphs. As an illustration of this section, we now consider a special family of distributive hypergraphs which were already considered in [Def23].

Definition 5.33. Let S be a **Schröder tree** (i.e. a rooted plane tree where each internal node has at least two children) with n leaves labeled by $[n]$ from left to right. Label each node of S by the set of leaves of its subtree. We say that the hypergraph \mathbb{I}_S formed by all singletons and the labels of the nodes of S is a **Schröder hypergraph**.

Example 5.34. All but the top left interval hypergraphs of Figure 6 are Schröder hypergraphs. The corresponding Schröder trees are given by:



Example 5.35. For instance, for the left comb C , the Schröder hypergraph \mathbb{I}_C consists of all singletons and all initial intervals. The hypergraphic polytope $\Delta_{\mathbb{I}_C}$ is the Pitman–Stanley polytope [SP02].

Proposition 5.36. *An interval hypergraph \mathbb{I} is a Schröder hypergraph if and only if $I \subseteq J$ or $I \supseteq J$ or $I \cap J = \emptyset$ for all $I, J \in \mathbb{I}$.*

Proof. Consider first two nodes i and j of a Schröder tree S , and let I and J be the corresponding labels. Then $I \subseteq J$ if i is a descendant of j , $I \supseteq J$ if i is an ancestor of j , and $I \cap J = \emptyset$ otherwise.

Conversely, consider an interval hypergraph \mathbb{I} where any two hyperedges are nested or disjoint. Then the inclusion poset on \mathbb{I} is a Schröder tree S with $\mathbb{I}_S = \mathbb{I}$. \square

Remark 5.37. Proposition 5.36 implies that Schröder hypergraphs are building sets, so that Schröder hypergraphic polytopes are nestohedra [FS05, Pos09]. In fact, the latter were already considered in the literature under the name **fertilotopes** [Def23].

Corollary 5.38. *Any Schröder hypergraph is a distributive interval hypergraph, hence the Schröder hypergraphic posets are distributive lattices.*

Proof. The distributivity condition of Definition 5.1 is clearly fulfilled as any $I \subseteq J$ or $I \supseteq J$ or $I \cap J = \emptyset$ for all $I, J \in \mathbb{I}$. \square

Proposition 5.39. *Given a Schröder tree S , the poset of join irreducible acyclic orientations on \mathbb{I}_S is isomorphic to a disjoint union of chains. More precisely, it has one chain for each node n of S , whose elements are the leaves of n and the leftmost leaves of the children of n , except the leftmost leaf below n .*

Proof. This description is a specialization of the description of the join irreducible poset of distributive interval hypergraphic posets from Section 5.1. \square

6. SEMIDISTRIBUTIVE INTERVAL HYPERGRAPHIC LATTICES

In this section, we prove Theorem C which we first introduce properly:

Definition 6.1. We say that an interval hypergraph \mathbb{I} is **join semidistributive** if it is closed under intersection and for all $[r, r'], [s, s'], [t, t'], [u, u'] \in \mathbb{I}$ such that $r < s \leq r' < s'$, $r < t \leq s' < t'$, $u < \min(s, t)$ and $s' < u'$, there is $[v, v'] \in \mathbb{I}$ such that $v < s$ and $s' < v' < t'$.

Definition 6.2. A lattice (L, \leq, \vee, \wedge) is **join semidistributive** if

$$a \vee b = a \vee c \quad \text{implies} \quad a \vee (b \wedge c) = a \vee b$$

for all $a, b, c \in L$. Equivalently, L is join semidistributive if and only if for any cover relation $a < b$ in L , the set $\{c \in L \mid a \vee c = b\}$ admits a unique minimal element $k_{a < b}$. The meet semidistributivity is defined dually. A lattice L is **semidistributive** if it is both meet and join semidistributive.

Remark 6.3. In general, the set $\{c \in L \mid a \vee c = b\}$ might have more than one minimal element. Note however that any minimal element of $\{c \in L \mid a \vee c = b\}$ is always join irreducible. Indeed, Assume that $a \vee c = b$ and $c = d \vee e$. Then $a \neq a \vee d$ or $a \neq a \vee e$ since $a < b = a \vee c = a \vee (d \vee e) = (a \vee d) \vee (a \vee e)$. Moreover, $a \vee d \leq b$ and $a \vee e \leq b$ since $a \leq b$ and $d \leq c \leq b$ and $e \leq c \leq b$. As $a < b$ is a cover relation of L , we conclude that $a \vee d = b$ or $a \vee e = b$ so that c is not minimal.

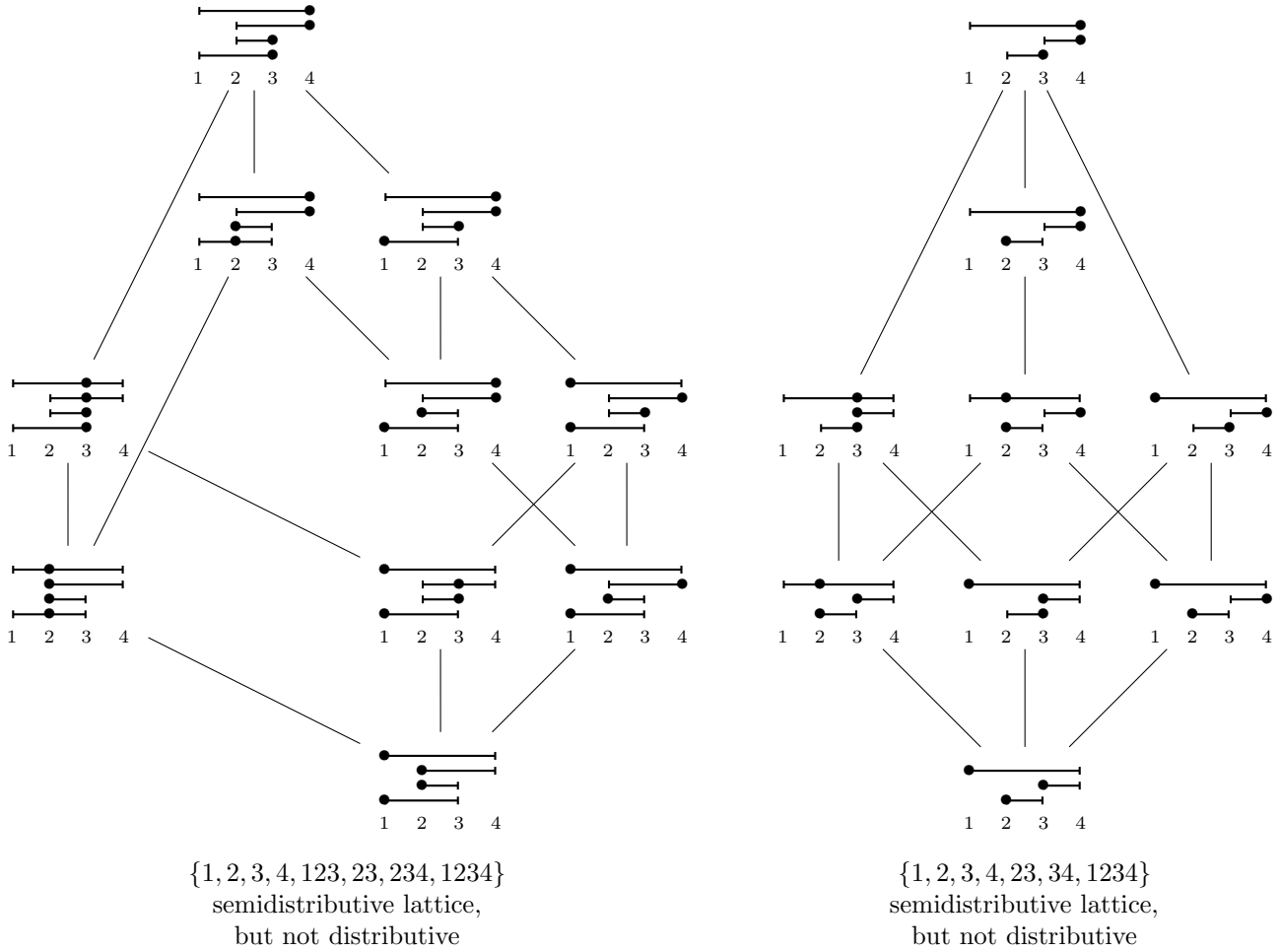


FIGURE 7. Two interval hypergraphic lattices which are semidistributive but not distributive.

Remark 6.4. Although we will not insist on this aspect in this paper, the join semidistributivity is equivalent to the existence of canonical join representations. A **join representation** of $b \in L$ is a subset $J \subseteq L$ such that $b = \bigvee J$. Such a representation is **irredundant** if $b \neq \bigvee J'$ for any strict subset $J' \subsetneq J$. The irredundant join representations of $b \in L$ are ordered by $J \leq J'$ if and only if for any $j \in J$ there is $j' \in J'$ with $j \leq j'$. The **canonical join representation** of b is the minimal irredundant join representation of b for this order when it exists. The lattice L is semidistributive if and only if any element of L admits a canonical join representation. Moreover, the canonical join representation of b is given by $b = \bigvee_{a < b} k_{a < b}$, where a ranges over all elements of L covered by b .

Theorem C. For an interval hypergraph \mathbb{I} on $[n]$ (with our convention that $\{i\} \in \mathbb{I}$ for all $i \in [n]$), the poset $P_{\mathbb{I}}$ is a join semidistributive lattice if and only if \mathbb{I} is join semidistributive. Under the symmetry of Proposition 2.12, a dual characterization holds for meet semidistributive.

Example 6.5. Figure 6 shows four distributive (hence semidistributive) interval hypergraphic lattices. Figure 7 shows two interval hypergraphic lattices which are semidistributive but not distributive. Figure 8 shows two interval hypergraphic lattices which are not semidistributive.

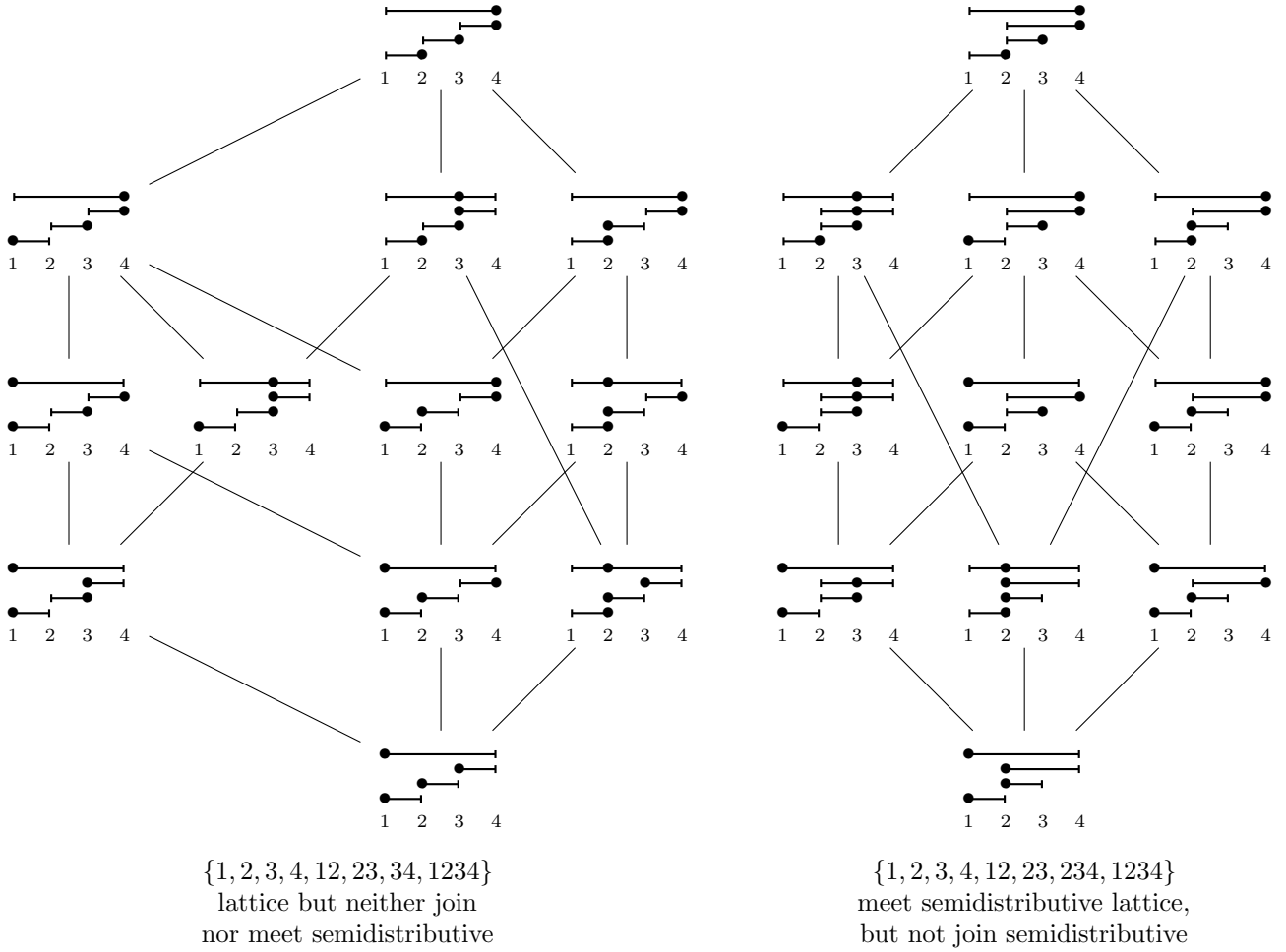


FIGURE 8. Two interval hypergraphic lattices which are not semidistributive.

Example 6.6. The hypergraphic poset of $\{1, 2, 3, 12, 23, 123\}$ is a semidistributive (but not distributive) lattice (it is a pentagon). In general, the hypergraphic poset of all intervals is the Tamari lattice, which is semidistributive but not distributive. See Figure 5.

6.1. All join irreducible acyclic orientations. In this section, we assume that \mathbb{I} is an interval hypergraph closed under intersection so that the hypergraph poset $P_{\mathbb{I}}$ is a lattice by Proposition 4.11, and we describe all the join irreducible elements of $P_{\mathbb{I}}$. The following notations are illustrated in Examples 6.12 to 6.17 below.

Notation 6.7. For $1 \leq i < j \leq n$ such that there exists $I \in \mathbb{I}$ with $\{i, j\} \subseteq I$, we let

$$J_{ij} = \bigcap_{\substack{I \in \mathbb{I} \\ \{i, j\} \subseteq I}} I$$

(note that $J_{ij} \in \mathbb{I}$ since \mathbb{I} is closed under intersection), and we set $\mu_{ij} := \min(J_{ij})$ and $\nu_{ij} := \max(J_{ij})$. By definition $\min(I) \leq \mu_{ij} \leq i < j \leq \nu_{ij} \leq \max(I)$ for any $I \in \mathbb{I}$ such that $\{i, j\} \subseteq I$.

Notation 6.8. Define $\mathcal{IJ}_{\mathbb{I}}$ to be the set of pairs (i, j) where $1 \leq i < j \leq n$ are such that there exists $I \in \mathbb{I}$ with $\{i, j\} \subseteq I$ and (i, j) satisfies the relation

$$i = \max \left([\mu_{ij}, j[\setminus \bigcup_{\substack{J \in \mathbb{I} \\ J \subseteq [\mu_{ij}, j[}} (J \setminus \{\min(J)\}) \right).$$

Notation 6.9. For $(i, j) \in \mathcal{I}\mathcal{J}_{\mathbb{I}}$, consider the acyclic orientation $A_{ij} := \mathcal{O}_{(\mu_{ij}, \mu_{ij}+1, \dots, j)}$, obtained as the image by the surjection map \mathcal{O} of Definition 2.13 of the cycle permutation

$$(\mu_{ij}, \mu_{ij} + 1, \dots, j) = 12 \cdots (\mu_{ij} - 1)j \mu_{ij} \cdots (j - 1)(j + 1) \cdots n$$

obtained from the identity permutation by placing j just before μ_{ij} . Note that that for all $J \in \mathbb{I}$

$$A_{ij}(J) = \begin{cases} j & \text{if } j \in J \text{ and } \min(J) \geq \mu_{ij}, \\ \min(J) & \text{otherwise.} \end{cases}$$

Remark 6.10. In Notations 5.7 and 5.8, we defined the interval J_j , the index μ_j , and the join irreducible A_j for $j \in \mathcal{J}_{\mathbb{I}}$. The set

$$X_j = [\mu_j, j[\setminus \bigcup_{\substack{J \in \mathbb{I} \\ J \subseteq [\mu_j, j[}} (J \setminus \{\min(J)\})$$

is nonempty as it contains at least μ_j . Let $\mu_i \leq i_j := \max X_j$. Then we have

$$\mu_{ij} = \mu_j \quad \text{and} \quad J_{ij} = J_j.$$

Furthermore, the pair $(i_j, j) \in \mathcal{I}\mathcal{J}_{\mathbb{I}}$ since it satisfies all the conditions of Notation 6.8. Comparing Notation 5.8 and Notation 6.9 we see that $A_j = A_{i_j j}$. Moreover, for all $(k, j) \in \mathcal{I}\mathcal{J}_{\mathbb{I}}$, we must have $k \leq i_j$. See Examples 6.12 to 6.17 below.

Notation 6.11. For $(i, j), (k, \ell) \in \mathcal{I}\mathcal{J}_{\mathbb{I}}$, we write $(i, j) \preccurlyeq (k, \ell)$ if and only if $j \leq \ell$ and $\ell \in J_{ij}$ and $\mu_{ij} \geq \mu_{k\ell}$. Note in particular that $i \geq k$ and $j = \ell$ implies $(i, j) \preccurlyeq (k, \ell)$.

Example 6.12. For the hypergraph $\mathbb{I} := \{1, 2, 3, 12, 23, 123\}$ of Examples 5.11 and 5.30 and Figure 5 (left), we have $\mathcal{I}\mathcal{J}_{\mathbb{I}} = \{(1, 2), (2, 3), (1, 3)\}$, and the join irreducible acyclic orientations are

$$A_{12} = A_2 = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \end{array} \quad A_{23} = A_3 = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \end{array} \quad \text{and} \quad A_{13} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \end{array}$$

and we have $A_{23} < A_{13}$. Note that $A_{12} = A_2$, $A_{23} = A_3$ from Section 5.1, while $A_{13} = A$ is the join irreducible acyclic orientation not covered by the description of Section 5.1.

Example 6.13. For the hypergraph $\mathbb{I} := \{1, 2, 3, 12, 23, 34, 123, 234, 1234\}$ of Example 5.12 and Figure 5 (right), we have $\mathcal{I}\mathcal{J}_{\mathbb{I}} = \{(1, 2), (2, 3), (3, 4), (1, 3), (2, 4), (1, 4)\}$, and the join irreducible acyclic orientations are

$$\begin{array}{ccc} A_{12} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array} & A_{23} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array} & A_{34} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \\ A_{13} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array} & A_{24} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array} & A_{14} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \end{array}$$

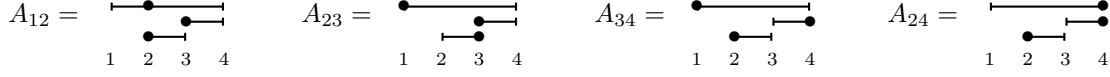
and we have $A_{23} < A_{13}$ and $A_{34} < A_{24} < A_{14}$.

Example 6.14. For the hypergraph $\mathbb{I} := \{1, 2, 3, 4, 123, 23, 234, 1234\}$ illustrated in Figure 7 (left), we have $\mathcal{I}\mathcal{J}_{\mathbb{I}} = \{(1, 2), (2, 3), (2, 4), (1, 4)\}$, and the join irreducible acyclic orientations are

$$A_{12} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \quad A_{23} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \quad A_{24} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \quad A_{14} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ 1 \quad 2 \quad 3 \quad 4 \end{array}$$

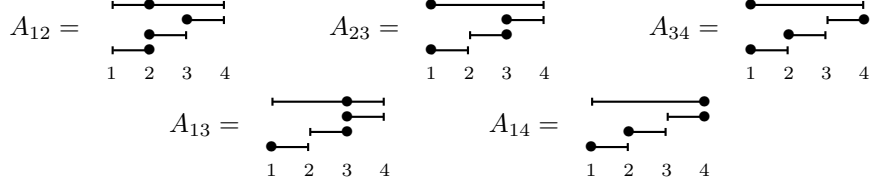
and we have $A_{24} < A_{14}$.

Example 6.15. For the hypergraph $\mathbb{I} := \{1, 2, 3, 4, 23, 34, 1234\}$ illustrated in Figure 7 (right), we have $\mathcal{I}\mathcal{J}_{\mathbb{I}} = \{(1, 2), (2, 3), (3, 4), (2, 4)\}$, and the join irreducible acyclic orientations are



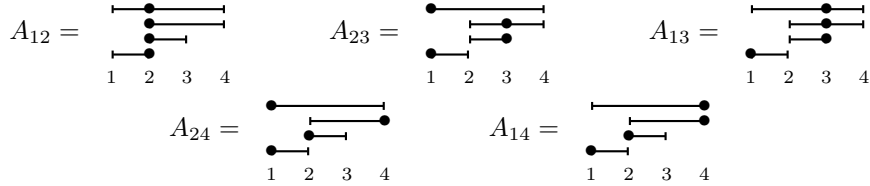
and we have $A_{12} < A_{24}$ and $A_{34} < A_{24}$.

Example 6.16. For the hypergraph $\mathbb{I} := \{1, 2, 3, 4, 12, 23, 34, 1234\}$ illustrated in Figure 8 (left), we have $\mathcal{I}\mathcal{J}_{\mathbb{I}} = \{(1, 2), (2, 3), (3, 4), (1, 3), (1, 4)\}$, and the join irreducible acyclic orientations are



and we have $A_{23} < A_{13}$ and $A_{34} < A_{14}$.

Example 6.17. For the hypergraph $\mathbb{I} := \{1, 2, 3, 4, 12, 23, 234, 1234\}$ illustrated in Figure 8 (right), we have $\mathcal{I}\mathcal{J}_{\mathbb{I}} = \{(1, 2), (2, 3), (3, 4), (1, 3), (1, 4)\}$, and the join irreducible acyclic orientations are



and we have $A_{23} < A_{13}$ and $A_{24} < A_{14}$.

Lemma 6.18. For $(i, j) \neq (k, \ell) \in \mathcal{I}\mathcal{J}_{\mathbb{I}}$, we have $A_{ij} \neq A_{k\ell}$.

Proof. Assume that $A_{ij} = A_{k\ell}$. As $j = A_{ij}(J_{ij}) = A_{k\ell}(J_{ij}) \in \{\ell, \mu_{ij}\}$ and $\mu_{ij} < j$, we obtain that $j = \ell = A_{k\ell}(J_{ij})$ hence that $\mu_{ij} \geq \mu_{k\ell}$. By symmetry, we have $\mu_{ij} = \mu_{k\ell}$. As $j = \ell$ and $\mu_{ij} = \mu_{k\ell}$, we obtain that $i = k$ by definition of $\mathcal{I}\mathcal{J}_{\mathbb{I}}$. We conclude that $(i, j) = (k, \ell)$. \square

Lemma 6.19. For $(i, j), (k, \ell) \in \mathcal{I}\mathcal{J}_{\mathbb{I}}$, we have $A_{ij} \leq A_{k\ell} \iff (i, j) \preceq (k, \ell)$. In particular $j = \ell$ implies that A_{ij} and $A_{k\ell}$ are comparable in $P_{\mathbb{I}}$.

Proof. By Proposition 3.13 we have $A_{ij} \leq A_{k\ell}$ if and only if $A_{ij}(I) \leq A_{k\ell}(I)$ for all $I \in \mathbb{I}$.

For the forward direction, assume that $A_{ij} \leq A_{k\ell}$. As $j = A_{ij}(J_{ij}) \leq A_{k\ell}(J_{ij}) \in \{\ell, \mu_{ij}\}$ and $\mu_{ij} < j$, we obtain that $j \leq \ell$ and $A_{k\ell}(J_{ij}) = \ell$, hence $\ell \in J_{ij}$ and $\mu_{ij} \geq \mu_{k\ell}$.

For the backward direction, assume that $j \leq \ell$, $\ell \in J_{ij}$ and $\mu_{ij} \geq \mu_{k\ell}$. Assume by means of contradiction that there is $I \in \mathbb{I}$ such that $A_{ij}(I) > A_{k\ell}(I)$. As $A_{ij}(I) \in \{j, \min(I)\}$ and $A_{k\ell} \in \{\ell, \min(I)\}$, and $j \leq \ell$, we must have $A_{ij}(I) = j$ and $A_{k\ell}(I) = \min(I)$. As $A_{ij}(I) = j$, we have $j \in I$ and $\mu_{ij} \leq \min(I)$, hence $\{i, j\} \subseteq [\mu_{ij}, j] \subseteq I$ so that $\ell \in J_{ij} \subseteq I$ by minimality of J_{ij} . As $A_{k\ell}(I) \neq \ell$ and $\ell \in I$, we obtain that $\min(I) < \mu_{k\ell}$. We thus obtain that $\mu_{ij} \leq \min(I) < \mu_{k\ell}$ contradicting our assumption that $\mu_{ij} \geq \mu_{k\ell}$. \square

Proposition 6.20. If \mathbb{I} is an interval hypergraph closed under intersection, then the acyclic orientation A_{ij} is join irreducible for any $(i, j) \in \mathcal{I}\mathcal{J}_{\mathbb{I}}$.

Proof. As $A_{ij} \neq \min(P_{\mathbb{I}})$, we just need to prove that it covers a single acyclic orientation in $P_{\mathbb{I}}$. We consider a cover relation $A -_{k\ell} \rightarrow A_{ij}$ in $P_{\mathbb{I}}$ and prove that $i = k$ and $j = \ell$.

By definition, there is $I \in \mathbb{I}$ such that $\{k, \ell\} \subseteq I$ and $A(I) = k$ while $A_{ij}(I) = \ell$. Since $A_{ij}(I) \in \{j, \min(I)\}$ and $A_{ij}(I) = \ell > k \geq \min(I)$, we obtain that $j = \ell$.

Assume now that $i < k$. Then there is $J \in \mathbb{I}$ with $J \subseteq [\mu_{ij}, j[$ and $k \in J \setminus \{\min(J)\}$. We obtain that $A(J) = A_{ij}(J) = \min(J) \in [\mu_{ij}, j[\subseteq I$ and $A(I) = k \in J$ and $A(J) \neq A(I)$, contradicting the acyclicity of A .

Assume now that $k < i$. Since $A^{-k\ell} \rightarrow A_{ij}$ is a cover relation, the decreasing analogue of Proposition 3.18 implies that

$$]k, \ell[\subseteq \bigcup_{\substack{J \in \mathbb{I} \\ A_{ij}(J) \in]k, \ell[}} (J \setminus \{A_{ij}(J)\}).$$

As $i \in]k, \ell[$, there is $J \in \mathbb{I}$ with $A_{ij}(J) \in]k, \ell[=]k, j[$ and $i \in J \setminus \{A_{ij}(J)\}$. As $A_{ij}(J) \in \{j, \min(J)\}$ by definition of A_{ij} , we obtain that $A_{ij}(J) = \min(J)$. If $j \leq \max(J)$, then $A_{ij}(I) = \ell \in J$ and $A_{ij}(J) \in]k, \ell[\subseteq I$ and $A_{ij}(I) \neq A_{ij}(J)$ contradict the acyclicity of A_{ij} . As \mathbb{I} is closed under intersection, we can thus assume that $J \subseteq [\mu_{ij}, j[$, up to intersecting J with J_{ij} . We conclude that $J \subseteq [\mu_{ij}, j[$ and $i \in J \setminus \{A_{ij}(J)\} = J \setminus \{\min(J)\}$, contradicting the definition of i in Notation 6.8. \square

Proposition 6.21. *If \mathbb{I} is an interval hypergraph closed under intersection, then the map $(i, j) \mapsto A_{ij}$ is a bijection from $\mathcal{I}\mathcal{J}_{\mathbb{I}}$ to the join irreducible acyclic orientations of \mathbb{I} .*

Proof. Consider a join irreducible acyclic orientation B of \mathbb{I} and let $A^{-ij} \rightarrow B$ be the only increasing flip which ends at B . Let $I \in \mathbb{I}$ be such that $A(I) = i$ and $B(I) = j$. Observe that $B(J) \in \{\min(J), j\}$ for any $J \in \mathbb{I}$ by Proposition 3.16.

We first prove that $(i, j) \in \mathcal{I}\mathcal{J}_{\mathbb{I}}$. Let

$$X := [\mu_{ij}, j[\setminus \bigcup_{\substack{J \in \mathbb{I} \\ J \subseteq [\mu_{ij}, j[}} (J \setminus \{\min(J)\}).$$

Since $B(I) = j$ and $\min(I) \leq \mu_{ij}$, Proposition 3.15 ensures that there is no $J \in \mathbb{I}$ such that $i \in J \setminus \{B(J)\}$ and $B(J) \in [\mu_{ij}, j[$, which yields $i \in X$ (using that $B(J) \in \{\min(J), j\}$). Moreover, for any $k \in]i, j[$, Proposition 3.18 ensures that there is $J \in \mathbb{I}$ such that $B(J) \in]i, j[$ and $k \in J \setminus \{B(J)\}$. Since $]i, j[\subseteq [\mu_{ij}, j[$ and $B(J) = \min(J)$, we obtain that $k \notin X$. We conclude that $i = \max(X)$, so that $(i, j) \in \mathcal{I}\mathcal{J}_{\mathbb{I}}$.

We now prove that $B = A_{ij}$. As already observed, we have $B(J) \in \{\min(J), j\}$ for any $J \in \mathbb{I}$. Assume first that there is $J \in \mathbb{I}$ such that $j \in J$ and $\min(J) \geq \mu_{ij}$, but $B(J) \neq j$. Then $B(J) = \min(J) \in [\mu_{ij}, j[\subseteq I \setminus \{B(I)\}$ and $B(I) = j \in J$, which contradicts the acyclicity of B . Assume now that there is $J \in \mathbb{I}$ such that $\min(J) < \mu_{ij}$ but $B(J) = j$, and consider such a J with $\min(J)$ minimal. For any $K \in \mathbb{I}$ with $A(K) \in [\min(J), j[$, we have $j \notin K$ by acyclicity of B , so that $A(K) = \min(K) \geq \min(J)$, thus $\min(J) \notin K \setminus \{\min(K)\}$. Proposition 3.15 thus ensures that the orientation C obtained from B by flipping j to $\min(J)$ is acyclic. Moreover, as $\min(J) < \mu_{ij}$, we have $\min(J) \notin J_{ij}$, so that $C(J_{ij}) = B(J_{ij}) = j > i = A(J_{ij})$. We obtain that $C < B$ but $C \not\leq A$ contradicting our assumption that B is join irreducible. We conclude that $B(J) = j$ if $j \in J$ and $\min(J) \geq \mu_{ij}$, and $B(J) = \min(J)$ otherwise, so that $B = A_{ij}$. \square

6.2. If \mathbb{I} is semidistributive then $P_{\mathbb{I}}$ is semidistributive. We now prove the backward direction of Theorem C.

Proposition 6.22. *If \mathbb{I} is a join (resp. meet) semidistributive interval hypergraph, then $P_{\mathbb{I}}$ is a join (resp. meet) semidistributive lattice.*

Proof. We prove the result for join semidistributivity, the result for meet semidistributivity follows by the symmetry of Proposition 2.12.

Assume by means of contradiction that \mathbb{I} is a semidistributive interval hypergraph for which $P_{\mathbb{I}}$ is not join semidistributive. By Remark 6.3, there exist a cover relation $A^{-pq} \rightarrow B$ of $P_{\mathbb{I}}$ and some $(i, j), (k, \ell) \in \mathcal{I}\mathcal{J}_{\mathbb{I}}$ such that A_{ij} and $A_{k\ell}$ are both minimal in $P_{\mathbb{I}}$ with $A \vee A_{ij} = B = A \vee A_{k\ell}$. Assume by symmetry that $j \leq \ell$. As $A_{ij} \not\leq A_{k\ell}$, we obtain by Lemma 6.19 that $j < \ell$, and that $\ell \notin J_{ij}$ or $\mu_{ij} < \mu_{k\ell}$.

As $A_{ij} \not\leq A$ but $A_{ij} \leq B$, there is $U := [u, u'] \in \mathbb{I}$ such that $A(U) < A_{ij}(U) \leq B(U)$ by Proposition 3.13. Since A and B only differ by the flip of p to q , this implies that $A(U) = p$ while $B(U) = q$. Moreover, by definition of A_{ij} , we must have $A_{ij}(U) = j$, hence $j \in U$ and $\mu_{ij} \leq u$. We conclude that $\mu_{ij} \leq u \leq p < j \leq q \leq u'$. Similarly, $\mu_{k\ell} \leq u \leq p < \ell \leq q \leq u'$.

Observe that for any $V \in \mathbb{I}$ such that $A(V) \in]p, q[$, we have

- $p < \min(V)$ as otherwise $A(U) = p \in V \setminus \{A(V)\}$ while $A(V) \in]p, q] \subseteq U \setminus \{A(U)\}$ would contradict the acyclicity of A ,
- if $j \in V$, then $A(V) = B(V) \geq A_{ij}(V) = j$.

Since $A \dashv_{pq} B$ is a cover relation of $P_{\mathbb{I}}$, Proposition 3.18 implies that there is $S := [s, s'] \in \mathbb{I}$ such that $A(S) \in]p, q]$ and $j \in S \setminus \{A(S)\}$. From the observation above, we have $p < s$ and $A(S) \geq j$. We claim that we can moreover assume that $s' < \ell$. Indeed, suppose that $s' \geq \ell$. As $\mu_{k\ell} \leq p < s \leq j < \ell \leq s'$, we obtain that $k < s \leq j < \ell$. Hence, there exists $I \in \mathbb{I}$ such that $j \in I \subseteq [k, \ell]$. As \mathbb{I} is closed under intersection, we can replace S by $S \cap I$ which proves the claim. Hence, we have found $S := [s, s'] \in \mathbb{I}$ with $p < s \leq j \leq A(S) \leq s' < \ell \leq q$, and we choose such an S so that s is minimal.

As $A(S) \in S \subseteq]p, q]$, Proposition 3.18 implies that there is $T := [t, t'] \in \mathbb{I}$ such that $A(T) \in]p, q]$ and $A(S) \in T \setminus \{A(T)\}$. We have $A(T) \geq j$, as otherwise $j \in [A(T), A(S)] \subseteq T$ and $A(T) < j$ would contradict the observation above. We deduce that $A(T) > s'$ as otherwise $A(T) \in [j, s'] \subseteq S$ and $A(S) \in T \setminus \{A(T)\}$ would contradict the acyclicity of A . Hence, we have found $T := [t, t'] \in \mathbb{I}$ with $p < t \leq s' < t'$, and we choose such a T so that t' is minimal.

As $\mu_{ij} \leq p < s$ and $j \in S$, we have $i < s$, from which we deduce that there is $R := [r, r'] \in \mathbb{I}$ such that $R \subseteq [\mu_{i,j}, j]$ and $s \in R \setminus \{r\}$. As $r' < j \leq s'$, we have $r < s \leq r' < s'$, and we choose such an R so that r is minimal. Assume for a moment that $t \leq r$. Then $t < s$, which by minimality of s in our choice of S implies that $\ell \in T$. If $\ell \notin J_{ij}$, then $J_{ij} \cap T$ contradicts the minimality of t' in our choice of T . Hence, $\ell \in J_{ij}$ which implies that $\mu_{ij} < \mu_{k\ell}$. We conclude that $i < k < j < \ell$, as $\mu_{k\ell} \leq p < t$ and $\ell \in T$. As \mathbb{I} is closed under intersection, $i < k < j < \ell$ implies that $(i, j) \in \mathcal{IJ}_{\mathbb{I}}$. As $i < k$, we have $A_{kj} < A_{ij} \leq B$. Moreover, $A_{kj}(J_{kj}) = j > \mu_{kj} = A(J_{kj})$, so that $A \not\leq A_{kj}$. Hence, $A_{kj} \vee A = B$ and $A_{kj} < A_{ij}$ contradicting the minimality of A_{ij} . We conclude that $r < t$.

We have thus found $R := [r, r']$, $S := [s, s']$, $T := [t, t']$ and $U := [u, u']$ in \mathbb{I} with $r < s \leq r' < s'$, $r < t \leq s' < t'$, and $u \leq p < \min(s, t)$ and $s' < q \leq u'$. As \mathbb{I} is join semidistributive, there is $V := [v, v'] \in \mathbb{I}$ such that $v < s$ and $s' < v' < t'$. As \mathbb{I} is closed under intersection, it contains $T \cap V = [\max(v, t), v']$. But $\max(v, t) \leq s' < v' < t'$ which contradicts the minimality of t' in our choice of T . \square

6.3. If $P_{\mathbb{I}}$ is semidistributive then \mathbb{I} is semidistributive. We now prove the forward direction of Theorem C.

Proposition 6.23. *If \mathbb{I} is an interval hypergraph such that $P_{\mathbb{I}}$ is a join (resp. meet) semidistributive lattice, then \mathbb{I} is join (resp. meet) semidistributive.*

Proof. We prove the result for join semidistributivity, the result for meet semidistributivity follows by the symmetry of Proposition 2.12.

First, we can assume that \mathbb{I} is closed under intersection, otherwise $P_{\mathbb{I}}$ is not even a lattice by Proposition 4.6. Assume now that there is $[r, r'], [s, s'], [t, t'], [u, u'] \in \mathbb{I}$ such that $r < s \leq r' < s'$, $r < t \leq s' < t'$, $u < \min(s, t)$ and $s' < u'$, and there is no $[v, v'] \in \mathbb{I}$ such that $v < s$ and $s' < v' < t'$. Let

$$j := \max \left([u, t'[\setminus \bigcup_{\substack{J \in \mathbb{I} \\ J \subseteq [u, t'[}} (J \setminus \{\min(J)\}) \right) \quad \text{and} \quad i := \mu_{jt'} := \max \{ \min(I) \mid \{j, t'\} \subseteq I \in \mathbb{I} \}.$$

Note that $i \leq j < s < s' < t'$. We further assume below that $i < j$, the argument for $i = j$ is similar and even slightly simpler.

Consider the three permutations

$$\begin{aligned} \pi_A &:= 1 \cdots (i-1) j i \cdots (j-1) (j+1) \cdots (s-1) t' s' s \cdots (s'-1) (s'+1) \cdots (t'-1) (t'+1) \cdots n \\ \pi_B &:= 1 \cdots (i-1) s' i \cdots (s'-1) (s'+1) \cdots n \\ \pi_C &:= 1 \cdots (i-1) t' i \cdots (t'-1) (t'+1) \cdots n \end{aligned}$$

(written in one line notation), and let $A := \mathcal{O}_{\pi_A}$, $B := \mathcal{O}_{\pi_B}$ and $C := \mathcal{O}_{\pi_C}$ be the corresponding acyclic orientations of \mathbb{I} .

Claim 1: $A \vee B = A \vee C$. Note that $\pi_A = \pi_A^\downarrow$. Indeed, π_A has only 3 descents ji , $t's'$ and $s's$. Let τ , σ and ρ denote the permutations obtained from π_A by exchanging ji , $t's'$ and $s's$, respectively. Then

$$\mathcal{O}_\tau(J_{jt'}) = i \neq j = X(J_{jt'}), \quad \mathcal{O}_\sigma([t, t']) = s' \neq t' = X([t, t']) \quad \text{and} \quad \mathcal{O}_\rho([s, s']) = s \neq s' = X([s, s']).$$

Hence, π_A is indeed the minimal permutation for A . Similarly, $\pi_B = \pi_B^\downarrow$ and $\pi_C = \pi_C^\downarrow$.

We therefore obtain from Proposition 4.11 that $A \vee B = \mathcal{O}_{\pi_A \vee \pi_B}$ and $A \vee C = \mathcal{O}_{\pi_A \vee \pi_C}$. As

$$\pi_A \vee \pi_B = 1 \cdots (i-1) t' s' j i \cdots (j-1) (j+1) \cdots (s'-1) (s'+1) \cdots (t'-1) (t'+1) \cdots n$$

$$\pi_A \vee \pi_C = 1 \cdots (i-1) t' j i \cdots (j-1) (j+1) \cdots (s-1) s' s \cdots (s'-1) (s'+1) \cdots (t'-1) (t'+1) \cdots n$$

and there is no $[v, v'] \in \mathbb{I}$ such that $v < s$ and $s' < v' < s'$ by assumption, we have $A \vee B = A \vee C$.

Claim 2: $A \geq B \wedge C$. For any interval I , we have

$$B(I) = \begin{cases} s' & \text{if } s' \in I \text{ and } i \leq \min(I), \\ \min(I) & \text{otherwise} \end{cases} \quad \text{and} \quad C(I) = \begin{cases} t' & \text{if } t' \in I \text{ and } i \leq \min(I), \\ \min(I) & \text{otherwise} \end{cases}$$

Hence, $(B \wedge C)(I) = \min(I)$ except maybe if $[s', t'] \subseteq I$ and $i \leq \min(I)$. For such I , we actually have that

- $(B \wedge C)(I) \leq j$ since $B(I) = s'$ and $C(I) = \min(I)$ for all $I \subseteq [u, t'[$ and using the definition of j ,
- $A(I) = j$ since $j \in I$ and $i \leq \min(I)$.

Hence, we obtain that $A(I) \geq (B \wedge C)(I)$ for all $I \in \mathbb{I}$ so that $A \geq B \wedge C$ by Proposition 3.13.

Claim 3: $A \vee B = A \vee C \neq A = A \vee (B \wedge C)$. The first equality holds by Claim 1. The last equality holds by Claim 2. The inequality holds since $A(J_{jt'}) = j$ while $(B \wedge C)(J_{jt'}) \geq C(J_{jt'}) = t'$. We conclude that $P_{\mathbb{I}}$ is not semidistributive. \square

Example 6.24. For the hypergraph $\mathbb{I} := \{1, 2, 3, 4, 12, 23, 34, 1234\}$ illustrated in Figure 8 (right), we have

$$\begin{array}{c|c|c|c|c|c|c|c|c|c} r & r' & s & s' & t & t' & u & u' & i & j \\ \hline 1 & 2 & 2 & 3 & 3 & 4 & 1 & 4 & 1 & 1 \end{array}$$

and

$$A = \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \text{---} \bullet \\ | \quad | \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \quad B = \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \text{---} \bullet \\ | \quad | \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \quad C = \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \text{---} \bullet \\ | \quad | \\ 1 \quad 2 \quad 3 \quad 4 \end{array}$$

so that

$$A \vee B = A \vee C = \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \text{---} \bullet \\ | \quad | \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \neq C = C \vee \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \text{---} \bullet \\ | \quad | \\ 1 \quad 2 \quad 3 \quad 4 \end{array} = A \vee (B \wedge C).$$

Example 6.25. For the hypergraph $\mathbb{I} := \{1, 2, 3, 4, 12, 23, 234, 1234\}$ illustrated in Figure 8 (right), we have

$$\begin{array}{c|c|c|c|c|c|c|c|c|c} r & r' & s & s' & t & t' & u & u' & i & j \\ \hline 1 & 2 & 2 & 3 & 2 & 4 & 1 & 4 & 1 & 1 \end{array}$$

and

$$A = \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \text{---} \bullet \\ | \quad | \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \quad B = \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \text{---} \bullet \\ | \quad | \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \quad C = \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \text{---} \bullet \\ | \quad | \\ 1 \quad 2 \quad 3 \quad 4 \end{array}$$

so that

$$A \vee B = A \vee C = \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \text{---} \bullet \\ | \quad | \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \neq A = A \vee \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \text{---} \bullet \\ | \quad | \\ 1 \quad 2 \quad 3 \quad 4 \end{array} = A \vee (B \wedge C).$$

7. QUOTIENT INTERVAL HYPERGRAPHIC LATTICES

In this section, we prove Theorem D which we first introduce properly:

Definition 7.1. We say that an interval hypergraph \mathbb{I} is **closed under initial** (resp. **final**) **subintervals** if $[i, k] \in \mathbb{I}$ implies $[i, j] \in \mathbb{I}$ (resp. $[j, k] \in \mathbb{I}$) for any $1 \leq i < j < k \leq n$.

Remark 7.2. Note that if an interval hypergraph \mathbb{I} containing all singletons $\{i\}$ for $i \in [n]$ is closed under initial (resp. final) subintervals, then it is also closed under intersection. Indeed, for any $1 \leq i < j < k < \ell \leq n$ with $[i, k] \in \mathbb{I}$ and $[j, \ell] \in \mathbb{I}$, we have $[j, k] \in \mathbb{I}$ since it is an initial (resp. final) interval of $[j, \ell]$ (resp. of $[i, k]$). Hence, we do not need to add the closed by intersection condition.

Definition 7.3. A map ϕ between two meet semilattices (L, \leq, \wedge) and (L', \leq', \wedge') is a **meet semilattice morphism** if $\phi(a \wedge b) = \phi(a) \wedge' \phi(b)$ for all $a, b \in L$. Equivalently, if

- each fiber F of ϕ is order convex (meaning that $a < b < c$ and $a, c \in F$ implies $b \in F$) and admits a unique minimal element,
- the map $\pi^\downarrow : L \rightarrow L$ sending an element a to the minimal element b with $\phi(a) = \phi(b)$ is order preserving.

The join semilattice morphisms are defined and characterized dually.

Theorem D. *For an interval hypergraph \mathbb{I} on $[n]$, the poset morphism \mathcal{O} from the weak order on \mathfrak{S}_n to the interval hypergraphic poset $P_{\mathbb{I}}$ is a meet (resp. join) semilattice morphism if and only if \mathbb{I} is closed under initial (resp. final) subintervals.*

Proof. We only prove the meet semilattice morphism, the result for the join semilattice morphism follows by the symmetry of Proposition 2.12.

Assume first that \mathbb{I} is not closed under initial subintervals, and let $1 \leq i < j < k \leq n$ be such that $[i, j] \notin \mathbb{I}$ while $[i, k] \in \mathbb{I}$. Note that we can assume that $k = j + 1$. Let $\omega := ijkX$, $\sigma := jikX$, $\tau := kjiX$, and $\rho := kjjX$, where X is an arbitrary permutation of $[n] \setminus \{i, j, k\}$. Since \mathbb{I} is closed under intersection and contains $[i, k]$ but not $[i, j]$ and $k = j + 1$, any $J \in \mathbb{I}$ containing $\{i, j\}$ also contains k . Hence, we have $\mathcal{O}_\tau = \mathcal{O}_\rho$, so that $\pi^\uparrow(\tau) \leq \rho$. Moreover, since $[i, k] \in \mathbb{I}$, we have $\mathcal{O}_\omega < \mathcal{O}_\sigma$. By Proposition 4.11, we obtain that $\mathcal{O}_\sigma \wedge \mathcal{O}_\tau = \mathcal{O}_{\pi^\uparrow(\sigma) \wedge \pi^\uparrow(\tau)} \geq \mathcal{O}_{\sigma \wedge \rho} = \mathcal{O}_\sigma > \mathcal{O}_\omega = \mathcal{O}_{\sigma \wedge \tau}$. Hence, \mathcal{O} is not a meet semilattice morphism.

Conversely, assume that \mathbb{I} is closed under initial subintervals. We prove by induction on the length of τ that $\sigma \leq \tau$ implies $\pi^\downarrow(\sigma) \leq \pi^\downarrow(\tau)$. If $\tau = \pi^\downarrow(\tau)$, we have $\pi^\downarrow(\sigma) \leq \sigma < \tau = \pi^\downarrow(\tau)$. If there is a permutation ρ such that $\sigma < \rho < \tau$, then we have $\pi^\downarrow(\sigma) \leq \pi^\downarrow(\rho)$ by induction, so that we just need to show that $\pi^\downarrow(\rho) \leq \pi^\downarrow(\tau)$. We can thus assume that $\sigma < \tau$ is a cover relation, with $\tau \neq \pi^\downarrow(\tau)$. Let $\tau' \triangleleft \tau$ be such that $\mathcal{O}_\tau = \mathcal{O}_{\tau'}$. Let $p, q \in [n - 1]$ be such that σ (resp. τ') is obtained from τ by exchanging its entries at positions p and $p + 1$ (resp. q and $q + 1$). We distinguish three cases:

If $|p - q| > 1$. Consider $\sigma' = \sigma \wedge \tau'$. As $\mathcal{O}_\tau = \mathcal{O}_{\tau'}$, we have $\min(\tau^{-1}(I)) \neq q$ or $\tau(q + 1) \notin I$ for any $I \in \mathbb{I}$. Since $|p - q| > 1$, it implies that $\min(\sigma^{-1}(I)) \neq q$ or $\sigma(q + 1) \notin I$. Hence we obtain that $\mathcal{O}_\sigma = \mathcal{O}_{\sigma'}$. Since $\sigma' < \tau' < \tau$, we obtain by induction that $\pi^\downarrow(\sigma') \leq \pi^\downarrow(\tau')$. As $\mathcal{O}_\sigma = \mathcal{O}_{\sigma'}$ and $\mathcal{O}_\tau = \mathcal{O}_{\tau'}$, we conclude that $\pi^\downarrow(\sigma) \leq \pi^\downarrow(\tau)$.

If $p = q - 1$. Then $\sigma = XjkjY$, $\tau = XkjiY$ and $\tau' = XkijY$ for some letters $1 \leq i < j < k \leq n$ and some words X, Y on $[n]$. Let $\sigma' := XjkiY$, $\sigma'' := XijkY$ and $\tau'' := XikjY$. If $\mathcal{O}_\sigma \neq \mathcal{O}_{\sigma'}$, then there exist $1 \leq u \leq i < k \leq v \leq n$ such that $[u, v] \in \mathbb{I}$ and $q = \min(\sigma^{-1}([u, v]))$. As \mathbb{I} is closed under final subintervals, we have $[u, j] \in \mathbb{I}$ and $q = \min(\tau^{-1}([u, j]))$, which contradicts that $\mathcal{O}_\tau = \mathcal{O}_{\tau'}$. If $\mathcal{O}_{\sigma'} \neq \mathcal{O}_{\sigma''}$, then there exists $1 \leq u \leq i < j \leq v \leq n$ such that $[u, v] \in \mathbb{I}$ and $p = \min(\sigma'^{-1}([u, v]))$. This would imply that $q = (\tau^{-1}([u, v]))$, contradicting that $\mathcal{O}_\tau = \mathcal{O}_{\tau'}$. Since $\sigma'' < \tau' < \tau$, we obtain by induction that $\pi^\downarrow(\sigma'') \leq \pi^\downarrow(\tau')$. As $\mathcal{O}_\sigma = \mathcal{O}_{\sigma'} = \mathcal{O}_{\sigma''}$ and $\mathcal{O}_\tau = \mathcal{O}_{\tau'}$, we conclude that $\pi^\downarrow(\sigma) \leq \pi^\downarrow(\tau)$.

If $p = q + 1$. Then $\sigma = XkijY$, $\tau = XkjiY$ and $\tau' = XjkjY$ for some letters $1 \leq i < j < k \leq n$ and some words X, Y on $[n]$. Let $\sigma' := XikjY$, $\sigma'' := XijkY$ and $\tau'' := XjikY$. If $\mathcal{O}_\sigma \neq \mathcal{O}_{\sigma'}$, then

there exist $1 \leq u \leq i < k \leq v \leq n$ such that $[u, v] \in \mathbb{I}$ and $q = \min(\sigma^{-1}([u, v]))$. This would imply that $p = \min(\tau^{-1}([u, v]))$, which contradicts that $\mathcal{O}_\tau = \mathcal{O}_{\tau'}$. If $\mathcal{O}_{\sigma'} \neq \mathcal{O}_{\sigma''}$, then there exists $i < u \leq j < k \leq v \leq n$ such that $[u, v] \in \mathbb{I}$ and $p = \min(\sigma'^{-1}([u, v]))$. This would imply that $q = \min(\tau^{-1}([u, v]))$, contradicting that $\mathcal{O}_\tau = \mathcal{O}_{\tau'}$. Since $\sigma'' < \tau' < \tau$, we obtain by induction that $\pi^\downarrow(\sigma'') \leq \pi^\downarrow(\tau')$. As $\mathcal{O}_\sigma = \mathcal{O}_{\sigma'} = \mathcal{O}_{\sigma''}$ and $\mathcal{O}_\tau = \mathcal{O}_{\tau'}$, we conclude that $\pi^\downarrow(\sigma) \leq \pi^\downarrow(\tau)$. \square

Corollary 7.4. *For an interval hypergraph \mathbb{I} on $[n]$, the poset morphism \mathcal{O} from the weak order on \mathfrak{S}_n to the interval hypergraphic poset $P_{\mathbb{I}}$ is a lattice morphism if and only if \mathbb{I} is closed under subintervals. In this case, $P_{\mathbb{I}}$ is a Cartesian product of Tamari lattices.*

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